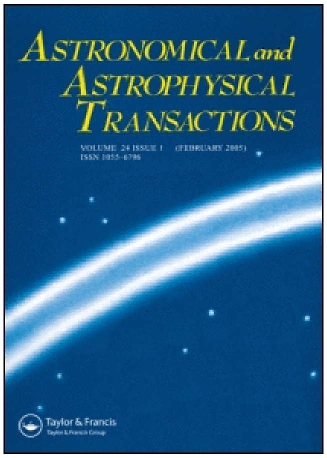

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ON HOMOGENEOUS AND ISOTROPIC COSMOLOGICAL MODELS WITH ARBITRARY NUMBERS OF TIME AND SPACE DIMENSIONS

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It was suggested in [1] that the dimensionality of our physical space may vary from integer three to non integer values on various scales. We gave there some arguments suggested by observational data that the real dimensionality of space may decrease from three to two when passing to cosmological scales. However, general relativity teaches us that space and time should be considered on equal grounds. Here we propose that time my have a noninteger dimensionality distinct from unity. We have no consistent theory to describe this proposition in full, but we can construct a Friedman-like cosmological model which is in principle solvable due to its simplicity and symmetry. We present here some interesting and useful cosmological solutions.

Consider a Riemannian space that is a topological product of m-dimensional flat "temporal" space and n-dimensional homogeneous and isotropic space (m and n are integers as yet). Let the metric interval have the form

$$ds^2 = c^2(dt_1^2 + dt_2^2 + \ldots + dt_m^2) - a^2(t)dl^2, \tag{1}$$

where

$$t = (t_1^2 + t_2^2 + \ldots + t_m^2)^{1/2}$$

is the radial coordinate in the m-dimensional space and a is the scale factor. Our propositions are just that light velocities are the same along all the time axes and that a depends only on radial temporal coordinate t. The latter plays here the role of time. The interval dl^2 is that of conformally connected n-dimensional space of constant curvature,

$$dl^2 = dr^2 + \tilde{\phi}^2(r)\, d\sigma_{(n)}^2,$$

where r is the ordinary radial coordinate. As usual,

$$\tilde{\phi}(r) = \begin{cases} \sin r, & k = +1, \\ r, & k = 0, \\ \operatorname{sh} r, & k = -1, \end{cases}$$

where k is the sign of the Gauss curvature; $d\sigma_{(n)}$ is the differential of the angular distance along a surface on an n-dimensional sphere. As for indices, we denote $A, B, \ldots = 1, \ldots, m$; $i, k, \ldots = 1, \ldots n$ and $\alpha, \beta = A, B \ldots \oplus i, k \ldots$.

The Christoffel symbols for the interval (1) are

$$\Gamma^A_{BC} = 0,\ \Gamma^k_{AB} = 0,\ \Gamma^B_{Aj} = 0,\ \Gamma^k_{Aj} = \frac{\dot a}{a}\frac{t_A}{t}\delta^k_j,\ \Gamma^A_{ij} = -\frac{\dot a}{a}\frac{t^A}{t}g_{ij}, \tag{2}$$

and purely spatial components are not required in explicit forms.

The pure temporal and spatial components of the Einstein tensor can be respectively represented in the forms

$$G_{AB} = f_1\frac{t_A t_B}{t^2} + f_2\left[\delta_{AB} - \frac{t_A t_B}{t^2}\right], \tag{3}$$

$$G_{ij} = f_3 g_{ij}, \tag{4}$$

where (the dot denotes differentiation with respect to ct)

$$\begin{aligned} f_1 &= n\left[\frac{m-1}{t}\frac{\dot a}{a} + \frac{n-1}{2}\left(\frac{\dot a^2}{a^2} + \frac{k}{a^2}\right)\right], \\ f_2 &= n\left[\frac{\ddot a}{a} + \frac{m-2}{t}\frac{\dot a}{a} + \frac{n-1}{2}\left(\frac{\dot a^2}{a^2} + \frac{k}{a^2}\right)\right], \\ f_3 &= (n-1)\left[\frac{\ddot a}{a} + \frac{m-1}{t}\frac{\dot a}{a} + \frac{n-2}{2}\left(\frac{\dot a^2}{a^2} + \frac{k}{a^2}\right)\right], \end{aligned}$$

Obviously, mixed components vanish: $G_{iA} = 0$.

To accomplish constructing our cosmological model, we have to fix the stress-energy tensor $T_{\alpha\beta}$ and to solve the Einstein equations

$$G_{\alpha\beta} = \varkappa T_{\alpha\beta}, \tag{5}$$

with $\varkappa$ some constant, and henceforth we set $\varkappa = 1$ and $c = 1$. The appearance of the Einstein tensor (3)–(5) suggests to choose $T_{\alpha\beta}$ in the form

$$T_{AB} = \varepsilon u_A u_B - p'(\delta_{AB} - u_A u_B), \tag{6}$$

$$T_{iA} = 0,$$

$$T_{ij} = -p g_{ij}, \tag{7}$$

which generalizes hydrodynamic stress-energy tensor and reduces to the latter for $m = 1$. In the general case, $u_A = t_A/t$ is the unit vector field in the temporal space, as usual, the spatial components of the $(m+n)$-dimensional vector field should be zero: $u_i = 0$, then

$$u_\alpha u^\alpha = 1.$$

In the case $m = 1, u_\alpha = \{1, 0, 0, 0\}$ is the common four-velocity vector of a matter fluid. We have introduced in (6) the additional parameter p' which plays the role of tension in the temporal space; for $m = 1$, p' gives no contribution to the single time component T_{00}.

Taking into account (3), (4) and (6), (7), equations (5) are

$$f_1 = \varepsilon, \tag{8a}$$

$$f_2 = -p', \tag{8b}$$

$$f_3 = -p. \tag{8c}$$

The conservation law $T^{\beta}_{\alpha;\beta} = 0$, where the covariant derivative is taken with the use of (2), has the form

$$\dot{\varepsilon} + \frac{m-1}{t}(\varepsilon + p') + n\frac{\dot{a}}{a}(\varepsilon + p) = 0. \tag{9}$$

Now, we should clarify two conceptional points. First, the set of three equations (8) and equation (9) is just the set of equations to be solved, but only three of them are independent. That is why, following [2], we impose the restriction

$$p = q\varepsilon, \tag{10}$$

where q is some real number. The case $q = 0$ corresponds to dust matter. The case $q = 1/n$ might correspond to pure radiation in a spacetime with a single axis because it would give zero trace of the stress-energy tensor. The relation $p = -\varepsilon$ was introduced in cosmology by Einstein, it gives rise to inflationary solutions which permit one to resolve the problems of horizon and flatness [3]. Let us stress that we do not fix the parameter p' *a priori* but Eq. (8b) is considered as an equation for p'.

Second, given the set of equations (8) and (9), we may forget that m and n are integers and consider them real numbers. The only restriction which is worth imposing is $n > 1$ and $m > 0$.

Let us multiply (8a) by q and take the sum with (8c) using (10); we then obtain the equation for the scale factor:

$$\frac{\ddot{a}}{a} + \frac{nq+n-1}{(n-1)}\frac{m-1}{t}\frac{\dot{a}}{a} + \left(\frac{n(q+1)}{2} - 1\right)\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 0,$$

or, equivalently,

$$\frac{\ddot{a}}{a} + \frac{m-1}{m_{\rm cr}-1}\frac{1}{t}\frac{\dot{a}}{a} + \left(\frac{1}{2}\frac{m_{\