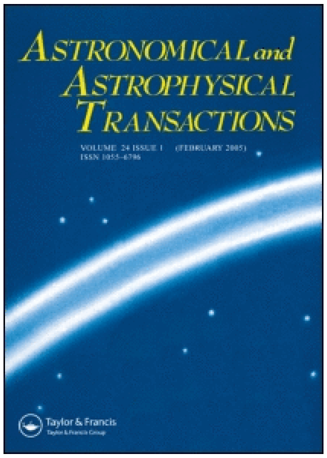

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Unperturbed chandler motion and perturbation theory of the rotation motion of deformable celestial bodies

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UNPERTURBED CHANDLER MOTION AND PERTURBATION THEORY OF THE ROTATION MOTION OF DEFORMABLE CELESTIAL BODIES

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New unperturbed motions are suggested for the study of the rotational motion of deformable celestial bodies. This motion describes the rotation of an isolated celestial body deformed by its own rotation. By some natural simplifications and by using special forms of canonical variables (similar to Andoyer's variables) the problem is reduced to the classical Euler–Poinsot problem for a rigid body, but with different moments of inertia.

The suggested unpertubed motion describes Chandler's pole motion and we shall call it Chandler or Euler–Chandler motion. The development of the unperturbed theory is described in this paper. The solution of the Chandler problem (Andoyer's variables, components of angular velocity of the body's axes, and their direction cosines) is presented in elliptical and θ-functions, and in the form of Fourier series in the angle-action variables. Similar Fourier series were obtained for products and squares of the diraction cosines. The coefficients of these series are expressed through full elliptical integrals of the first, second and third kinds with modulus which is the defining function of the action variables. It is the principal peculiarity of these series.

As an illustration we give a application of this unperturbed theory to the study of the Earth's rotation (the principal properties of the Earth's rotation and perturbations).

So, the unperturbed motion describes the following phenomena of the Earth's rotation:

(1) Chandler's motion of the pole of the Earth's axis of rotation;

(2) the ellipticity of the trajectory of the Earth's pole;

(3) the non-uniformity of the pole motion along the elliptical trajectory;

(4) the variation with Chandler's period of the modulus of the Earth's angular velocity.

Theory of the perturbed rotational motion of the Earth is constructed on the basis of the special forms of equations of the rotation of a deformable body (in angle-action variables and their modifications for the Chandler–Euler problem). For the construction of the perturbing function in these variables we use the Hamiltonian expression in Andoyer variables of Getino Ferrandiz paper (1991), in which the theory of the Earth's rotation was developed. In that paper we obtained the full trigonometric development of the second harmonic of the force function of the Earth–Moon system (and also for Earth–Sun system) in angle-action variables. The analytical formulae for perturbations of the first order in the Earth's rotation on the basis of these equations and developments were obtained.

Secular perturbations in the Earth's rotation due to second harmonics of the force function were studied (the definition of the constant of precession; constant additives to the angular velocities of the Chandler and axial motions of the Earth).

All the results of this paper are presented in analytical form and are applicable for studies of the perturbed rotational motions of other celestial bodies (Venus, asteroids, satellites etc.).

KEY WORDS Chandler's motion, Earth's rotation, perturbation theory, angle-action variation

1 INTRODUCTION

Let us study the rotational motion of a weakly deformable elastic celestial body due to the attraction of other bodies. Due to the perturbing influence of the other bodies and due to its own rotation the body undergoes tidal and centrifugal deformation as a body with concentric distribution of density. Deformations of the body are described by a classical solution (Takeuchi, 1950).

Let $Oxyz$ be a Cartesian reference system with axes directed along the principal axes of inertia for the undeformed state. We will neglect small effects due to displacement of the point O relative of the mass centre. Let $\overline{\omega}$ be the vector of the angular velocity of the reference system $Oxyz$ with components p, q and r (in the axes) w.r.t. the principal reference system $OXYZ$ with axes fixed in space. Let A_0, B_0 and C_0 be then principal moments of inertia of the body (for its indeformed state) about the axes Ox, Oy and Oz.

We will describe the rotation of the body by Andoyer's variables (Getino and Ferrandiz, 1990):

$$L,\ G,\ H,\ l,\ g,\ h, \tag{1}$$

which are connected with the angular moment vector $\overline{G}$ of the rotational motion of the body.

Here L and H are projections of the vector $\overline{G}$ on the axes Oz and OZ of the corresponding reference systems. Let ρ and θ be angles, formed by the vector $\overline{G}$ with these axes. Then $L = G\cos\theta$, $H = G\cos\rho$. Geometrical values of the other variables (1) are described in many papers (e.g. Getino and Ferrandiz, 1990, 1991a).

Andoyer's canonical variables (1) can be related to the immovable plane OXY of the main reference system, and to the moving plane (E) with a given motion. In the last case the Hamiltonian of the problem has some additional terms (Kinoshita, 1977):

$$R_E = H(1-\cos i)\frac{d\alpha}{dt} + \sqrt{G^2-H^2}\left[\sin i\cos(h-\alpha)\frac{d\alpha}{dt} - \sin(h-\alpha)\frac{di}{dt}\right], \tag{2}$$

Here i and α are inclination and longitude of the ascending node of the moving plane w.r.t. the reference system $OXYZ$. These angles are known functions of time. For example, in Earth's rotation theory (Kinoshita, 1977) the angles $\alpha = \Pi_1$,

$i = \pi_1$ define the position of the ecliptic of the data with respect to the ecliptic plane in a given epoch. They are presented by following formulae:

$$\sin\pi_1 \sin\Pi_1 = pt + p't^2 + p''t^3 = 5.''341t + 0.''1935t^2 - 0.''00019t^3$$
$$\sin\pi_1 \cos\Pi_1 = qt + q't^2 + q''t^3 = -46.''838t + 0.''563t^2 + 0.''00035t^3. \quad (3)$$

The full expression of the Hamiltonian of the problem of the rotation of a deformable body in the variables (1) was obtained by Getino and Ferrandiz (1991a) and involves the terms

$$\mathcal{H} = T_0 + T_r + T_t + R_E + E_t + E_r - U - U_t - U_r \quad (4)$$

where T_0, T_r and T_t are the kinetic energies caused by the rotation of the body and its centrifugal and tidal deformation; R_E is caused by the motion of the main reference system; E_t and E_r are the energies of the tidal and rotational deformation accumulated in the elastic body; U is the force function of the Newtonian interaction of the body with the perturbing bodies; U_t and U_r are additional force functions caused by tidal and centrifugal deformation.

For these term in (4) the necessary trigonometric expressions and developments in Andoyer's variables were obtained by Getino and Ferrandiz (1991a,b) and Kinoshita (1977).

Application of the equations of rotational motion in Andoyer's variables, for example in the theory Earth's rotation , is connected with certain mathematical difficulties in the case of small values of the angle θ and causes some artificial constructions for the description of the pole motion of the vector $\overline{G}$ (or the angular velocity vector $\overline{\omega}$).

On other hand it is well known that the main component of the Earth's pole motion is Chandler's motion and the deviation from this motion is defined by modern observations with higher accuracy (about 1 mm of the Earth's surface by a Chandler's amplitude of a few metres). From observations it follows that Chandler's motion is excited and is damped within 25–40 years IERS (1993).

At the same time for definit and sufficiently long time intervals, we can assume that the character of the Chandler's motion changes weakly (for example in the period 1983 y to 1995 y; IERS (1993)).

The above statements are the basis for the construction and application a new unperturbed rotational motion of the deformable body. This motion is Chandler's rotational motion of a weakly deformed celestial body.

The perturbation theory on the basis of this unperturbed motion is realized by compact and elegant analytical formulas, although with addition of terms of elliptic functions and elliptic integrals.

More important unperturbed motion can be used to construct rotation theories of other celestial bodies (Venus, Mars, asteroids etc.). So, for the Earth the angle θ between the angular moment vector $\overline{G}$ and the polar inertia axis is small ($\theta \sim 10^{-6}$). But for Venus it is equal to about $\sim 13°$. There are data that some asteroids have big angles θ.

The analytical theory of perturbed rotational motion constructed is first adopted to the study of the motion of these bodies. However, the application of this theory to the study of Earth's rotation also has important positive value. For example, the description and interpretation of the main Chandler effects in unperturbed motion; a full analytical description of the perturbation effects in the Earth's pole motion and in its precession and nutation.

The purpose of this paper is to construct a theory of the unperturbed rotational motion deformed by own rotation, which describes Chandler's motion of the pole of its rotation axis and to give its application for the description of the Chandler effects in the Earth's rotation.

Then we introduce angle-action variables and construct a new analytical theory of the perturbed rotational motion of the deformed body on the basis of the above unperturbed motion, and give its application to Earth's rotation theory.

The assumed unperturbed motion corresponds to classical Euler's rotational motion of the rigid body but with changed moments of inertia. This is the main feature of our approach to the problem. It lets us simplify our study and analytical constructions and gives description of the kinematical and dynamical effects in unperturbed and perturbed motions. Also this approach lets us use a wide set of investigations of the unperturbed Euler's motion. In particular here we use some well-known results of Sadov (1970) and Kinoshita (1972) and the author's very wide results of the study of Euler's problem from the unpublished Saragossa course of lecture (Barkin, 1992) "Introduction in the rotation theory of the celestial bodies", delivered for group of celestial mechanists at Saragossa University in 1992–1993.

2 HAMILTONIAN OF THE UNPERTURBED MOTION

The Hamiltonian of the unperturbed rotational motion is easily constructed on the basis of the expressions of the kinetic energies T_0, T_r from the Hamiltonian of the perturbed motion (4) (Getino and Ferrandiz, 1991a):

$$\begin{aligned} T_0 &= \frac{1}{4}G^2\left(\frac{1}{A}+\frac{1}{B}\right)+\frac{1}{4}L^2\left(\frac{2}{C}-\frac{1}{A}-\frac{1}{B}\right) \\ &+ \frac{1}{4}\left(G^2-L^2\right)\left(\frac{1}{B}-\frac{1}{A}\right)\cos 2l \end{aligned} \tag{5}$$

$$\begin{aligned} T_r &= \frac{3}{4}D_r\sin\theta_r\cos\theta_r\frac{G^2}{C}\sin 2\theta\left[\frac{1}{A}+\left(\frac{1}{B}-\frac{1}{A}\right)\cos l_r\cos l\right] \\ &+ \frac{3}{4}D_r\sin^2\theta_r\frac{G^2-L^2}{AB}\sin 2l_r\sin 2l \end{aligned} \tag{6}$$

where D_r is some parameter of the elastic rotational deformation of the body (for the Earth $D_r = -2.845379\times 10^{41}(g^2)$; Getino and Ferrandiz, 1991a).

Here expression (5) is the kinetic energy of the rotation of the body with inertia moments A, B and C, and (6) is the kinetic energy connected with the centrifugal deformation.

Here θ_r and l_r are angles analogous to Andoyer's variables θ and l defined not for the vector $\overline{G}$, but for the angular velocity vector $\overline{\omega}$.

For bodies similar to the Earth the angles θ and θ_r are small. For the Earth these angles are $\sim 10^{-6}$ and we can approximately assume $\theta_r \simeq \theta$ and $\cos(l - l_r) \simeq 1$.

Using this assumption we relate the second term T_r (it has order θ^4) to the perturbation function $\mathcal{H}_1$. From the first part of (6) we relate the function $\mathcal{H}_1$ to the following small terms:

$$\frac{3}{8} D_r \frac{G^2}{AC}(\sin 2\theta_r \sin 2\theta - 4\sin^2\theta) + \frac{3}{8} D_r \frac{G^2}{C}\left(\frac{1}{B} - \frac{1}{A}\right) \times [\sin 2\theta_r \sin 2\theta \cos l_r \cos l - 4\sin^2\theta\cos^2 l]. \tag{7}$$

The main terms of the function T_r (6):

$$T_r^{(0)} = \frac{3}{2} D_r \frac{G^2}{AC}\sin^2\theta + \frac{3}{2} D_r \frac{G^2}{C}\left(\frac{1}{B} - \frac{1}{A}\right)\sin^2\theta\cos^2 l \tag{8}$$

with the function T_0 form the Hamiltonian of the unperturbed rotational motion:

$$\mathcal{H}_0 = T_0 + T_r^{(0)}.$$

We can present this Hamiltonian in the following way:

$$\mathcal{H}_0 = \frac{G^2 - L^2}{2}\left(\frac{\sin^2 l}{\tilde{A}} + \frac{\cos^2 l}{\tilde{B}}\right) + \frac{L^2}{2\tilde{C}} \tag{9}$$

where

$$\frac{1}{\tilde{A}} = \frac{1}{A}\left(1 + \frac{3D_r}{C}\right)$$
$$\frac{1}{\tilde{B}} = \frac{1}{B}\left(1 + \frac{3D_r}{C}\right)$$
$$\frac{1}{\tilde{C}} = \frac{1}{C}. \tag{10}$$

Here the moments A, B and C are given by the sum of their values A_0, B_0 and C_0 in the absence of deformation and the corresponding corrections for centrifugal deformation:

$$A = A_0 + D_r(1 - 3\sin^2\theta_r \sin^2 l_r)$$
$$B = B_0 + D_r(1 - 3\sin^2\theta_r \sin^2 l_r)$$
$$C = C_0 + D_r(-2 + 3\sin^2\theta_r). \tag{11}$$

Finally in the unperturbed motion we save only the constant corrections $\Delta A = \Delta B = D_r$, $\Delta C = -2D_r$, and we refer additional small terms $\sim \sin^2\theta_r$ to the perturbation function $\mathcal{H}_1$.

So, in the unperturbed motion the deformed body is characterized by the constant moments of inertia:

$$\overline{A} = A_0 + D_r$$
$$\overline{B} = B_0 + D_r$$
$$\overline{C} = C_0 - 2D_r. \tag{12}$$

From the expression for $\mathcal{H}_0$ it follows that in the unperturbed motion the body rotates as a body with changed moments of inertia $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, which are defined by

$$\frac{1}{\tilde{A}} = \frac{1}{A}\left(1 + \frac{3D_r}{\overline{C}}\right)$$
$$\frac{1}{\tilde{B}} = \frac{1}{B}\left(1 + \frac{3D_r}{\overline{C}}\right)$$
$$\frac{1}{\tilde{C}} = \frac{1}{C}. \tag{13}$$

For small values of D_r and $\overline{C} \simeq \overline{A}$ (for the Earth $D_r \simeq 10^{-3}$) from equations (13) we obtain the relations:

$$\tilde{A} = \overline{A} - 3D_r = A_0 - 2D_r$$
$$\tilde{B} = \overline{B} - 3D_r = B_0 - 2D_r$$
$$\tilde{C} = \overline{C} = C_0 - 2D_r. \tag{14}$$

For Earth's model with elastic mantle studied by Getino and Ferrandiz (1991a) the following numerical values of the deformation parameter D_r and of the corrections to the moments of inertia (12) were obtained:

$$\Delta C = -2\Delta A = -2\Delta B = -2D_r = 5.690758 \times 10^{41} \text{ g cm}^2. \tag{15}$$

They are in agrement with the same numerical values of Mank and MacDonald:

$$\Delta C = -2\Delta A = -2\Delta B = \frac{2k_2 R_\oplus^5 \omega_0^2}{9G} = 5.609173 \times 10^{41} \text{ g cm}^2. \tag{16}$$

Here $k_2 = 0.3$ is Love's number, $R_\oplus$ is Earth's radius, ω_0 is the angular velocity of the Earth, and G is the gravitational constant.

As a result the Hamiltonian of the unperturbed motion is reduced to the form

$$\mathcal{H}_0 = \frac{G^2 - L^2}{2}\left(\frac{\sin^2 l}{\tilde{A}} + \frac{\cos^2 l}{\tilde{B}}\right) + \frac{L^2}{2\tilde{C}} \tag{17}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ have constant values (14).

The Hamiltonian (17) corresponds to the Euler–Poinsot problem but with changed moments of inertia $\tilde{A}$, $\tilde{B}$, $\tilde{C}$. We will call the problem with Hamiltonian (17) Chandler–Euler's problem or Chandler's problem.

The solutions of Euler's problem were studied by many authors (for example Sadov, 1970; Kinoshita, 1972) in particular by the Hamilton–Jacobi method.

Here we will use a wide set of results concerning the integration and study of Euler's problem presented in the Saragossa manuscript (Barkin, 1992) with some natural additions and creations.

From them we find:

(1) the solution of Euler's problem by the Hamiltonian–Jacobi method and the introduction of the angle-action variables I_i, φ_i;

(2) the solution in elliptical and θ-functions (Andoyer's variables, the components of the angular velocity p, q, r, direction cosines of the body axes b_{ij}, their products, squares, etc.);

(3) the dynamical and kinematical properties of the unperturbed motion;

(4) Fourier series for the canonical Andoyer variables;

(5) Fourier series for the components of the angular velocity, for direction cosines of the body's axes b_{ij}, and also for their mutual products and squares, etc.

These results were used for a full and detailed description of Chandler's motion and for the construction of an analytical theory of the perturbed rotational motion of a deformable body (in the case of arbitrary values of the parameters of the problem).

As an illustration all the obtained analytical results are used for the description of the corresponding effects in the Earth's rotation. However, we must point out that the constructed theory is sufficiently universal and can be applied to the construction of the rotational theory of the other celestial bodies (Venus, asteroids, satellites, etc.) and for arbitrary values of the angle θ.

We can formulate the results on the construction of the Hamiltonian of the unperturbed motion in the form theorem.

Theorem. The rotation of an isolated celestial body deformed by its own rotation in the first approximation is described by to Euler–Poinsot solution for some rigid body. The values of the principal and central moments of inertia for this body are equal to the principal central moments of inertia of the body in the absence of deformation, but increased by the constant correction

$$\Delta C = -2D_r > 0.$$

At same time in this approximations the moments of inertia of the body, deformed by its own rotation, are constant and equal to

$$\overline{A} = A_0 + D_r$$
$$\overline{B} = B_0 + D_r$$
$$\overline{C} = C_0 - 2D_r$$

3 GENERAL FORMULAE OF THE UNPERTURBED ROTATIONAL MOTION

Now that the problem of the unperturbed rotational motion of a deformable body has been reduced to the Euler–Poinsot problem, we can use known results of the investigations of this classical problem. Here we shall give the general formulae of the solution of this problem (without detailed introduction but in their final form).

Firstly the formulae of the transformation from Andoyer's variables to "angle-action" variables are very interesting and important, as are the expressions of the components of the angular velocity of the body p, q, r and of the direction cosines of its axes b_{ij} through the angle-action and time variables (we will denote the angle-action variables as I_i, φ_i $(i = 1, 2, 3)$).

These variables were introduced (for Euler–Poinsot problem) by different authors. From among these authors the papers of Sadov (1970) and Kinoshita (1972) stand out. Here side by side with the Sadov and Kinoshita results we shall use the results, concerning this problem, from the course of lectures by Barkin (1992). In this course a wide set of results on the investigation of the unperturbed Eulerian motion was obtained. These include: Fourier series in the angle-action variables for canonical Andoyer variables, for products and squares of the direction cosines b_{ij} and components of the angular velocity p, q, r (and also for their higher orders); geometrical and dynamical interpretation of the properties of the unperturbed motion, etc. These results are the basis of the given study.

First we give here the formulae which express the canonical Andoyer variables in the angle-action variables I_i, φ_i:

$$
\begin{aligned}
L &= I_2 \frac{\kappa}{\sqrt{\kappa^2+\lambda^2}}\, dn\, u \\
G &= I_2, \quad H = I_3 \\
l &= \arcsin\left[\frac{cn\, u}{\sqrt{1+\kappa^2 sn^2\, u}}\right] \\
g &= \varphi_2 + \frac{\sqrt{(1+\kappa^2)(\kappa^2+\lambda^2)}}{\kappa}\left[\frac{2\varphi_1}{\pi}\Pi(\frac{\pi}{2}, \kappa^2, \lambda) - \Pi(am\, u, \kappa^2, \lambda)\right] \\
h &= \varphi_3.
\end{aligned} \tag{18}
$$

Here $sn\, u$, $cn\, u$, $dn\, u$ are the elliptical Jacobi functions, $\Pi(\frac{\pi}{2}, \kappa^2, \lambda)$ and $\Pi(am\, u, \kappa^2, \lambda)$ are complete and incomplete elliptical integrals of the third kind, and u is a linear function of time:

$$u = \frac{2K(\lambda)}{\pi}\varphi_1. \tag{19}$$

The vatiables "angle" are

$$
\begin{aligned}
\varphi_1 &= n_1 t + \varphi_1^{(0)} \\
\varphi_2 &= n_2 t + \varphi_2^{(0)}
\end{aligned} \tag{20}
$$

where $\varphi_1^{(0)}$, $\varphi_2^{(0)}$ are initial values of these variables.

The frequencies of the Chandler–Euler motion are

$$n_1 = \frac{I_2(A-C)}{2AC}\frac{\pi\kappa}{\sqrt{(1+\kappa^2)(\kappa^2+\lambda^2)}K(\lambda)}$$
$$n_2 = \frac{I_2}{C}\left(1 - \frac{A-C}{A}\frac{\Pi(\frac{\pi}{2},\kappa^2,\lambda)}{K(\lambda)}\right) \tag{21}$$

where $K(\lambda)$ is an elliptical full integral of the first kind, λ is the modulus of the elliptical functions, which is defined by the initial conditions of the problem:

$$\lambda^2 = \kappa^2\frac{A^2}{C^2}\frac{p_0^2}{r_0^2}$$
$$\kappa^2 = \frac{C(A-B)}{A(B-C)} \tag{22}$$

p_0, r_0 ($q_0 = 0$) are initial values of the unperturbed components of the angular velocity p, r (and q), respectively. Here A, B and C are constant and "reduced" moments of inertia for the deformable body (see 14) $A = \tilde{A}$, $B = \tilde{B}$ $C = \tilde{C}$ (for simplicity we shall nense further omit the sign "~" for all parameters of the problem: A, B, C, κ, λ.

In the unperturbed motion I_i ($i = 1,2,3$), $\varphi_1^{(0)}$, $\varphi_2^{(0)}$ and φ_3 are constant.

The constant values of the variables G, H, h in the unperturbed motion are the mean of the conservation of the angular moments $\overline{G}$ of the deformable body. The angle ρ between the axis Oz fixed in the space and vector $\overline{G}$ has the constant value $\rho = \rho_0$:

$$\cos\rho_0 = \frac{I_3}{I_2}, \quad \sin\rho = \frac{\sqrt{I_2^2 - I_3^2}}{I_2}. \tag{23}$$

The modulus λ of the elliptical functions and integrals (18), (21) is defined as a function of the variables I_1, I_2 as the result of inversion of the following equation:

$$\frac{I_1}{I_2} = \Lambda(\lambda) \tag{24}$$

where

$$\Lambda(\lambda) = \frac{2\kappa\sqrt{1+\kappa^2}}{\pi\sqrt{\kappa^2+\lambda^2}}\left\{\frac{\kappa^2+\lambda^2}{\kappa^2}\Pi(\frac{\pi}{2},\kappa^2,\lambda) - \frac{\lambda^2}{\kappa^2}K(\lambda)\right\}. \tag{25}$$

In addition to (18)–(25) we give here the other formulae of the unperturbed motion:

$$\cos\theta = \frac{\kappa}{\sqrt{\kappa^2+\lambda^2}}dn\ u,$$
$$\sin\theta = \frac{\lambda\sqrt{1+\kappa^2 sn^2 u}}{\sqrt{\kappa^2+\lambda^2}},$$
$$\sin l = \frac{cn\ u}{\sqrt{1+\kappa^2 sn^2 u}},$$

$$\cos l = -\frac{\sqrt{1+\kappa^2}sn\ u}{\sqrt{1+\kappa^2 sn^2 u}},$$
$$\tan\theta = \frac{\lambda\sqrt{1+\kappa^2 sn^2 u}}{kdn\ u},$$
$$\tan l = -\frac{cn\ u}{\sqrt{1+\kappa^2}sn\ u}. \qquad (26)$$

Expressions for the direction cosines of the body's axes $Oxyz$ (relative to the intermediate reference system $OG_1G_2G_3$, connected with the vector $\overline{G}$) in the elliptical functions and integrals are:

$$b_{11} = \frac{-1}{\sqrt{1+\kappa^2 sn^2\ u}}\left\{\sqrt{1+\kappa^2}sn\ u\cos g + \frac{\kappa}{\sqrt{\kappa^2+\lambda^2}}dn\ u\, cn\ u\sin g\right\},$$
$$b_{21} = \frac{-1}{\sqrt{1+\kappa^2 sn^2\ u}}\left\{\sqrt{1+\kappa^2}sn\ u\sin g - \frac{\kappa}{\sqrt{\kappa^2+\lambda^2}}cn\ u\, dn\ u\cos g\right\},$$
$$b_{31} = \frac{\lambda}{\sqrt{\kappa^2+\lambda^2}}cn\ u,$$
$$b_{12} = \frac{-1}{\sqrt{1+\kappa^2 sn^2\ u}}\left\{cn\ u\ \cos g - \frac{\kappa\sqrt{1+\kappa^2}}{\sqrt{\kappa^2+\lambda^2}}sn\ u\, dn\ u\sin g\right\},$$
$$b_{22} = \frac{-1}{\sqrt{1+\kappa^2 sn^2\ u}}\left\{cn\ u\ \sin g + \frac{\kappa\sqrt{1+\kappa^2}}{\sqrt{\kappa^2+\lambda^2}}sn\ u\, dn\ u\cos g\right\},$$
$$b_{32} = -\frac{\lambda\sqrt{1+\kappa^2}}{\sqrt{\kappa^2+\lambda^2}}sn\ u,$$
$$b_{13} = \frac{\lambda}{\sqrt{\kappa^2+\lambda^2}}\sqrt{1+\kappa^2 sn^2 u}\sin g$$
$$b_{23} = -\frac{\lambda}{\sqrt{\kappa^2+\lambda^2}}\sqrt{1+\kappa^2 sn^2 u}\cos g$$
$$b_{33} = \frac{\kappa}{\sqrt{\kappa^2+\lambda^2}}dn\ u, \qquad (27)$$

where

$$g = \varphi_2 + \frac{\sqrt{(1+\kappa^2)(\kappa^2+\lambda^2)}}{\kappa}\left[\frac{2\varphi_1}{\pi}\Pi(\frac{\pi}{2},\kappa^2,\lambda) - \Pi(am\ u,\kappa^2,\lambda)\right],$$
$$u = \frac{2\varphi_1 K(\lambda)}{\pi}. \qquad (28)$$

For the components and modulus of the angular velocity of the body we have following formulae[†]

$$p = \frac{G\lambda}{A\sqrt{\kappa^2+\lambda^2}}cn\ u,$$

[†]Here we use the formal components of the angular velocity of a rigid body with moments of inertia A, B, C, but their values, for example in Earth's rotation theory, are very close to the real components of the angular velocity of the deformable body.

$$q = -\frac{\lambda G\sqrt{1+\kappa^2}}{B\sqrt{\kappa^2+\lambda^2}} sn\ u,$$
$$r = \frac{G\kappa}{C\sqrt{\kappa^2+\lambda^2}} dn\ u. \tag{29}$$

$$\omega = \frac{G}{A\sqrt{\kappa^2+\lambda^2}}\sqrt{\lambda^2+\frac{A^2}{C^2}\kappa^2-\lambda^2\left(1+\frac{A^2}{C^2}\kappa^2\right)sn^2\,u}. \tag{30}$$

The theory of the unperturbed motion includes in particular such important questions as the construction of Fourier series for different functions of the Andoyer variables (for example, for components of the angular velocity, for direction cosines b_{ij}, for their products and squares etc.). For the construction of similar series by Sadov (1970) and Barkin (1992) method based on the application of the apparatus of θ-functions of complex argument and on the theory of residues, was developed.

The fourier series in angle-action variables for direction cosines b_ij are defined by the formulae (Sadov, 1970; Barkin, 1992):

$$b_{11} = \frac{-\pi}{2K\sqrt{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\left\{\frac{\sin[(2m+1)\varphi_1+\varphi_2]}{sh[(2m+1)d-\sigma]}+\frac{\sin[(2m+1)\varphi_1-\varphi_2]}{sh[(2m+1)d+\sigma]}\right\}$$
$$b_{21} = \frac{\pi}{2K\sqrt{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\left\{\frac{\cos[(2m+1)\varphi_1+\varphi_2]}{sh[(2m+1)d-\sigma]}+\frac{\cos[(2m+1)\varphi_1-\varphi_2]}{sh[(2m+1)d+\sigma]}\right\}$$
$$b_{31} = \frac{\pi}{K\sqrt{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\left\{\frac{\cos[(2m+1)\varphi_1]}{sh[(2m+1)d]}\right\}$$
$$b_{12} = \frac{-\pi}{2K}\sqrt{\frac{1+\kappa^2}{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\left\{\frac{\cos[(2m+1)\varphi_1+\varphi_2]}{ch[(2m+1)d-\sigma]}+\frac{\cos[(2m+1)\varphi_1-\varphi_2]}{ch[(2m+1)d+\sigma]}\right\}$$
$$b_{22} = \frac{-\pi}{2K}\sqrt{\frac{1+\kappa^2}{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\left\{\frac{\sin[(2m+1)\varphi_1+\varphi_2]}{ch[(2m+1)d-\sigma]}-\frac{\sin[(2m+1)\varphi_1-\varphi_2]}{ch[(2m+1)d+\sigma]}\right\}$$
$$b_{32} = \frac{-\pi}{K}\sqrt{\frac{1+\kappa^2}{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\left\{\frac{\sin[(2m+1)\varphi_1]}{sh[(2m+1)d]}\right\}$$
$$b_{13} = \frac{\pi\kappa}{2K\sqrt{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\left\{\frac{\sin[2m\varphi_1+\varphi_2]}{sh(2md-\sigma)}+\frac{\sin[2m\varphi_1-\varphi_2]}{sh(2md+\sigma)}\right\}(1+\delta_{m0})^{-1}$$
$$b_{23} = \frac{-\pi\kappa}{2K\sqrt{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\left\{\frac{\cos[2m\varphi_1+\varphi_2]}{sh(2md-\sigma)}+\frac{\cos[2m\varphi_1-\varphi_2]}{sh(2md+\sigma)}\right\}(1+\delta_{m0})^{-1}$$
$$b_{33} = \frac{\pi\kappa}{K\sqrt{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\left\{\frac{\cos[2m\varphi_1]}{sh(2md)}\right\}(1+\delta_{m0})^{-1}, \tag{31}$$

($\delta_{00} = 1$, $\delta_{m0} = 0$ for $m \geq 1$).

Here the coefficients of the Fourier series are presented in terms of hyperbolic functions with arguments $\kappa_1 d + \kappa_2 \sigma$ ($\kappa_1,\ \kappa_2 \in N$), where d and σ are auxiliary arguments:

$$d = \frac{\pi K'}{2K} \tag{32}$$

$$\sigma = \frac{\pi}{2K} F(\arctan \frac{\kappa}{\lambda}, \sqrt{1-\lambda^2}), \tag{33}$$

where $K' = K(\lambda')$, $\lambda' = \sqrt{1-\lambda^2}$; and F is incomplete elliptical integral of the first kind.

The Fourier series for the components of the angular velocity $p,\ q,\ r$ have a similar structure:

$$p = \frac{\pi G}{AK\sqrt{\kappa^2+\lambda^2}} \sum_{m=0}^{\infty} \left\{ \frac{\cos[(2m+1)\varphi_1]}{ch[(2m+1)d]} \right\}$$

$$q = -\frac{\pi G\sqrt{1+\kappa^2}}{BK\sqrt{\kappa^2+\lambda^2}} \sum_{m=0}^{\infty} \left\{ \frac{\sin[(2m+1)\varphi_1]}{sh[(2m+1)d]} \right\}$$

$$r = \frac{\pi G\kappa}{CK\sqrt{\kappa^2+\lambda^2}} \sum_{m=0}^{\infty} \left\{ \frac{\cos[2m\varphi_1]}{ch(2md)} \right\} (1+\delta_{m0})^{-1}. \tag{34}$$

On basis of formulae (31)–(34) a wide set of other formulae for the unperturbed motion are obtained; for example, the Fourier series for the derivatives of the direction cosines and projections of the angular velocity with respect to time ($\dot{b}_{ij},\ \dot{p},\ \dot{q},\ \dot{r}$; Barkin, 1992); formulae for the directions cosines r_{ij} (and their derivatives r_{ij}) of the body's axes relative to the main refernce system $OXYZ$ etc. (Barkin, 1992).

In this paper we shall use the Fourier series for the products and squares of the direction cosines b_{ij} for the construction of the corresponding trigonometric development of the force function of the Newtonian attraction between the Earth and the Moon (also the Earth and Sun).

The Fourier series for the products and squares of the direction cosines are given in the Appendix. Here we point only the structure of these series:

$$b_{ij}b_{nk} = \sum b_{m_1,m_2}^{(i,j;n,k)} \left\{ \begin{array}{l} \cos(m_1\varphi_1 + m_2\varphi_2) \\ \sin(m_1\varphi_1 + m_2\varphi_2), \end{array} \right. \tag{35}$$

where the developments are set out only in terms of sines or only in terms of cosines. The coefficients $b_{m_1,m_2}^{(i,j;n,k)}$ are expressed through the parameters λ, κ in terms of the complete elliptical integral of the first, second and third kinds K, E and Π, and of the hyperbolic functions of the arguments d and σ (32), (33).

Here we give only secular components of the series (35):

$$b_{0.0}^{(1.1;1.1)} = \frac{K(1+\kappa^2) - E}{2K(\kappa^2+\lambda^2)}$$

$$b_{0.0}^{(1.2;1.2)} = \frac{1}{2}\left[1 + \frac{(1+\kappa^2)(E-K)}{K(\kappa^2+\lambda^2)}\right]$$

$$
\begin{aligned}
b_{0.0}^{(1.3;1.3)} &= \frac{1}{2}\left[1 - \frac{\kappa^2 E}{K(\kappa^2+\lambda^2)}\right] \\
b_{0.0}^{(2.1;1.2)} &= -\frac{\pi\kappa}{4K\sqrt{\kappa^2+\lambda^2}} \\
b_{0.0}^{(2.1;2.1)} &= \frac{K(1+\kappa^2)-E}{2K(\kappa^2+\lambda^2)} \\
b_{0.0}^{(2.2;1.1)} &= \frac{\pi\kappa}{4K\sqrt{\kappa^2+\lambda^2}} \\
b_{0.0}^{(2.2;2.2)} &= \frac{1}{2}\left[1 + \frac{(1+\kappa^2)(E-K)}{K(\kappa^2+\lambda^2)}\right] \\
b_{0.0}^{(2.3;2.3)} &= \frac{1}{2}\left[1 - \frac{\kappa^2 E}{K(\kappa^2+\lambda^2)}\right] \\
b_{0.0}^{(3.1;3.1)} &= \frac{E-\lambda'^2 K}{K(\kappa^2+\lambda^2)} \\
b_{0.0}^{(3.2;3.2)} &= -\frac{(1+\kappa^2)(E-K)}{K(\kappa^2+\lambda^2)} \\
b_{0.0}^{(3.3;3.3)} &= \frac{\kappa^2 E}{K(\kappa^2+\lambda^2)}.
\end{aligned} \tag{36}
$$

Expressions (36) and the full series (35) are satisfied by known geometrical relations between the products and squares of the direction cosines (Barkin, 1992).

Similar Fourier series for canonical Andoyer variables are (Barkin, 1992):

$$
\begin{aligned}
L &= G\frac{\pi\kappa}{K\sqrt{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\frac{\cos 2m\varphi_1}{ch\,2md}(1+\delta_{m0})^{-1}, \\
G &= I_2, \quad H = I_3 \\
l &= \varphi_1 + \sum_{m=1}^{\infty}\frac{ch\ m\sigma}{m\ ch(2md)}\sin 2m\varphi_1 \\
g &= \varphi_2 + \sum_{m=1}^{\infty}\frac{sh\ m\sigma}{m\ sh(2md)}\sin 2m\varphi_1 \\
h &= \varphi_3
\end{aligned} \tag{37}
$$

($\delta_{00} = 1,\ \delta_{m0}$ for $m \geq 1$).

$$
\begin{aligned}
d &= \frac{\pi K(\lambda')}{2K(\lambda)} \\
\sigma &= \frac{\pi}{2K(\lambda)}F(\arctan\frac{\kappa}{\lambda}, \lambda') \\
\lambda' &= \sqrt{1-\lambda^2}.
\end{aligned} \tag{38}
$$

4 MAIN PROPERTIES OF CHANDLER MOTION. UNPERTURBED CHANDLER MOTION OF THE EARTH

In this paragraph we shall consider the application of the Chandler unperturbed theory for the explanation of some properties of the Earth's rotation. For this purpose we first define the principal dynamical parameters κ, λ, etc.

Let us define the constant values of the moments of inertia $\overline{A}$, $\overline{B}$, $\overline{C}$ (12), corresponding to the reference system $Oxyz$ (to the principal axes of the Earth for its underformed state). In practice, they are some average value of the moments of inertia A, B and C, which can be calculated by the formulae:

$$\begin{aligned}\overline{A} &= \overline{C}[1-(2C_{22}-C_{20})J]\\ \overline{B} &= \overline{C}[1+(C_{20}-2C_{22})J],\end{aligned} \tag{39}$$

where C_{20}, C_{22} are the coefficients of the geopotential in the principal axes of the Earth. On the basis of the model of the geopotential (SE 3) we found:

$$\begin{aligned}C_{20} &= -1082.6370\times 10^{-6}\\ C_{22} &= 1.7711\times 10^{-6}.\end{aligned} \tag{40}$$

In (39)

$$J=\frac{mR^2}{C}=3.024086=(0.3306784)^{-1} \tag{41}$$

(where m and R are the Earth's mass and radius).

Here we took into account the fact that the principal Earth's axis Ox has swivelled to the west by an angle 14.5° relative to the corresponding Greenwich's axis.

As a result for value $\overline{C}=8.110000\times 10^{44}$ g cm^2 from (40), (41), (39) we obtain:

$$\begin{aligned}\overline{A} &= 8.083361\times 10^{44}\ \text{g cm}^2\\ \overline{B} &= 8.083535\times 10^{44}\ \text{g cm}^2\end{aligned} \tag{42}$$

We can see that $\overline{C}>\overline{B}>\overline{A}$ and the axis Ox corresponds to the smaller of these moments.

In this paper we shall use two values of the parameter D_r (see Table 1) for the definition of the parameters κ, λ for Chandler's motion. The first of them was obtained by Getino and Ferrandiz (1991):

$$D_r = -2.845379\times 10^{41}\ \text{g cm}^2 \tag{43}$$

$$\begin{aligned}\frac{\Delta C}{mR^2} &= 0.234178\times 10^{-3}\\ \frac{\Delta C}{C} &= 0.708176\times 10^{-3}.\end{aligned} \tag{44}$$

An other "empirical" value for D_r was obtained from the optimal correspondence of the theoretical Chandler period of the pole motion and its observational value:

$$D_r = -2.623000\times 10^{41}\ \text{g cm}^2 \tag{45}$$

and (15), (41):

$$\frac{\Delta C}{mR^2} = 0.215876$$
$$\frac{\Delta C}{C} = 0.652829. \tag{46}$$

We choose the initial conditions of the pole motion of the Earth's axis on the basis of the observational data (Getino and Ferrandiz, 1991a):

$$\frac{p_0}{r_0} = 0.''252273 = 1.233054 \times 10^{-6}, \quad q_0 = 0 \tag{47}$$

at the moment of time 0.0 h 15 September 1990.

For the modulus of the angular velocity of the Earth we have the following value (Barkin *et al.*, 1995):

$$\omega_0 = 7.292115 \times 10^{-5}\frac{1}{s} = 0.230117 \times 10^{6}\frac{1}{cy}. \tag{48}$$

For the initial values (47) and (48) and for the parameters (43), (45) the value of the principal parameters of the unperturbed motion κ^2, λ^2 were obtained as for Eulerian motion (for a rigid body, with the assumption $D_r = 0$) and for Chandlerian motion (for values D_r (43) and (45)).

The Eulerian values were calculated by the formulae:

$$\kappa_E^2 = \frac{\overline{C}(\overline{A}-\overline{B})}{\overline{A}(\overline{B}-\overline{C})}$$
$$\lambda_E^2 = \kappa_E^2 \frac{\overline{A}^2 p_0^2}{\overline{C}^2 r_0^2}. \tag{49}$$

Their numerical value are:

$$\kappa_E^2 = 0.659639 \times 10^{-2}$$
$$\lambda_E^2 = 0.980256 \times 10^{-14}. \tag{50}$$

For Chandler motion the parameters κ^2, λ^2 are defined by the following formulae (see the Introduction):

$$\kappa_{ch}^2 = \frac{\tilde{C}(\tilde{A}-\tilde{B})}{\tilde{A}(\tilde{B}-\tilde{C})} = \frac{\overline{C}(\overline{A}-\overline{B})}{(\overline{A}-3D_r)(\overline{B}-\overline{C}-3D_r)} \tag{51}$$

$$\lambda_{ch}^2 = \frac{(\overline{A}-3D_r)(\overline{A}-\overline{B})}{\overline{C}(\overline{B}-\overline{C}-3D_r)}\left(\frac{p_0}{r_0}\right)^2 = \kappa_{ch}^2 \frac{\tilde{A}^2}{\tilde{C}^2}\frac{p_0^2}{r_0^2}. \tag{52}$$

The calculated values of the parameters (51), (52) and κ_{ch}, λ_{ch} and many other geometrical and dynamical characteristics are presented in Table 1 in colomns CH I and CH II (for two variants of the above values of the parameter D_r).

Table 1. Main parameters and characteristics of the unperturbed rotational motion of the Earth (E – Euler's motion; CH I and CH II – Chandlers motions)

Parameters	*E*	*CH I*	*CH II*
D_r	0	-2.845379×10^{41} g cm^2	-2.623×10^{41} g cm^2
$-3D_r$	0	8.536137×10^{41} g cm^2	7.8700×10^{41} g cm^2
A	8.083361×10^{44} g cm^2	8.091897×10^{44} g cm^2	8.091231×10^{44} g cm^2
B	8.083535×10^{44} g cm^2	8.092071×10^{44} g cm^2	8.091405×10^{44} g cm^2
C	8.110000×10^{44} g cm^2	8.110000×10^{44} g cm^2	8.110000×10^{44} g cm^2
$(B-A)/C$	0.021455×10^{-3}	0.021273×10^{-3}	0.0000213
$(C-A)/C$	3.284710×10^{-3}	0.002232	0.002314
$(C-B)/C$	3.263255×10^{-3}	0.002211	0.002293
κ^2	0.659639×10^{-2}	0.964274×10^{-2}	0.931200×10^{-2}
λ^2	0.980256×10^{-14}	1.442481×10^{-14}	1.386506×10^{-14}
κ	0.812182×10^{-1}	0.981975×10^{-1}	0.964987×10^{-1}
λ	0.990079×10^{-7}	1.201033×10^{-7}	1.177500×10^{-7}
e	0.080680	0.097514	0.095835
a	$0.''252273$	$0.''252273$	$0.''252273$
	1.223054×10^{-6}	1.223054×10^{-6}	1.223054×10^{-6}
b	$0.''253104$	$0.''253486$	$0.''253445$
	1.227081×10^{-6}	1.228937×10^{-6}	1.228735×10^{-6}
n_1/ω_0	-3.285009×10^{-3}	-2.226285×10^{-3}	-2.308643×10^{-3}
n_2/ω_0	0.996715	0.997774	0.997691
T_{ch}	$304.413145d$	$449.178794d$	$433.154906d$
$\lvert\dot{l}\rvert_{\max}/\omega_0$	3.295534×10^{-3}	2.237267×10^{-3}	2.309046×10^{-3}
$\lvert\dot{l}\rvert_{\min}/\omega_0$	3.273938×10^{-3}	2.215899×10^{-3}	2.287742×10^{-3}
$\frac{\lvert\dot{l}\rvert_{\max}-\lvert\dot{l}\rvert_{\min}}{\omega_0}$	0.021596×10^{-3}	0.021368×10^{-3}	0.021304×10^{-3}
$T_{ch,\min}$	$303.440957d$	$446.973919d$	$433.079392d$
$T_{ch,\max}$	$305.442571d$	$451.284106d$	$437.112226d$
$\Delta T_{ch} = T_{ch,max} - T_{ch,\min}$	$2.001614d$	$4.310187d$	$4.032845d$
$L_{\max}/G$	$1{-}0.743025 \times 10^{-12}$	$1{-}0.747961 \times 10^{-12}$	$1{-}0.744473 \times 10^{-12}$
L_{min}/G	$1{-}0.747926 \times 10^{-12}$	$1{-}0.755173 \times 10^{-12}$	$1{-}0.751406 \times 10^{-12}$
$(L_{\max} - L_{\min})/G$	0.490128×10^{-14}	0.721241×10^{-14}	0.693253×10^{-14}

Main Properties of Chandler Motion (with Application to the Earth)

As a result of the analysis of the formulae for unperturbed rotation motion given in the Introduction the following properties were established (Barkin, 1992):

(1) The geodograph of the projection of the angular velocity $\overline{\omega}$ on the plane parallel of the equatorial plane Oxy is an ellipse with semiaxes

$$a = \omega_0 \frac{C\lambda}{A\sqrt{\kappa^2+\lambda^2}}, \quad b = \omega_0 \frac{C\lambda\sqrt{1+\kappa^2}}{B\sqrt{\kappa^2+\lambda^2}}, \quad \left(\omega_0 = \frac{G}{C}\right) \tag{53}$$

and with eccentricity

$$e = \sqrt{1 - \left(\frac{B}{A}\right)^2 \frac{1}{1+\kappa^2}}. \tag{54}$$

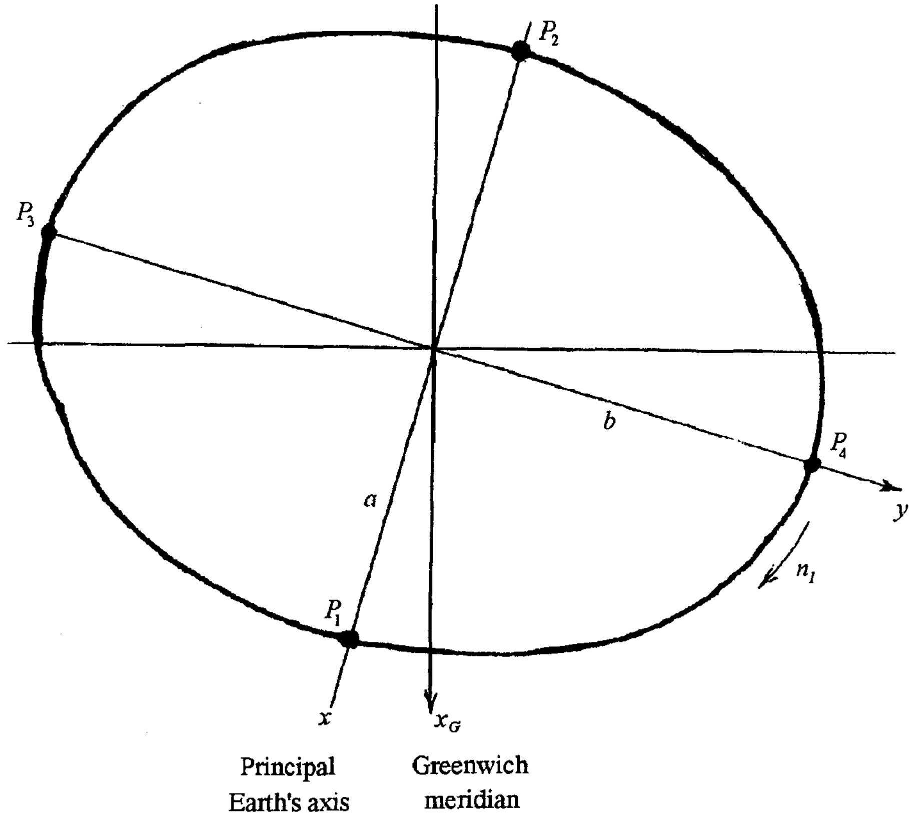

Figure 1 Chandler motion of the Earth's pole.

The minor axis of this ellipse corresponds to the smaller moment of inertia A, and the major axis to the bigger moment of inertia of the equatorial moments (B). The semiaxes a, b (Figure 1) are defined by the initial conditions of the problem:

$$a = p_0, \quad b = \frac{\dot{q}_0}{\omega_0}.$$

(2) The motion of the end of the projection $\overline{\omega}$ along this ellipse takes place in a straight direction ($n_1 < 0$ for $C > B > A$, $0 < \lambda^2 < 1$) with variable velocity $\dot{l}$. The maximal value of the modulus of this velocity is

$$|\dot{l}|_{\max} = \frac{(C-B)}{B}\omega_0 \frac{\kappa(1+\kappa^2)}{\sqrt{\kappa^2+\lambda^2}} \tag{55}$$

and takes place at the moments of the passing of the minor axis of the ellipse (at the points P_1, P_2 in Figure 1), and the minimal value of the $|\dot{l}|$

$$|\dot{l}|_{\min} = \frac{(C-B)}{B}\omega_0 \frac{\kappa(1-\lambda^2)}{\sqrt{\kappa^2+\lambda^2}} \tag{56}$$

takes place at the moments of the passing of the major axis of the ellipse at the point P_3, P_4 in Figure 1). The difference of these velocities is

$$\Delta \dot{l} = |\dot{l}|_{\max} - |\dot{l}|_{\min} = \frac{B-C}{B}\omega_0 \kappa \sqrt{\kappa^2+\lambda^2}. \qquad (57)$$

The corresponding extremal values of the periods for the velocities (55), (56) are:

$$T_{\max} = \frac{2\pi}{|\dot{l}|_{\min}} = \frac{2\pi}{\omega_0}\frac{B}{(C-B)}\frac{\sqrt{\kappa^2+\lambda^2}}{\kappa(1-\lambda^2)} \qquad (58)$$

$$T_{\min} = \frac{2\pi}{|\dot{l}|_{\max}} = \frac{2\pi}{\omega_0}\frac{B}{(C-B)}\frac{\sqrt{\kappa^2+\lambda^2}}{\kappa(1+\kappa^2\lambda^2)}. \qquad (59)$$

The difference of these periods is defined by the formulae:

$$\Delta T = T_{\max} - T_{\min} = \frac{2\pi}{\omega_0}\frac{B}{(C-B)}\frac{\sqrt{\kappa^2+\lambda^2}}{\kappa}\left[\frac{1}{\sqrt{1-\lambda^2}} - \frac{1}{1+\kappa^2}\right]. \qquad (60)$$

(3) The averaged values of the components of the angular velocity $\overline{\omega}$ are defined by the formulae:

$$\langle p \rangle = 0, \quad \langle q \rangle = 0, \quad \langle r \rangle = \frac{G\kappa}{C\sqrt{\kappa^2+\lambda^2}}. \qquad (61)$$

(4) The modulus of the angular velocity is a periodic function of time with semi-Chandler period:

$$\omega = \frac{G\kappa}{C\sqrt{\kappa^2+\lambda^2}}\sqrt{1+\lambda^2\frac{C^2}{A^2\kappa^2} - \lambda^2\left(\frac{C^2}{A^2\kappa^2}+1\right)sn^2 u} \qquad (62)$$

$$u = \frac{2K(\lambda)}{\pi}\varphi_1, \quad \varphi_1 = n_1 t + \varphi_{10}. \qquad (63)$$

The extremal values $\omega_{\max}$ and $\omega_{\min}$ are achived at the moments when the vector $\overline{\omega}$ is situated in the reference planes Oxz and Oyz of the coordinate system $Oxyz$, respectively:

$$\omega_{\max} = \frac{G\kappa}{C\sqrt{\kappa^2+\lambda^2}}\sqrt{1+\lambda^2\frac{C^2}{A^2\kappa^2}}$$
$$\omega_{\min} = \frac{G\kappa}{C}\frac{\sqrt{1-\lambda^2}}{\sqrt{\kappa^2+\lambda^2}} \qquad (64)$$

and

$$\Delta\omega = \omega_{\max} - \omega_{\min} = \frac{G\kappa}{C\sqrt{\kappa^2+\lambda^2}}\frac{\lambda^2\left(\frac{C^2}{A^2\kappa^2}+1\right)}{\left[\sqrt{1+\lambda^2\frac{C^2}{A^2\kappa^2}}+\lambda'\right]}. \qquad (65)$$

(5) The external values of the projection of the vector $\overline{G}$ on the polar axis of inertia of the Earth Oz are defined by the formulae:

$$L_{\max} = \frac{G\kappa}{\sqrt{\kappa^2+\lambda^2}}, \quad L_{\min} = \frac{G\kappa\sqrt{1-\lambda^2}}{\sqrt{\kappa^2+\lambda^2}}. \tag{66}$$

These values are achieved when the vector $\overline{G}$ is situated in the coordinate planes Oxz and Oyz, respectively.

Here

$$\frac{\Delta L}{G} = \frac{L_{\max} - L_{\min}}{G} = \frac{\kappa\lambda^2}{\sqrt{\kappa^2+\lambda^2}(1+\sqrt{1-\lambda^2})}. \tag{67}$$

The corresponding extermal values of the angle θ between the vector $\overline{G}$ and the axis of the body Oz are defined by the formulae:

$$\theta_{\max} = \arccos\left(\frac{\kappa\sqrt{1-\lambda^2}}{\sqrt{\kappa^2+\lambda^2}}\right)$$
$$\theta_{\min} = \arccos\left(\frac{\kappa}{\sqrt{\kappa^2+\lambda^2}}\right). \tag{68}$$

(6) The parameter λ is considered as the conditional excentricity of the phase curves of the Chandler problem on the phase plane of the Andoyer variables L, l (Barkin, 1992):

$$\lambda = \sqrt{1 - \frac{L^2_{\min}}{L^2_{\max}}}. \tag{69}$$

(7) The extremal values of the velocity $\dot{L}$ are

$$\dot{L}_{\max,\min} = \pm\frac{1}{2}\left(\frac{1}{A} - \frac{1}{B}\right)\frac{\lambda^2\sqrt{1+\kappa^2}}{\lambda^2+\kappa^2} \tag{70}$$

and

$$\Delta\dot{L} = \dot{L}_{\max} - \dot{L}_{\min} = \left(\frac{1}{A} - \frac{1}{B}\right)\frac{\lambda^2\sqrt{1+\kappa^2}}{\lambda^2+\kappa^2}. \tag{71}$$

(8) The direction cosines of the body b_{ij} (31) (in the intermediate coordinate system, connected with the angular moment $\overline{G}$) are conditionally periodic functions of the variables φ_1, φ_2 with frequencies:

$$n_1 = \frac{G}{2C}\frac{(A-C)}{A}\frac{\pi\kappa}{\sqrt{(1+\kappa^2)(\kappa^2+\lambda^2)}K(\lambda)} \tag{72}$$

$$n_2 = \frac{G}{C}\left(1 - \frac{A-C}{A}\frac{\Pi(\frac{\pi}{2},\kappa^2,\lambda)}{K(\lambda)}\right). \tag{73}$$

The frequency n_1 gives the Chandler motion of the Earth's pole with period

$$T = \frac{2\pi}{n_1} = \frac{2\pi}{\omega_0}\frac{A\sqrt{(1+\kappa^2)(\kappa^2+\lambda^2)}}{(A-C)\pi\kappa}K(\lambda) \tag{74}$$

and the frequency n_2 is the mean angular velocity of the change of the variable g. The corresponding period is:

$$T = \frac{2\pi}{n_2} = \frac{2\pi}{\omega_0}\frac{AK(\lambda)}{[AK(\lambda)+(C-A)\Pi(\frac{\pi}{2},\kappa^2,\lambda)]}. \tag{75}$$

The above formulated properties of the unperturbed rotational motion take place for arbitrary values of the parameters $\kappa^2 > 0$ and $0 < \lambda^2 < 1$.

In the case of the Earth the parameter λ has a very small value ($\sim 10^{-7}$), which is why for the above dynamical and kinematical characteristics we obtain the corresponding approximate formulae (saving only the main terms with respect λ in their development). We have:

$$\begin{aligned}
n_1 &= \omega_0\left(1-\frac{C}{A}\right)\frac{1}{\sqrt{1+\kappa^2}}\\
n_1 &= \omega_0\left[1-\left(1-\frac{C}{A}\right)\frac{1}{\sqrt{1+\kappa^2}}\right]\\
T_{ch} &= \frac{2\pi}{n_1}\\
T_2 &= \frac{2\pi}{n_2}\\
\omega_0 &= n_1+n_2\\
L_{\max} &= G\left\{1-\frac{1}{2}\left(\frac{\lambda}{\kappa}\right)^2+\dots\right\}\\
L_{\min} &= G\left\{1-\frac{1}{2}\left(\frac{\lambda}{\kappa}\right)^2-\frac{1}{2}\lambda^2+\dots\right\}\\
\frac{\Delta L}{G} &= \frac{1}{2}\lambda^2\\
\frac{|\dot{l}|_{\max}}{\omega_0} &= \frac{C-B}{B}(1+\kappa^2)\\
\frac{|\dot{l}|_{\min}}{\omega_0} &= \frac{C-B}{B}\\
\frac{\Delta\dot{l}}{\omega_0} &= \frac{C-B}{B}\kappa^2\\
T_{ch,\max} &= \frac{2\pi}{\omega_0}\frac{B}{(C-B)(1+\kappa^2)}\\
T_{ch,\min} &= \frac{2\pi}{\omega_0}\frac{B}{(C-B)}.
\end{aligned} \tag{76}$$

In Table 1 we present the values of the parameters (76) for three variants of the unperturbed rotational motion of the Earth:

(1) for Euler's motion (column E);

(2), (3) for Chandler's motion for two possible values of the elastic parameter D_r (columns CH I and CH II).

5 THE EQUATIONS OF THE PERTURBED ROTATIONAL MOTION OF A DEFORMABLE BODY

In accordance with Hamilton–Jacobi method the equations in the angle-action variables I_i, φ_i will be canonical with Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$:

$$\frac{d\varphi_i}{dt} = \frac{\partial\mathcal{H}_0}{\partial I_i} + \frac{\partial\mathcal{H}_1}{\partial I_i}$$
$$\frac{dI_i}{dt} = -\frac{\partial\mathcal{H}_0}{\partial\varphi_i} - \frac{\partial\mathcal{H}_1}{\partial\varphi_i}, \quad (i = 1, 2, 3) \tag{77}$$

where the unperturbed Hamiltonian (17) $\mathcal{H}_0$ in angle-action variables is defined as (Sadov, 1970; Barkin, 1992):

$$\mathcal{H}_0 = \frac{I_2^2}{2A}\left[1 + \frac{A-C}{C}\frac{\kappa^2}{(\lambda^2+\kappa^2)}\right]. \tag{78}$$

Here λ is a function of the variables I_2, I_3 (24), (25). The perturbation function $\mathcal{H}_1 = \mathcal{H} - \mathcal{H}_0$ must be represented as a function of the angle-action variables by using a wide list of formulae of the unperturbed Chandler motion (Barkin, 1992).

In this paper we shall give the full solution of the problem of the calculation of the first-order perturbation in the rotational motion of the deformable body caused by the second harmonic of the force function of the perturbing body. We shall also give some applications of our method for the analysis of secular and periodic perturbations in the Earth's rotation due to the attraction of the Moon and the Sun.

For this purpose we shall obtain a trigonometric development of the force function of the Earth–Moon system (and the Earth–Sun system) in angle-action variables (see Section 2).

Equations (77) let us use different and effective mathematical methods for the investigation (for example based on canonical transformations), but here we shall use the non-canonical form of equations in the variables:

$$G = I_2, \quad \lambda = \lambda(I_1/I_2), \quad \rho = \rho(I_3/I_2), \quad \varphi = \varphi_1, \quad \psi = \varphi_2, \quad h = \varphi_3. \tag{79}$$

The corresponding equations were obtained by Barkin (1992):

$$\frac{dG}{dt} = -\frac{\partial\mathcal{H}_1}{\partial\psi}$$

$$\frac{d\lambda}{dt} = \frac{1}{GJ(\lambda)}\frac{\partial\mathcal{H}_1}{\partial\varphi} - \frac{\Lambda(\lambda)}{GJ(\lambda)}\frac{\partial\mathcal{H}_1}{\partial\psi}$$
$$\frac{d\rho}{dt} = -\frac{1}{G}\cot\rho\frac{\partial\mathcal{H}_1}{\partial\psi} + \frac{1}{G}\csc\rho\frac{\partial\mathcal{H}_1}{\partial h}$$
$$\frac{d\varphi}{dt} = \Omega(G,\lambda) - \frac{1}{GJ(\lambda)}\frac{\partial\mathcal{H}_1}{\partial\lambda}$$
$$\frac{dh}{dt} = -\frac{1}{G}\csc\rho\frac{\partial\mathcal{H}_1}{\partial\rho}$$
$$\frac{d\psi}{dt} = \omega(G,\lambda) + \frac{\Lambda(\lambda)}{GJ(\lambda)}\frac{\partial\mathcal{H}_1}{\partial\lambda} + \frac{1}{G}\cot\rho\frac{\partial\mathcal{H}_1}{\partial\rho} + \frac{\partial\mathcal{H}_1}{\partial G} \tag{80}$$

where we use special functions from the theory of the unperturbed motion:

$$J(\lambda) = \frac{2\kappa\lambda\sqrt{1+\kappa^2}}{\pi(\kappa^2+\lambda^2)^{3/2}}K(\lambda), \tag{81}$$

$$\Lambda(\lambda) = \frac{2\sqrt{1+\kappa^2}}{\pi\kappa\sqrt{\kappa^2+\lambda^2}}\left\{(\kappa^2+\lambda^2)\Pi(\frac{\pi}{2},\kappa^2,\lambda) - \lambda^2K(\lambda)\right\}; \tag{82}$$

Ω and ω are frequencies of the unperturbed Chandler motion:

$$\Omega(G,\lambda) = \frac{\pi G}{2K(\lambda)}\frac{A-C}{AC}\frac{\kappa}{\sqrt{(1+\kappa^2)(\kappa^2+\lambda^2}} \tag{83}$$

$$\omega(G,\lambda) = \frac{G}{C}\left(1 - \frac{A-C}{A}\frac{\Pi(\frac{\pi}{2},\kappa^2,\lambda)}{K(\lambda)}\right). \tag{84}$$

Of course, for analytical investigations we must have $\mathcal{H}_1$ as an explicit function of the variables (79) and time, for example, in the form of the Fourier series:

$$\mathcal{H}_1 = \mathcal{H}_1(G,\lambda,\rho,\varphi,\psi,h,t) = \mathcal{H} - \mathcal{H}_0 \tag{85}$$

Saving on the right-hand sides of equations (4), (80)–(85) only the terms R_E and U, which describe the influence of the movement of the ecliptical plane E and the Newtonian influence of the Moon's and Sun's attraction on the Earth's rotation, we obtained the following form of the equations of the rotational motion:

$$\dot{G} = \frac{\partial U}{\partial\psi}$$
$$\dot{\lambda} = -\frac{1}{GJ(\lambda)}\frac{\partial U}{\partial\varphi} + \frac{\Lambda(\lambda)}{GJ(\lambda)}\frac{\partial U}{\partial\psi}$$
$$\dot{\rho} = \sin\pi_1\sin(h-\Pi_1)\frac{d\Pi_1}{dt} + \cos(h-\Pi_1)\frac{d\pi_1}{dt} + \frac{1}{G}\cot\rho\frac{\partial U}{\partial\psi} - \frac{1}{G}\csc\rho\frac{\partial U}{\partial h}$$
$$\dot{\varphi} = \Omega(G,\lambda) + \frac{1}{GJ(\lambda)}\frac{\partial U}{\partial\lambda}$$
$$\dot{\psi} = \omega(G,\lambda) + \csc\rho\left[\sin\Pi_1\cos(h-\Pi_1)\frac{d\Pi_1}{dt} - \sin(h-\Pi_1)\frac{d\pi_1}{dt}\right]$$

$$- \frac{\Lambda(\lambda)}{GJ(\lambda)}\frac{\partial U}{\partial \lambda} - \frac{1}{G}\cot\rho\frac{\partial U}{\partial \rho}$$

$$\dot{h} = (1-\cos\pi_1)\frac{d\Pi_1}{dt} - \cot\rho\left[\sin\pi_1\cos(h-\Pi_1)\frac{d\Pi_1}{dt} - \sin(h-\Pi_1)\frac{d\pi_1}{dt}\right] + \frac{1}{G}\csc\rho\frac{\partial U}{\partial \rho} \tag{86}$$

where

$$U = U(\lambda, \rho, \varphi, \psi, h, t) \tag{87}$$

and the angles π_1, Π_1 are defined by formulae (3).

6 DEVELOPMENT OF THE SECOND HARMONIC OF THE FORCE FUNCTION FOR THE EARTH–MOON SYSTEM IN THE ANGLE-ACTION VARIABLES

In this paper we shall concentrate our attention on the study of the perturbations in the Earth's rotation due to the second harmonic of the force function for Newtonian attraction of the Moon (and the Sun):

$$U = \frac{fm_L}{2r^3}(A+B+C-3I). \tag{88}$$

Here f is the gravitational constant, m_L is the Moon's mass, r is the distance between the centres of mass $O_\oplus$ (the Earth) and O_L (the Moon). I is the moment of inertia of the Earth about the line $\overline{O_\oplus O_L}$ passing through the mass centres:

$$I = A\alpha_1^2 + B\alpha_2^2 + C\alpha_3^2 - F\alpha_1\alpha_2 - E\alpha_1\alpha_3 - D\alpha_2\alpha_3 \tag{89}$$

where A, B, C and F, E, D are axial and centrifugal components of the inertia tensor of the Earth with respect to reference system $O_\oplus xyz$; $\alpha_1\alpha_2$ and α_3 are cosines of the angles which the line $O_\oplus O_L$ forms with the body's axes $O_\oplus x$, $O_\oplus y$ and $O_\oplus z$.

Let r, φ, λ be the spherical coordinates of the mass centre of the Moon relative to the geocentric ecliptic coordinate system $O_\oplus XYZ$ of the data. We shall relate the Andoyer variables to our reference system.

On the basis of the simple geometrical relations for direction cosines we obtain:

$$\alpha_i = [b_{1i}\cos(\lambda-h) + b_{2i}\cos\rho\sin(\lambda-h) - b_{3i}\sin\rho\sin(\lambda-h)\cos\varphi + (b_{2i}\sin\rho + b_{3i}\cos\rho)\sin\varphi \tag{90}$$

where b_{ij} are the direction cosines of the axes $O_\oplus xyz$ relative to the intermediate coordinate system, which is connected with the vector of the angular moment $\overline{G}$.

For our unperturbed motion these direction cosines are defined by formulae (27), (28) (in terms of elliptic functions) or by formulae (31)–(33) (in the form of the Fourier series in angle-action variables).

In this paper we shall consider only the main perturbations in the rotation caused by the constant parts of the tensor inertia components, assuming that

$$A = \overline{A},\ B = \overline{B},\ C = \overline{C},\ F = E = D = 0. \tag{91}$$

For squares of the direction cosines α_i, we have the following expressions:

$$\begin{aligned}
\alpha_i^2 = & \frac{1}{2}(b_{1i}^2 + b_{2i}^2 \cos^2 \rho + b_{3i}^2 \sin^2 \rho - 2b_{2i}b_{3i} \cos \rho) \cos^2 \varphi \\
& + \frac{1}{2}(b_{1i}^2 - b_{2i}^2 \cos^2 \rho - b_{3i}^2 \sin^2 \rho + 2b_{2i}b_{3i} \cos \rho) \cos^2 \varphi \cos 2(\lambda - h) \\
& + (b_{1i}b_{2i} \cos \rho - b_{1i}b_{3i} \sin \rho) \cos^2 \varphi \sin 2(\lambda - h) \\
& + (b_{2i}^2 \sin^2 \rho + b_{2i}b_{3i} \sin 2\rho + b_{3i}^2 \cos^2 \rho) \sin^2 \varphi \\
& + 2(b_{1i}b_{2i} \sin \rho - b_{1i}b_{3i} \cos \rho) \sin \varphi \cos \varphi \cos(\lambda - h) \\
& + [(b_{2i}^2 - b_{3i}^2) \sin 2\rho + 2b_{2i}b_{3i} \cos 2\rho] \sin \varphi \cos \varphi \sin(\lambda - h).
\end{aligned} \tag{92}$$

The spherical coordinates of the perturbing body (the Moon) in the ecliptic coordinate system $O_\oplus x_E y_E z_E$ are represented as known trigonometric series. From the standpoint of the averaging method, $\lambda_m = \lambda - h$, which appears in the disturbing function, can be considered to be the longitude referred to the mean equinox of the data and the ecliptic of the data.

For presentation of the coordinates λ_m, φ and r of the disturbing bodies (the Moon and the Sun) on the basis of Brown's (1919) theory and Newcomb's (1895) theory, in this paper we shall follow Kinoshita (1992). In particular we shall use Kinoshita's developments in Poinsot series for the following functions of the spherical coordinates of the perturbing body:

$$\begin{aligned}
\frac{1}{2}\left(\frac{a}{r}\right)^3 (1 - 3\sin^2 \varphi) &= \sum_\nu A_\nu^{(0)} \cos \Theta_\nu \\
\left(\frac{a}{r}\right)^3 \sin \varphi \cos \varphi \sin \lambda_m &= \sum_\nu A_\nu^{(1)} \cos \Theta_\nu \\
\left(\frac{a}{r}\right)^3 \sin \varphi \cos \varphi \cos \lambda_m &= \sum_\nu \overline{A}_\nu^{(1)} \sin \Theta_\nu \\
\left(\frac{a}{r}\right)^3 \cos^2 \varphi \cos 2\lambda_m &= \sum_\nu A_\nu^{(2)} \cos \Theta_\nu \\
\left(\frac{a}{r}\right)^3 \cos^2 \varphi \sin 2\lambda_m &= \sum_\nu \overline{A}_\nu^{(2)} \sin \Theta_\nu
\end{aligned} \tag{93}$$

where

$$\begin{aligned}
\Theta_\nu &= i_1 l_L + i_1 l_\odot + i_3 F + i_4 D + i_5 \Omega \\
i &= i_1, i_2, i_3, i_4, i_5 \\
l_L &= \text{the mean anomaly of the Moon}
\end{aligned}$$

$$
\begin{aligned}
l_\odot &= \text{the mean anomaly of the Sun}\\
F &= L_L - \Omega, \quad D = L_L - L_\odot\\
\Omega &= \text{the mean longitude of the node of the Moon}\\
l_L &= \text{the mean longitude of the Moon}\\
L_\odot &= \text{the mean longitude of the Sun}\\
i_5 &\geq 0.
\end{aligned} \tag{94}
$$

The longitude h^* of the plane normal to the angular momentum is included in Ω in the form $\Omega_0 - h^*$ (Kinoshita, 1992). From additional terms due to long-period and planetary perturbations only four terms, which were densed by Woolard, should be added to the right-hand sides of equations (93), (94). The coefficients

$$A_\nu^{(j)} = A_{\nu;0}^{(j)} + A_{\nu;1}^{(j)} t \tag{95}$$

in which t is measured in Julian centuries from 1900. Numerical values of $A_{\nu;0}^{(j)}$ and $A_{\nu;1}^{(j)}$ 95 that produce nutational terms with amplitudes $\leq 0.''0001$ were given by Kinoshita (1972) (Table 1).

Kinoshita pointed out an important property of the trigonometric series (93):

$$\overline{A}_\nu^{(1)} = -A_\nu^{(1)}, \quad \overline{A}_\nu^{(2)} = A_\nu^{(2)}.$$

With the help of formulae (92), (93) we obtain the following development:

$$
\begin{aligned}
\left(\frac{a}{r}\right)^3 \alpha_i^2 = \sum_{\|\nu\|\geq 0} & (b_{2i}^2 R_{22}^\nu + b_{3i}^2 R_{33}^\nu + b_{2i} b_{3i} R_{23}^\nu) \cos\Theta_\nu\\
&+ (b_{1i} b_{2i} R_{12}^\nu + b_{1i} b_{3i} R_{13}^\nu) \sin\Theta_\nu
\end{aligned} \tag{96}
$$

where R_{ij}^ν are functions only of the inclination ρ:

$$
\begin{aligned}
R_{22}^\nu &= -A_\nu^{(0)} \sin^2\rho - \frac{1}{2}(1+\cos^2\rho) A_\nu^{(2)} + A_\nu^{(1)} \sin 2\rho\\
R_{33}^\nu &= -A_\nu^{(0)} \cos^2\rho - \frac{1}{2}(1+\sin^2\rho) A_\nu^{(2)} - A_\nu^{(1)} \sin 2\rho\\
R_{23}^\nu &= -A_\nu^{(0)} \sin 2\rho + \frac{1}{2} A_\nu^{(2)} \sin 2\rho + 2A_\nu^{(1)} \cos 2\rho\\
R_{12}^\nu &= A_\nu^{(2)} \cos\rho - 2\sin\rho A_\nu^{(1)}\\
R_{13}^\nu &= -A_\nu^{(2)} \sin\rho - 2\cos\rho A_\nu^{(1)}.
\end{aligned} \tag{97}
$$

For the force function U we have following expression:

$$U = \frac{3}{2} n^2 (\overline{A} - \overline{B}) \left(\frac{a}{r}\right)^3 (\alpha_2^2 + \delta\alpha_3^2) \tag{98}$$

where $n^2 = f m_L / a^3$, a is the mean semi-major axis of the Moon's orbit, and

$$\delta = \frac{\overline{A} - \overline{C}}{\overline{A} - \overline{B}} > 0. \tag{99}$$

Substituting (96), (97) in (98) we obtain:

$$U = \frac{3}{2}n^2(\overline{A} - \overline{B}) \sum_{\|\nu\|\geq 0} \{[b_{22}^2 + \delta b_{23}^2)R_{22}^\nu + (b_{32}^2 + \delta b_{33}^2)R_{33}^\nu + (b_{22}b_{32} + \delta b_{23}b_{33})R_{23}^\nu] \cos\Theta_\nu + [(b_{12}b_{22} + \delta b_{13}b_{23})R_{12}^\nu + (b_{12}b_{32} + \delta b_{13}b_{33})R_{13}^\nu] \sin\Theta_\nu\}. \quad (100)$$

From the theory of unperturbed rotational motion we have the Fourier series in angle-action variables I_i, φ_i for the products and squares of the direction cosines (see Appendix, and Barkin, 1992). On the basis of these formulae we have the following series:

$$b_{22}^2 + \delta b_{23}^2 = \sum_{m=1}^{\infty} \{A_{2m,-2}\cos 2(m\varphi_1 - \varphi_2) + A_{2m,2}\cos 2(m\varphi_1 + \varphi_2) + A_{2m,0}\cos 2m\varphi_1\} + A_{0,2}\cos 2m\varphi_2 + A_{0,0};$$

$$b_{32}^2 + \delta b_{33}^2 = \sum_{m=1}^{\infty} \{-2A_{2m,0}\cos 2m\varphi_1\} + B_{0,0};$$

$$b_{22}b_{32} + \delta b_{23}b_{33} = \sum_{m=1}^{\infty} \{C_{2m,-1}\cos(2m\varphi_1 - \varphi_2) + C_{2m,1}\cos(2m\varphi_1 + \varphi_2)\} + C_{0,1}\cos\varphi_2;$$

$$b_{12}b_{22} + \delta b_{13}b_{23} = \sum_{m=1}^{\infty} \{A_{2m,-2}\sin 2(m\varphi_1 - \varphi_2) - A_{2m,2}\sin 2(m\varphi_1 + \varphi_2)\} - A_{0,2}\sin 2\varphi_2;$$

$$b_{12}b_{32} + \delta b_{13}b_{33} = \sum_{m=1}^{\infty} \{C_{2m,-1}\sin(2m\varphi_1 - \varphi_2) - C_{2m,1}\sin(2m\varphi_1 + \varphi_2)\} - C_{0,1}\sin\varphi_2. \quad (101)$$

For the coefficients of the series (101) we obtain an expressions in the following compact form in terms of elliptical integrals:

$$A_{2m,2\varepsilon} = Q\frac{m(D-1) - 2\varepsilon P}{sh\, 2(md - \varepsilon\sigma)}$$

$$A_{2m,0} = 2Q\frac{m(1-D)}{sh\,(2md)}$$

$$A_{0,2} = 2Q\frac{P}{sh\,(2\sigma)}$$

$$C_{2m,\varepsilon} = -2Q\frac{-\varepsilon m(D-1) + P}{ch\,(2md - \varepsilon\sigma)}$$

$$C_{0,1} = \frac{2QP}{ch\,\sigma} \quad (102)$$

where $\varepsilon = \pm 1$,

$$\begin{aligned}
Q &= \frac{\pi^2(1+\kappa^2)}{4K^2(\kappa^2+\lambda^2} \\
P &= DM_1 - M_3 \\
M_1 &= \frac{\Pi}{\pi\kappa}\sqrt{(1+\kappa^2)(\kappa^2+\lambda^2)} \\
M_3 &= \frac{\sqrt{\kappa^2+\lambda^2}}{\pi\kappa\sqrt{1+\kappa^2}}[\Pi(1+\kappa^2)-K] \\
P &= \frac{\sqrt{\kappa^2+\lambda^2}}{\pi\kappa\sqrt{1+\kappa^2}}[(1+D)(1+\kappa^2)\Pi-K] \\
D &= \frac{\delta\kappa^2}{1+\kappa^2} = \frac{\overline{A}-\overline{C}}{\overline{A}-\overline{B}}\frac{C}{B}
\end{aligned} \tag{103}$$

For secular terms $A_{0,0}$ and $B_{0,0}$ in (101) we have the formulae:

$$\begin{aligned}
B_{0,0} &= \frac{(1+\kappa^2)(K-E)+\delta\kappa^2 E}{K(\kappa^2+\lambda^2)} \\
A_{0,0} &= \frac{1}{2}(1+\delta) + \frac{[(1+\kappa^2)(E-K)-\delta\kappa^2 E]}{2K(\kappa^2+\lambda^2)} \\
B_{0,0} &= -2A_{0,0} + 1 + \delta.
\end{aligned} \tag{104}$$

Substituting formulae (101), (102) in (96) we obtain the final trigonometric development of the force function U in angle-action variables:

$$U = \sum_{\|\nu\|\geq 0}\sum_{k_1}\sum_{k_2} U_{\nu;k_1,k_2}(\lambda,\rho)\cos(\Theta_\nu + k_1\varphi_1 + k_2\varphi_2) \tag{105}$$

where the indexes $k_1 = 0, \pm 2m$; $k_2 = 0, \pm 2$; $m = 1, 2, \ldots$; the coefficients $U_{\nu;k_2,k_2}$ are defened by the formulae:

$$\begin{aligned}
U_{\nu;2\varepsilon m,2\mu} &= \frac{3}{2}n^2(\overline{A}-\overline{B})A_{2m,2\varepsilon\mu}R_{\nu;2\mu} \\
U_{\nu;2\varepsilon m,\mu} &= \frac{3}{2}n^2(\overline{A}-\overline{B})C_{2m,\varepsilon\mu}R_{\nu;\mu} \\
U_{\nu;2\varepsilon m,0} &= \frac{3}{2}n^2(\overline{A}-\overline{B})C_{2m,\varepsilon}R_{\nu;0} \\
U_{\nu;0,\mu} &= \frac{3}{2}n^2(\overline{A}-\overline{B})C_{0,1}R_{\nu;\mu} \\
U_{\nu;0,2\mu} &= \frac{3}{2}n^2(\overline{A}-\overline{B})A_{0,2}R_{\nu;2\mu} \\
U_{\nu;0,0} &= \frac{1}{4}n^2(\overline{A}-\overline{B})(1+\delta-3B_{0,0})R_{\nu;0}
\end{aligned} \tag{106}$$

where

$$R_{\nu;2\mu} = -A_\nu^{(0)}\sin^2\rho + 2\sin\rho(\cos\rho-\mu)A_\nu^{(1)} - \frac{1}{2}A_\nu^{(2)}(1-\mu\cos\rho)^2$$

$$R_{\nu;\mu} = -A_\nu^{(0)} \sin 2\rho + 2(-1 - \mu\cos\rho + 2\cos^2\rho)A_\nu^{(1)} + \sin\rho(\cos\rho - \mu)A_\nu^{(2)}$$
$$R_{\nu;0} = -A_\nu^{(0)}(1 - 3\cos^2\rho) + 3\sin 2\rho A_\nu^{(1)} + \frac{3}{2}\sin^2\rho A_\nu^{(2)}. \quad (107)$$

We shall use the development of the second harmonic of the force function (105)–(107) in angle-action variables for the construction of the perturbed theory of the Earth's rotation and, in particular, for the analysis of the main secular perturbations and effects in the precession and Chandler's motion of the Earth.

7 SECULAR PERTURBATIONS

Differential equations for secular perturbations are obtained by averaging the right sides of the equations (86), (87) on all fast variables of the problem:

$$\dot G = 0, \quad \dot\lambda = 0$$
$$\dot\rho = \sin\pi_1 \sin(h - \Pi_1)\frac{d\Pi_1}{dt} + \cos(h - \Pi_1)\frac{d\Pi_1}{dt}$$
$$\dot\varphi = \Omega(G,\lambda) + \frac{1}{GJ(\lambda)}\frac{\partial\langle U\rangle}{\partial\lambda}$$
$$\dot\psi = \omega(G,\lambda) + \csc\rho\left[\sin\Pi_1\cos(h - \Pi_1)\frac{d\Pi_1}{dt} - \sin(h - \Pi_1)\frac{d\Pi_1}{dt}\right]$$
$$- \frac{\Lambda(\lambda)}{GJ(\lambda)}\frac{\partial\langle U\rangle}{\partial\lambda} - \frac{1}{G}\cot\rho\frac{\partial\langle U\rangle}{\partial\rho}$$
$$\dot h = (1 - \cos\pi_1)\frac{d\Pi_1}{dt} - cot\,\rho\left[\sin\pi_1\cos(h - \Pi_1)\frac{d\Pi_1}{dt} - \sin(h - \Pi_1)\frac{d\pi_1}{dt}\right]$$
$$+ \frac{1}{G}\csc\rho\frac{\partial\langle U\rangle}{\partial\rho} \quad (108)$$

where $\langle U\rangle$ is the average value of the force function (105)–(107):

$$\langle U\rangle = \frac{1}{4}n^2(\overline{A} - \overline{B})(1 + \delta - 3B_{0,0})R_{0,0} \quad (109)$$
$$B_{0,0} = \frac{(1+\kappa^2)(K - E) + \delta\kappa^2 E}{K(\kappa^2 + \lambda^2}$$
$$R_{0,0} = -A_0^{(0)}(1 - 3\cos^2\rho) + 3\sin 2\rho A_0^{(1)} + \frac{3}{2}\sin^2\rho A_0^{(2)}$$
$$\Lambda(\lambda) = \frac{2\sqrt{1+\kappa^2}}{\pi\kappa\sqrt{\kappa^2+\lambda^2}}\{(\kappa^2 + \lambda^2)\Pi(\frac{\pi}{2}, \kappa^2, \lambda) - \lambda^2 K(\lambda)\}$$
$$J(\lambda) = \frac{2\kappa\lambda\sqrt{1+\kappa^2}}{\pi(\kappa^2+\lambda^2)^{3/2}}K(\lambda). \quad (110)$$

Numerical values of the coefficients are

$$A^{(0)} = 4963033 \times 10^{-7}$$

$$A^{(1)} = -207 \times 10^{-7}$$
$$A^{(2)} = -1 \times 10^{-7} \tag{111}$$

and the angles π_1, Π_1 are defined as functions of time by formulae (3).

The solutions of these equations can be found in the form of series with respect degrees of time.

However, in this paper we shall analyse only secular perturbations of first order in the body rotation due to the Newtonian attraction of the Moon (and the Sun). These perturbations by constant values of the variables G, λ, ρ are defined by the formulae:

$$\begin{aligned}\langle \dot{h} \rangle &= \frac{1}{G} \csc \rho \frac{\partial \langle U \rangle}{\partial \rho} \\ \langle \dot{\varphi} \rangle &= \frac{1}{GJ(\lambda)} \frac{\partial \langle U \rangle}{\partial \lambda} \\ \langle \dot{\psi} \rangle &= -\frac{\Lambda(\lambda)}{GJ(\lambda)} \frac{\partial \langle U \rangle}{\partial \lambda} - \frac{1}{G} \cot \rho \frac{\partial \langle U \rangle}{\partial \rho}.\end{aligned} \tag{112}$$

On the basis of these relations we obtain the following equations:

$$\begin{aligned}\langle \dot{\psi} \rangle &= -\Lambda(\lambda) \langle \dot{\varphi} \rangle - \cos \rho \langle \dot{h} \rangle \\ \langle \dot{\psi} + \dot{\varphi} \rangle &= (1 - \Lambda(\lambda)) \langle \dot{\varphi} \rangle - \cos \rho \langle \dot{h} \rangle.\end{aligned} \tag{113}$$

Substituting expression $\langle U \rangle$ (109)–(111) in the right sides (112), after some algebra, we obtain the following expressions for the secular velocities $\langle \dot{\varphi} \rangle$ and $\langle \dot{h} \rangle$:

$$\begin{aligned}\langle \dot{\varphi} \rangle &= \frac{3\pi n^2 (\overline{B} - \overline{A}) R_{0,0}}{8G\kappa\sqrt{1+\kappa^2}\lambda'^2 K^3 \sqrt{\kappa^2+\lambda^2}} [A_{11}K^2 + A_{12}KE + A_{22}E^2] \\ \langle \dot{h} \rangle &= \frac{3}{2G} n^2 (\overline{A} - \overline{B})(1 + \delta - 3B_{0,0}) \\ &\times \left[\left(-A_0^{(0)} + \frac{1}{2} A_0^{(2)} \right) \cos \rho + \cos 2\rho \csc \rho A_0^{(1)} \right]\end{aligned} \tag{114}$$

where

$$\begin{aligned}A_{11} &= \lambda'^2 [\lambda^2 (\delta\kappa^2 + 1 + \kappa^2) + \kappa^2 (\delta\kappa^2 - 1 - \kappa^2)] \\ A_{12} &= 2\lambda'^2 \kappa^2 (\delta\kappa^2 - 1 - \kappa^2) \\ A_{22} &= (1 + \kappa^2 - \delta\kappa^2)(\kappa^2 + \lambda^2).\end{aligned} \tag{115}$$

Formulae (114), (115) and (113) define the secular effects on the rotation of the body in the case of arbitary values of the parameter $0 < \lambda^2 < 1$. However, in the case of the Earth the parameter $\lambda \simeq 10^{-7}$ is small. Therefore, here we also obtain simplified formulae for perturbations (113), (114) saving only the main terms.

So the approximate averaging value of the force function is

$$\langle U \rangle = \frac{1}{4} n^2 (\overline{A} - \overline{B})(1 - 2\delta) \left[1 - \frac{3}{2} \frac{\lambda^2}{\kappa^2} \left(1 + \frac{\kappa^2 (1 - \delta)}{1 - 2\delta} \right) \right] R_{0,0} \tag{116}$$

and for the functions $J(\lambda), \Lambda(\lambda)$:

$$J(\lambda) = \frac{\lambda\sqrt{1+\kappa^2}}{\kappa^2}\left(1+\lambda^2\left(\frac{1}{4}-\frac{3}{2\kappa^2}\right)\right)$$
$$\Lambda(\lambda) = 1-\lambda^2\frac{\sqrt{1+\kappa^2}}{2\kappa^2}. \tag{117}$$

Neglecting small terms for the secular velocities (114) we obtain the following reduced formulae:

$$\langle\dot{\varphi}\rangle = \frac{3n^2(\overline{B}-\overline{A})}{4G\sqrt{1+\kappa^2}}[1-2\delta+\kappa^2(1-\delta)]R_{0,0}$$
$$\langle\dot{h}\rangle = \frac{3}{2G}n^2(\overline{A}-\overline{B})(1-2\delta)\left[1-\frac{3\lambda^2}{2\kappa^2}\left(1+\frac{\kappa^2(1-\delta)}{1-2\delta}\right)\right]$$
$$\times\left[\left(-A_0^{(0)}+\frac{1}{2}A_0^{(2)}\right)\cos\rho+\cos 2\rho\csc\rho A_0^{(1)}\right]. \tag{118}$$

For corresponding values of the velocities $\langle\dot{\psi}+\dot{\varphi}\rangle$, $\langle\dot{\psi}\rangle$ we have expressions:

$$\langle\dot{\psi}+\dot{\varphi}\rangle = \lambda^2\frac{\sqrt{1+\kappa^2}}{2\kappa^2}\langle\dot{\varphi}\rangle-\cos\rho\langle\dot{h}\rangle \tag{119}$$

$$\langle\dot{\psi}\rangle = \left(-1+\lambda^2\frac{\sqrt{1+\kappa^2}}{2\kappa^2}\right)\langle\dot{\varphi}\rangle-\cos\rho\langle\dot{h}\rangle. \tag{120}$$

From these formulae it follows that the values of the velocity $\langle\dot{\varphi}\rangle$ for Eulerian motion (for $\kappa=\kappa_E$) and for Chandlerian motion (for $\kappa=\kappa_{ch}$) are different, but for the velocities $\langle\dot{h}\rangle$, $\langle\dot{\psi}\rangle$, $\langle\dot{\psi}+\dot{\varphi}\rangle$ similar differences are small (of the order $\frac{\lambda^2}{\kappa^2}\sim 10^{-12}$ from main values).

8 FIRST-ORDER PERTURBATIONS

By setting $\mathcal{H}_1\equiv 0$ from equations (80) we obtain the solution describing the unperturbed rotation of the Earth:

$$G=G_0,\quad \rho=\rho_0,\quad \lambda=\lambda_0$$
$$\varphi=\Omega(G_0,\lambda_0)t+\varphi_0,\quad \psi=\omega(G_0,\lambda_0)t+\psi_0,\quad h=h_0 \tag{121}$$

where

$$G_0,\rho_0,\lambda_0,\varphi_0,\psi_0,h_0 \tag{122}$$

are initial conditions of the problem.

In this paper we will obtain only analytical formulae for perturbations of the first order in the Earth's rotation. Therefore, the numerical values of the initial conditions (122) are omitted here.

The first-order periodic perturbations in the Earth's rotation caused by the Moon's attraction are defined by the integrals:

$$
\begin{aligned}
\tilde{G}_1 &= \int \frac{\partial U}{\partial \psi} dt \\
\tilde{\lambda}_1 &= -\frac{1}{GJ(\lambda)} \int \frac{\partial U}{\partial \varphi} dt + \frac{\Lambda(\lambda)}{GJ(\lambda)} \int \frac{\partial U}{\partial \psi} dt \\
\tilde{\rho}_1 &= \frac{1}{G} \cot \rho \int \frac{\partial U}{\partial \psi} dt - \frac{1}{G} \csc \rho \int \frac{\partial U}{\partial h} dt \\
\tilde{\varphi}_1 &= \frac{\partial \Omega}{\partial G} \int G_1 dt + \frac{\partial \Omega}{\partial \lambda} \int \lambda_1 dt + \frac{1}{GJ(\lambda)} \int \frac{\partial U}{\partial \lambda} dt \\
\tilde{\psi}_1 &= \frac{\partial \omega}{\partial G} \int G_1 dt + \frac{\partial \omega}{\partial \lambda} \int \lambda_1 dt - \frac{\Lambda(\lambda)}{GJ(\lambda)} \int \frac{\partial U}{\partial \lambda} dt - \frac{1}{G} \cot \rho \int \frac{\partial U}{\partial \rho} dt \\
\tilde{h}_1 &= \frac{1}{G} \csc \rho \int \frac{\partial U}{\partial \rho} dt \qquad (123) \\
U &= \sum_{\|\nu\| \geq 0} \sum_{k_1} \sum_{k_2} U_{\nu;k_1,k_2}(\lambda, \rho) \cos(\Theta_\nu + k_1 \varphi_1 + k_2 \varphi_2) \qquad (124)
\end{aligned}
$$

where the indexes $k_1 = 0, \pm 2m$; $k_2 = 0, \pm 2$; $m = 1, 2, \ldots$; and the coefficients $U_{\nu;k_1,k_2}$ are defined by formulae (102), (103).

The variables on the right-hand sides of equations (123) and the expressions under the integral are taken to be unperturbed values. Also we save only periodic terms in these expressions under integral.

The results of the calculation of integrals (123), (124) are finally obtained by the following expressions for first-order perturbations in the Earth's rotation caused by Moon's (Sun's) attraction:

$$
\begin{aligned}
\tilde{G}_1 &= \sum G^{(1)}_{\nu;k_1,k_2}(\rho, \lambda) \cos(\Theta_\nu + k_1 \varphi_1 + k_2 \varphi_2) \\
\tilde{\lambda}_1 &= \sum \lambda^{(1)}_{\nu;k_1,k_2}(\rho, \lambda) \cos(\Theta_\nu + k_1 \varphi_1 + k_2 \varphi_2) \\
\tilde{\rho}_1 &= \sum \rho^{(1)}_{\nu;k_1,k_2}(\rho, \lambda) \cos(\Theta_\nu + k_1 \varphi_1 + k_2 \varphi_2) \\
\tilde{\varphi}_1 &= \sum \varphi^{(1)}_{\nu;k_1,k_2}(\rho, \lambda) \sin(\Theta_\nu + k_1 \varphi_1 + k_2 \varphi_2) \\
\tilde{\psi}_1 &= \sum \psi^{(1)}_{\nu;k_1,k_2}(\rho, \lambda) \sin(\Theta_\nu + k_1 \varphi_1 + k_2 \varphi_2) \\
\tilde{h}_1 &= \sum h^{(1)}_{\nu;k_1,k_2}(\rho, \lambda) \sin(\Theta_\nu + k_1 \varphi_1 + k_2 \varphi_2) \qquad (125)
\end{aligned}
$$

where

$$
\begin{aligned}
G^{(1)}_{\nu;k_1,k_2} &= \frac{k_2 U_{\nu;k_1,k_2}}{N_\nu + k_1 \Omega + k_2 \omega} \\
\lambda^{(1)}_{\nu;k_1,k_2} &= \frac{[-k_1 + k_2 \Lambda(\lambda)]}{N_\nu + k_1 \Omega + k_2 \omega} U_{\nu;k_1,k_2} \\
\rho^{(1)}_{\nu;k_1,k_2} &= \frac{1}{G} \csc \rho \frac{(\cos \rho k_2 - i_5)}{N_\nu + k_1 \Omega + k_2 \omega} U_{\nu;k_1,k_2}
\end{aligned}
$$

$$
\begin{aligned}
\varphi^{(1)}_{\nu;k_1,k_2} &= \frac{\{\frac{\partial\Omega}{\partial G}k_2 + [-k_1 + k_2\Lambda(\lambda)]\frac{\partial\Omega}{\partial\lambda}\}U_{\nu;k_1,k_2}}{(N_\nu + k_1\Omega + k_2\omega)^2} \\
&+ \frac{\frac{\partial U_{\nu;k_1,k_2}}{\partial\lambda}}{GJ(\lambda)(N_\nu + k_1\Omega + k_2\omega)} \\
\psi^{(1)}_{\nu;k_1,k_2} &= \frac{\{\frac{\partial\omega}{\partial G}k_2 + [-k_1 + k_2\Lambda(\lambda)]\frac{\partial\omega}{\partial\lambda}\}U_{\nu;k_1,k_2}}{(N_\nu + k_1\Omega + k_2\omega)^2} \\
&- \frac{1}{G}\left\{\frac{\Lambda(\lambda)}{J(\lambda)}\frac{\partial U_{\nu;k_1,k_2}}{\partial\lambda} + \cot\rho\frac{\partial U_{\nu;k_1,k_2}}{\partial\rho}\right\}\frac{1}{N_\nu + k_1\Omega + k_2\omega} \\
h^{(1)}_{\nu;k_1,k_2} &= \frac{1}{G}\csc\rho\frac{\frac{\partial U_{\nu;k_1,k_2}}{\partial\rho}}{N_\nu + k_1\Omega + k_2\omega}.
\end{aligned}
\tag{126}
$$

After some algebra we obtain the following set of formulae for derivatives of the frequencies Ω, ω and of the coefficients $U_{\nu;k_1,k_2}$.

The derivatives of the frequencies Ω, ω with respect to λ and G are:

$$
\begin{aligned}
\frac{\partial\Omega}{\partial\lambda} &= \frac{C-A}{AC}\frac{\kappa}{\sqrt{1+\kappa^2}}\frac{[-\lambda'^2\kappa^2K + (\kappa^2+\lambda^2)E]}{K^2\lambda\lambda'^2(\kappa^2+\lambda^2)^{3/2}} \\
\frac{\partial\omega}{\partial\lambda} &= \frac{C-A}{AC}\frac{[\lambda^2KE + \kappa^2\lambda'^2\Pi K - (\kappa^2+\lambda^2)\Pi E]}{K^2\lambda\lambda'^2(\kappa^2+\lambda^2)} \\
\frac{\partial\Omega}{\partial G} &= \frac{\Omega}{G}, \quad \frac{\partial\omega}{\partial G} = \frac{\omega}{G}.
\end{aligned}
\tag{127}
$$

Here K, E and Π are elliptic full integrals of the first, second and third kinds. The derivatives of the coefficients $U_{\nu;k_1,k_2}$ are:

$$
\begin{aligned}
\frac{\partial U_{\nu;2\varepsilon m,2\mu}}{\partial\rho} &= \frac{3}{2}n^2(\overline{A}-\overline{B})A_{2m;2\varepsilon\mu}R'_{\nu;2\mu} \\
\frac{\partial U_{\nu;2\varepsilon m,\mu}}{\partial\rho} &= \frac{3}{2}n^2(\overline{A}-\overline{B})C_{2m;\varepsilon\mu}R'_{\nu;\mu} \\
\frac{\partial U_{\nu;2\varepsilon m,0}}{\partial\rho} &= \frac{3}{2}n^2(\overline{A}-\overline{B})C_{2m;\varepsilon}R'_{\nu;0} \\
\frac{\partial U_{\nu;0,\mu}}{\partial\rho} &= \frac{3}{2}n^2(\overline{A}-\overline{B})C_{0,1}R'_{\nu;\mu} \\
\frac{\partial U_{\nu;0,2\mu}}{\partial\rho} &= \frac{3}{2}n^2(\overline{A}-\overline{B})A_{0,2}R'_{\nu;2\mu} \\
\frac{\partial U_{\nu;0,0}}{\partial\rho} &= \frac{1}{4}n^2(\overline{A}-\overline{B})(1+\delta-3B_{0,0})R'_{\nu;0}
\end{aligned}
\tag{128}
$$

where

$$
\begin{aligned}
R'_{\nu;2\mu} &= -A^{(0)}_\nu\sin 2\rho + 2(\cos 2\rho - \mu\cos\rho)A^{(1)}_\nu - A^{(2)}_\nu(1-\mu\cos\rho)\mu\sin\rho \\
R'_{\nu;\mu} &= -2A^{(0)}_\nu\cos 2\rho + 2(\mu\sin\rho - \sin 2\rho)A^{(1)}_\nu + (\cos 2\rho - \mu\cos\rho)A^{(2)}_\nu \\
R'_{\nu;0} &= (-3A^{(0)}_\nu + \frac{3}{2}A^{(2)}_\nu)\sin 2\rho + 6A^{(1)}_\nu\cos 2\rho.
\end{aligned}
\tag{129}
$$

The derivatives $\frac{\partial U_{\nu;k_1,k_2}}{\partial\lambda}$.

$$\begin{aligned}
\frac{\partial U_{\nu;2\varepsilon m,2\mu}}{\partial\lambda} &= \frac{3}{2}n^2(\overline{A}-\overline{B})\frac{\partial A_{2m,2\varepsilon\mu}}{\partial\lambda}R_{\nu,2\mu}\\
\frac{\partial U_{\nu;2\varepsilon m,\mu}}{\partial\lambda} &= \frac{3}{2}n^2(\overline{A}-\overline{B})\frac{\partial C_{2m,\varepsilon\mu}}{\partial\lambda}R_{\nu,\mu}\\
\frac{\partial U_{\nu;2\varepsilon m,0}}{\partial\lambda} &= \frac{3}{2}n^2(\overline{A}-\overline{B})\frac{\partial C_{2m,\varepsilon}}{\partial\lambda}R_{\nu,0}\\
\frac{\partial U_{\nu;0,\mu}}{\partial\lambda} &= \frac{3}{2}n^2(\overline{A}-\overline{B})\frac{\partial C_{0,1}}{\partial\lambda}R_{\nu,\mu}\\
\frac{\partial U_{\nu;0,2\mu}}{\partial\lambda} &= \frac{3}{2}n^2(\overline{A}-\overline{B})\frac{\partial A_{0,2}}{\partial\lambda}R_{\nu,2\mu}\\
\frac{\partial U_{\nu;0,0}}{\partial\lambda} &= -\frac{3}{4}n^2(\overline{A}-\overline{B})\frac{\partial B_{0,0}}{\partial\lambda}
\end{aligned}\tag{130}$$

where

$$\begin{aligned}
\frac{\partial A_{2m,2\varepsilon\mu}}{\partial\lambda} &= \left\{\frac{1}{Q}\frac{\partial Q}{\partial\lambda} - \frac{2\varepsilon\mu\frac{\partial P}{\partial\lambda}}{[m(D-1)-2\varepsilon\mu P]} - 2\coth 2(md-\varepsilon\mu\sigma)(md_\lambda-\varepsilon\mu\sigma_\lambda)\right\}\\
&\quad\times A_{2m;2\varepsilon\mu};\\
\frac{\partial C_{2m,\varepsilon\mu}}{\partial\lambda} &= \left\{\frac{1}{Q}\frac{\partial Q}{\partial\lambda} + \frac{1}{P}\frac{\partial P}{\partial\lambda} - \tanh(2md-\varepsilon\mu\sigma)(2md_\lambda-\varepsilon\mu\sigma_\lambda)\right\}\times C_{2m;\varepsilon\mu};\\
\frac{\partial C_{0,1}}{\partial\lambda} &= \left\{\frac{1}{Q}\frac{\partial Q}{\partial\lambda} + \frac{1}{P}\frac{\partial P}{\partial\lambda} - \tanh\sigma\sigma_\lambda\right\}\times C_{0,1};\\
\frac{\partial A_{0,2}}{\partial\lambda} &= \left\{\frac{1}{Q}\frac{\partial Q}{\partial\lambda} + \frac{1}{P}\frac{\partial P}{\partial\lambda} - 2\coth(2\sigma)\sigma_\lambda\right\}\times A_{0,2};\\
\frac{\partial B_{0,0}}{\partial\lambda} &= \frac{A_{11}K^2 + A_{12}KE + A_{22}E^2}{\lambda\lambda'^2K^2(\kappa^2+\lambda^2)^2}.
\end{aligned}\tag{131}$$

Here

$$\begin{aligned}
A_{11} &= \lambda'^2[\lambda^2(1+\kappa^2+\delta\kappa^2) + \kappa^2(\delta\kappa^2-1-\kappa^2)]\\
A_{12} &= 2\lambda'^2\kappa^2(\delta\kappa^2-1-\kappa^2)\\
A_{22} &= (1+\kappa^2-\delta\kappa^2)(\kappa^2+\lambda^2).
\end{aligned}\tag{132}$$

For our unperturbed motion (see the Introduction)

$$\begin{aligned}
\sigma &= \frac{\pi}{2K}F(\arctan\frac{\kappa}{\lambda},\lambda')\\
d &= \frac{\pi K'}{2K}
\end{aligned}\tag{133}$$

and for their derivatives relative to λ in (131) $(\sigma_\lambda, d_\lambda)$ we found the following representation:

$$\frac{\partial\sigma}{\partial\lambda} = -\frac{\pi^2}{4K^2\lambda\lambda'^2}\Lambda_0(n,\lambda) \tag{134}$$

where Λ_0 is the lambda function,

$$\begin{aligned} \Lambda_0 &= \frac{1}{\pi}[F(n,\lambda)(E(\lambda)-K(\lambda))+K(\lambda)E(n,\lambda)] \\ n &= \arctan\frac{\kappa}{\lambda} \end{aligned} \tag{135}$$

and

$$\frac{\partial d}{\partial\lambda} = -\frac{\pi}{2\lambda\lambda'^2K^2}[E'K+EK'-KK'] = -\frac{\pi}{2\lambda\lambda' K^2} \tag{136}$$

(in the last formula we used Legandre's relation).

For the derivatives of P and Q in (131) we have the following formulae:

$$\begin{aligned} \frac{\partial Q}{\partial\lambda} &= \frac{\pi^2(1+\kappa^2}{2K^3(\kappa^2+\lambda^2)^2}\frac{[\lambda'^2\kappa^2K - E(\kappa^2+\lambda^2)]}{\lambda\lambda'^2} \\ \frac{\partial P}{\partial\lambda} &= D\frac{\partial M_1}{\partial\lambda} - \frac{\partial M_3}{\partial\lambda} \\ \frac{\partial M_1}{\partial\lambda} &= \frac{\lambda\sqrt{1+\kappa^2}E}{\pi\kappa\lambda'^2\sqrt{\kappa^2+\lambda^2}} \\ \frac{\partial M_3}{\partial\lambda} &= \frac{\kappa(K\lambda^2-E\lambda'^2)}{\pi\lambda\lambda'^2\sqrt{(1+\kappa^2)(\kappa^2+\lambda^2)}}. \end{aligned} \tag{137}$$

9 CONCLUDING REMARKS

(1) A new method of the construction of the theory of rotational motion for a weakly deformable body was proposed. The basis of this method is a new unperturbed rotational motion (Chandler motion) and angle-action variables. We reduced the unperturbed problem to the Euler–Poinsot problem for a rigid body with special moments of inertia.

(2) The development of the force function of the problem about the perturbed Earth rotation was obtained in angle-action variables.

(3) In the Earth's rotation the following phenomena were described:

  (a) Chandler's direct motion of the pole of the Earth;

  (b) ellipticity of the trajectory of the Earth's pole;

  (c) non-uniformity of the pole motion along are elliptical trajectory.

(4) Explicit formulae for first-order perturbations due to the force function of the Earth–Moon system were obtained (secular and periodic effects).

All the results of this paper are presented in analytical form and are applicable for the study of Solar system bodies (Venus, the asteroids (in particular, for double asteroids), satellites with irregular forms, comet's cores, etc.).

The initial version of this analytical theory (for rigid body) was accepted at the ASS/AIAA Astrodynamics Conference in 1993 (Barkin, 1993). An improvement of the theory, to include the elastic properties of the body, were taken into account, and this was influenced by the well-known Getino, Ferrandiz papers (1990, 1991a). This was reflected in the joint paper (Barkin *et al.*, 1995) and also in the report (Barkin, 1996).

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Appendix. Fourier Series For Products And Squares Of The Direction Cosines b_{ij}

1.

$$b_{11}^2 = \frac{-\pi^2}{4K^2(\kappa^2+\lambda^2)}\left\{ \sum_{m=1}^{\infty}\left[\frac{m+2M_4}{sh\,2(md+\sigma)}\cos 2(m\varphi_1-\varphi_2) + \frac{m-2M_4}{sh\,2(md-\sigma)}\cos 2(m\varphi_1+\varphi_2) + \frac{2m}{sh\,2md}\cos 2m\varphi_1\right] + \frac{2M_4}{sh\,2\sigma}\cos 2\varphi_2\right\} + b_{0,0}^{(1.1;1.1)},$$

2.

$$b_{12}b_{11} = \frac{\pi^2\sqrt{1+\kappa^2}}{4K^2(\kappa^2+\lambda^2)}\left\{ \sum_{m=1}^{\infty}\left[\frac{m+M_3+M_4}{ch\,2(md+\sigma)}\sin 2(m\varphi_1-\varphi_2) + \frac{m-M_3-M_4}{ch\,2(md-\sigma)}\sin 2(m\varphi_1+\varphi_2) + \frac{2m}{ch\,(2md)}\sin 2m\varphi_1\right] - \frac{M_3+M_4}{ch(2\sigma)}\sin 2\varphi_2\right\},$$

3.

$$b_{12}^2 = \frac{\pi^2(1+\kappa^2)}{4K^2(\kappa^2+\lambda^2)}\left\{ \sum_{m=1}^{\infty}\left[\frac{m+2M_3}{sh\,2(md+\sigma)}\cos 2(m\varphi_1-\varphi_2) + \frac{m-2M_3}{sh\,2(md-\sigma)}\cos 2(m\varphi_1+\varphi_2) + \frac{2m}{sh\,(2md)}\cos 2m\varphi_1\right] + \frac{2M_3}{sh\,(2\sigma)}\cos 2\varphi_2\right\} + b_{0,0}^{(1.2;1.2)},$$

4.

$$b_{13}b_{11} = -\frac{\pi^2\kappa}{8K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\left\{\frac{2m+1+2M_1+2M_4}{sh[(2m+1)d+2\sigma]}\cos[(2m+1)\varphi_1-2\varphi_2] + \frac{2m+1-2M_1-2M_4}{sh[(2m+1)d-2\sigma]}\cos[(2m+1)\varphi_1+2\varphi_2] + \frac{2(2m+1)}{sh(2m+1)d}\cos(2m+1)\varphi_1\right\},$$

5.

$$b_{13}b_{12} = \frac{\pi^2\kappa\sqrt{1+\kappa^2}}{8K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\left\{\frac{2m+1+2M_1+2M_3}{ch[(2m+1)d+2\sigma]}\sin[(2m+1)\varphi_1-2\varphi_2] + \frac{2m+1-2M_1-2M_3}{ch[(2m+1)d-2\sigma]}\sin[(2m+1)\varphi_1+2\varphi_2] + \frac{2(2m+1)}{ch(2m+1)d}\sin(2m+1)\varphi_1\right\},$$

6.

$$b_{13}^2 = \frac{\pi^2\kappa^2}{4K^2(\kappa^2+\lambda^2)}\left\{ \sum_{m=1}^{\infty}\left[-\frac{m+2M_1}{sh\,2(md+\sigma)}\cos 2(m\varphi_1-\varphi_2) - \frac{m-2M_1}{sh\,2(md-\sigma)}\cos 2(m\varphi_1+\varphi_2) - \frac{2m}{sh\,(2md)}\cos 2m\varphi_1\right] - \frac{2M_1}{sh\,(2\sigma)}\cos 2\varphi_2\right\} + b_{0,0}^{(1.3;1.3)},$$

7.

$$b_{21}b_{11} = \frac{-\pi^2}{4K^2(\kappa^2+\lambda^2)}\left\{ \sum_{m=1}^{\infty}\left[-\frac{m+2M_4}{sh\,2(md+\sigma)}\sin 2(m\varphi_1-\varphi_2) + \frac{m-2M_4}{sh\,2(md-\sigma)}\sin 2(m\varphi_1+\varphi_2) + \frac{2M_4}{sh\,(2\sigma)}\sin 2\varphi_2\right\}$$

8.

$$b_{21}b_{12} = \frac{\pi^2\sqrt{1+\kappa^2}}{4K^2(\kappa^2+\lambda^2)}\left\{ \sum_{m=1}^{\infty}\left[\frac{m+M_3+M_4}{ch\,2(md+\sigma)}\cos 2(m\varphi_1-\varphi_2) - \frac{m-M_3-M_4}{ch\,2(md-\sigma)}\cos 2(m\varphi_1+\varphi_2)\right]\right.$$

$$+\frac{M_3+M_4}{ch\,(2\sigma)}\cos 2\varphi_2\Bigg\} - \frac{\pi\kappa}{2K\sqrt{\kappa^2+\lambda^2}}\sum_{m=1}^{\infty}\left\{\frac{\cos 2m\varphi_1}{ch\,(2md)}\right\} + b_{0,0}^{(2.1,1.2)},$$

9.

$$b_{21}b_{13} = \frac{\pi^2\kappa}{8K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{\frac{2m+1+2M_1+2M_4}{sh[(2m+1)d+2\sigma]}\sin[(2m+1)\varphi_1-2\varphi_2]$$

$$-\frac{2m+1-2M_1-2M_4}{sh[(2m+1)d-2\sigma]}\sin[(2m+1)\varphi_1+2\varphi_2]\Bigg\} - \frac{\pi\sqrt{1+\kappa^2}}{K\sqrt{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\frac{\sin[(2m+1)\varphi_1]}{sh\,[(2m+1)d]},$$

10.

$$b_{21}^2 = \frac{\pi^2}{4K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[\frac{m+2M_4}{sh\,2(md+\sigma)}\cos 2(m\varphi_1-\varphi_2) + \frac{m-2M_4}{sh\,2(md-\sigma)}\cos 2(m\varphi_1+\varphi_2)$$

$$-\frac{2m}{sh\,2md}\cos 2m\varphi_1\Bigg] + \frac{2M_4}{sh\,(2\sigma)}\cos 2\varphi_2\Bigg\} + b_{0,0}^{(2.1;2.1)},$$

11.

$$b_{22}b_{11} = \frac{\pi^2\sqrt{1+\kappa^2}}{4K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[\frac{m+M_3+M_4}{ch\,2(md+\sigma)}\cos 2(m\varphi_1-\varphi_2) - \frac{m-M_3-M_4}{ch\,2(md-\sigma)}\cos 2(m\varphi_1+\varphi_2)\Bigg]$$

$$+\frac{M_3+M_4}{ch\,(2\sigma)}\cos 2\varphi_2\Bigg\} + \frac{\pi\kappa}{2K\sqrt{\kappa^2+\lambda^2}}\sum_{m=1}^{\infty}\frac{\cos 2m\varphi_1}{ch\,(2md)} + b_{0,0}^{(2.2,1.1)}$$

12.

$$b_{22}b_{12} = \frac{\pi^2(1+\kappa^2)}{4K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[-\frac{m+2M_3}{sh\,2(md+\sigma)}\sin 2(m\varphi_1-\varphi_2) + \frac{m-2M_3}{sh\,2(md-\sigma)}\sin 2(m\varphi_1+\varphi_2)\Bigg]$$

$$+\frac{2M_3}{sh\,(2\sigma)}\sin 2\varphi_2\Bigg\},$$

13.

$$b_{22}b_{13} = \frac{\pi^2\kappa\sqrt{1+\kappa^2}}{8K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{\frac{2m+1+2M_1+2M_3}{ch\,[(2m+1)d+2\sigma]}\cos[(2m+1)\varphi_1-2\varphi_2]$$

$$-\frac{2m+1-2M_1-2M_3}{ch\,[(2m+1)d-2\sigma]}\cos[(2m+1)\varphi_1+2\varphi_2]\Bigg\} - \frac{\pi}{2K\sqrt{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\frac{\cos(2m+1)\varphi_1}{ch\,(2m+1)d},$$

14.

$$b_{22}b_{21} = \frac{\pi^2\sqrt{1+\kappa^2}}{4K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[-\frac{m+M_3+M_4}{ch\,2(md+\sigma)}\sin 2(m\varphi_1-\varphi_2) - \frac{m-M_3-M_4}{ch\,2(md-\sigma)}\sin 2(m\varphi_1+\varphi_2)$$

$$+\frac{2m}{ch\,(2md)}\sin 2m\varphi_1\Bigg] + \frac{M_3+M_4}{ch(2\sigma)}\sin 2\varphi_2\Bigg\},$$

15.

$$b_{22}^2 = \frac{\pi^2(1+\kappa^2)}{4K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[-\frac{m+2M_3}{sh\,2(md+\sigma)}\cos 2(m\varphi_1-\varphi_2) - \frac{m-2M_3}{sh\,2(md-\sigma)}\cos 2(m\varphi_1+\varphi_2)$$

$$+\frac{2m}{sh\,(2md)}\cos 2m\varphi_1\Bigg] - \frac{2M_3}{sh\,(2\sigma)}\cos 2\varphi_2\Bigg\} + b_{0,0}^{(2.2;2.2)},$$

16.

$$b_{23}b_{11} = \frac{\pi^2\kappa}{8K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{\frac{2m+1+2M_1+2M_4}{sh[(2m+1)d+2\sigma]}\sin[(2m+1)\varphi_1-2\varphi_2]$$

$$-\frac{2m+1-2M_1-2M_4}{sh[(2m+1)d-2\sigma]}\sin[(2m+1)\varphi_1+2\varphi_2]\Bigg\}+\frac{\pi\sqrt{1+\kappa^2}}{K\sqrt{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\frac{\sin[(2m+1)\varphi_1]}{sh\,[(2m+1)d]},$$

17.

$$b_{23}b_{12}=\frac{\pi^2\kappa\sqrt{1+\kappa^2}}{8K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{\frac{2m+1+2M_1+2M_3}{ch\,[(2m+1)d+2\sigma]}\cos[(2m+1)\varphi_1-2\varphi_2]$$

$$-\frac{2m+1-2M_1-2M_3}{ch\,[(2m+1)d-2\sigma]}\cos[(2m+1)\varphi_1+2\varphi_2]\Bigg\}+\frac{\pi}{2K\sqrt{\kappa^2+\lambda^2}}\sum_{m=0}^{\infty}\frac{\cos[(2m+1)\varphi_1]}{ch[(2m+1)d]},$$

18.

$$b_{23}b_{13}=\frac{\pi^2\kappa^2}{4K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[\frac{m+2M_1}{sh\,2(md+\sigma)}\sin 2(m\varphi_1-\varphi_2)$$

$$-\frac{m-2M_1}{sh\,2(md-\sigma)}\sin 2(m\varphi_1+\varphi_2)\Bigg]-\frac{2M_1}{sh\,2\sigma}\sin 2\varphi_2\Bigg\},$$

19.

$$b_{23}b_{21}=\frac{\pi^2\kappa}{8K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{\frac{2m+1+2M_1+2M_4}{sh[(2m+1)d+2\sigma]}\cos[(2m+1)\varphi_1-2\varphi_2]$$

$$+\frac{2m+1-2M_1-2M_4}{sh[(2m+1)d-2\sigma]}\cos[(2m+1)\varphi_1+2\varphi_2]-\frac{2(2m+1)}{sh(2m+1)d}\cos[(2m+1)\varphi_1]\Bigg\},$$

20.

$$b_{23}b_{22}=\frac{\pi^2\kappa\sqrt{1+\kappa^2}}{8K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{-\frac{2m+1+2M_1+2M_3}{ch\,[(2m+1)d+2\sigma]}\sin[(2m+1)\varphi_1-2\varphi_2]$$

$$-\frac{2m+1-2M_1-2M_3}{ch\,[(2m+1)d-2\sigma]}\sin[(2m+1)\varphi_1+2\varphi_2]+\frac{2(2m+1)}{ch(2m+1)d}\sin[(2m+1)\varphi_1]\Bigg\},$$

21.

$$b_{23}^2=\frac{\pi^2\kappa^2}{4K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[\frac{m+2M_1}{sh\,2(md+\sigma)}\cos 2(m\varphi_1-\varphi_2)+\frac{m-2M_1}{sh\,2(md-\sigma)}\cos 2(m\varphi_1+\varphi_2)$$

$$-\frac{2m}{sh\,(2md)}\cos 2m\varphi_1\Bigg]+\frac{2M_1}{sh\,(2\sigma)}\cos 2\varphi_2\Bigg\}+b_{0,0}^{(2.3;2.3)},$$

22.

$$b_{31}b_{11}=-\frac{\pi^2}{2K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[\frac{m+M_4}{ch\,(2md+\sigma)}\sin(2m\varphi_1-\varphi_2)+\frac{m-M_4}{ch\,(2md-\sigma)}\sin(2m\varphi_1+\varphi_2)\Bigg]$$

$$-\frac{M_4}{ch\,\sigma}\sin\varphi_2\Bigg\},$$

23.

$$b_{31}b_{12}=\frac{\pi^2(1+\kappa^2)}{2K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[\frac{m+2M_3}{sh\,(2md+\sigma)}\sin(2m\varphi_1-\varphi_2)-\frac{m-2M_3}{sh\,(2md-\sigma)}\sin(2m\varphi_1+\varphi_2)\Bigg]$$

$$+\frac{2M_3}{sh\,\sigma}\sin\varphi_2\Bigg\},$$

24.

$$b_{31}b_{13}=\frac{\pi^2\kappa^2}{4K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{-\frac{2m+1+2M_1}{ch[(2m+1)d+\sigma]}\sin[(2m+1)\varphi_1-\varphi_2]$$

$$+\frac{2m+1-2M_1}{ch[(2m+1)d-\sigma]}\sin[(2m+1)\varphi_1+\varphi_2]\Bigg\},$$

25.

$$b_{31}b_{21} = \frac{\pi^2}{2K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[-\frac{m+M_4}{ch\,(2md+\sigma)}\cos(2m\varphi_1-\varphi_2)$$
$$+\frac{m-M_4}{ch\,(2md-\sigma)}\cos(2m\varphi_1+\varphi_2)\Bigg]-\frac{M_4}{ch\,\sigma}\cos\varphi_2\Bigg\},$$

26.

$$b_{31}b_{22} = \frac{\pi^2(1+\kappa^2)}{2K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[\frac{m+2M_3}{sh\,(2md+\sigma)}\cos(2m\varphi_1-\varphi_2)$$
$$+\frac{m-2M_3}{sh\,(2md-\sigma)}\cos(2m\varphi_1+\varphi_2)\Bigg]+\frac{2M_3}{sh\,\sigma}\cos\varphi_2\Bigg\},$$

27.

$$b_{31}b_{23} = \frac{\pi^2\kappa}{4K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{-\frac{2m+1+2M_1}{ch[(2m+1)d+\sigma]}\cos[(2m+1)\varphi_1-\varphi_2]$$
$$+\frac{2m+1-2M_1}{ch[(2m+1)d-\sigma]}\cos[(2m+1)\varphi_1+\varphi_2]\Bigg\},$$

28.

$$b_{31}^2 = \frac{\pi^2}{K^2(\kappa^2+\lambda^2)}\sum_{m=1}^{\infty}\Bigg\{\frac{m}{sh(2md)}\cos 2m\varphi_1\Bigg\}+b_{0,0}^{(3.1;3.1)},$$

29.

$$b_{32}b_{11} = -\frac{\pi^2\sqrt{1+\kappa^2}}{2K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[\frac{m+M_4}{sh\,(2md+\sigma)}\cos(2m\varphi_1-\varphi_2)$$
$$+\frac{m-M_4}{sh\,(2md-\sigma)}\cos(2m\varphi_1+\varphi_2)\Bigg]+\frac{M_4}{sh\,\sigma}\cos\varphi_2\Bigg\},$$

30.

$$b_{32}b_{12} = \frac{\pi^2(1+\kappa^2)}{2K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[\frac{m+M_3}{ch\,(2md+\sigma)}\sin(2m\varphi_1-\varphi_2)$$
$$+\frac{m-M_3}{ch\,(2md-\sigma)}\sin(2m\varphi_1+\varphi_2)\Bigg]-\frac{M_3}{ch\,\sigma}\sin\varphi_2\Bigg\},$$

31.

$$b_{32}b_{13} = -\frac{\pi^2\kappa\sqrt{1+\kappa^2}}{4K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{\frac{2m+1+2M_1}{sh[(2m+1)d+\sigma]}\cos[(2m+1)\varphi_1-\varphi_2]$$
$$+\frac{2m+1-2M_1}{sh[(2m+1)d-\sigma]}\cos[(2m+1)\varphi_1+2\varphi_2]\Bigg\},$$

32.

$$b_{32}b_{21} = \frac{\pi^2\sqrt{1+\kappa^2}}{2K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[\frac{m+M_4}{sh\,(2md+\sigma)}\sin(2m\varphi_1-\varphi_2)\Bigg]$$
$$-\frac{m-M_4}{sh\,(2md-\sigma)}\sin(2m\varphi_1+\varphi_2)\Bigg]-\frac{M_4}{sh(\sigma)}\sin\varphi_2\Bigg\},$$

33.

$$b_{32}b_{22} = \frac{\pi^2(1+\kappa^2)}{2K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[\frac{m+M_3}{ch\,(2md+\sigma)}\cos(2m\varphi_1-\varphi_2)\Bigg]$$

$$-\frac{m-M_3}{ch\,(2md-\sigma)}\cos(2m\varphi_1+\varphi_2)\Bigg]+\frac{M_3}{ch\,\sigma}\cos\varphi_2\Bigg\},$$

34.

$$b_{32}b_{23}=\frac{\pi^2\kappa\sqrt{1+\kappa^2}}{4K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{\frac{2m+1+2M_1}{sh[(2m+1)d+\sigma]}\sin[(2m+1)\varphi_1-\varphi_2] -\frac{2m+1-2M_1}{sh[(2m+1)d-\sigma]}\sin[(2m+1)\varphi_1+\varphi_2]\Bigg\},$$

35.

$$b_{32}b_{31}=-\frac{\pi^2\sqrt{1+\kappa^2}}{K^2(\kappa^2+\lambda^2)}\sum_{m=1}^{\infty}\Bigg\{\frac{m}{sh(2md)}\sin(2m\varphi_1)\Bigg\},$$

36.

$$b_{32}^2=\frac{-\pi^2(1+\kappa^2)}{K^2(\kappa^2+\lambda^2)}\sum_{m=1}^{\infty}\Bigg\{\frac{m}{sh(2md)}\cos 2m\varphi_1\Bigg\}+b_{0,0}^{(3.2;3.2)},$$

37.

$$b_{33}b_{11}=-\frac{\pi^2\kappa}{4K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{\frac{2m+1+2M_4}{ch[(2m+1)d+\sigma]}\sin[(2m+1)\varphi_1-\varphi_2] +\frac{2m+1-2M_4}{ch[(2m+1)d-\sigma]}\sin[(2m+1)\varphi_1+\varphi_2]\Bigg\},$$

38.

$$b_{33}b_{12}=-\frac{\pi^2\kappa\sqrt{1+\kappa^2}}{4K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{\frac{2m+1+2M_3}{sh[(2m+1)d+\sigma]}\cos[(2m+1)\varphi_1-\varphi_2] +\frac{2m+1-2M_3}{sh[(2m+1)d-\sigma]}\cos[(2m+1)\varphi_1+\varphi_2]\Bigg\},$$

39.

$$b_{33}b_{13}=-\frac{\pi^2\kappa^2}{2K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[\frac{m+M_1}{ch\,2(md+\sigma)}\sin(2m\varphi_1-\varphi_2) +\frac{m-M_1}{ch\,(2md-\sigma)}\sin(2m\varphi_1+\varphi_2)\Bigg]-\frac{M_1}{ch\,\sigma}\sin\varphi_2\Bigg\},$$

40.

$$b_{33}b_{21}=-\frac{\pi^2\kappa}{4K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{\frac{2m+1+2M_4}{ch[(2m+1)d+\sigma]}\cos[(2m+1)\varphi_1-\varphi_2] -\frac{2m+1-2M_4}{ch[(2m+1)d-\sigma]}\cos[(2m+1)\varphi_1+\varphi_2]\Bigg\},$$

41.

$$b_{33}b_{22}=\frac{\pi^2\kappa\sqrt{1+\kappa^2}}{4K^2(\kappa^2+\lambda^2)}\sum_{m=0}^{\infty}\Bigg\{\frac{2m+1+2M_3}{sh[(2m+1)d+\sigma]}\sin[(2m+1)\varphi_1-\varphi_2] -\frac{2m+1-2M_3}{sh[(2m+1)d-\sigma]}\sin[(2m+1)\varphi_1+\varphi_2]\Bigg\},$$

42.

$$b_{33}b_{23}=\frac{\pi^2\kappa^2}{2K^2(\kappa^2+\lambda^2)}\Bigg\{\sum_{m=1}^{\infty}\Bigg[-\frac{m+M_1}{ch\,2(md+\sigma)}\cos[(2m\varphi_1-\varphi_2] +\frac{m-M_1}{ch(2md-\sigma)}\cos[(2m\varphi_1+\varphi_2)\Bigg]-\frac{M_1}{ch\,\sigma}cos\varphi_2\Bigg\},$$

43.

$$b_{33}b_{23} = \frac{\pi^2\kappa}{2K^2(\kappa^2+\lambda^2)} \sum_{m=0}^{\infty} \left\{ \frac{2m+1}{sh[(2m+1)d+\sigma]} \cos[(2m+1)\varphi_1] \right\},$$

44.

$$b_{33}b_{32} = -\frac{\pi^2\kappa}{2K^2(\kappa^2+\lambda^2)} \sum_{m=0}^{\infty} \left\{ \frac{2m+1}{ch[(2m+1)d+\sigma]} \sin[(2m+1)\varphi_1] \right\},$$

45.

$$b_{33}^2 = \frac{\pi^2\kappa^2}{K^2(\kappa^2+\lambda^2)} \sum_{m=1}^{\infty} \left\{ \frac{m}{sh\,2(md)} \cos 2m\varphi_1 \right\} + b_{0,0}^{(3.3;3.3)},$$

where

$$\begin{aligned} M_1 &= \frac{\Pi}{\pi\kappa}\sqrt{(1+\kappa^2)(\kappa^2+\lambda^2)} \\ M_2 &= \frac{\sqrt{1+\kappa^2}}{\pi\kappa\sqrt{\kappa^2+\lambda^2}}[\Pi(\kappa^2+\lambda^2) - K\lambda^2] \\ M_3 &= \frac{\sqrt{1+\kappa^2}}{\pi\kappa\sqrt{1+\kappa^2}}[\Pi(1+\kappa^2) - K] \\ M_4 &= \frac{\Pi - K}{\pi\kappa}\sqrt{(1+\kappa^2)(\kappa^2+\lambda^2)}. \end{aligned}$$

The secular components of the products and squares of the direction cosines are defined by (36).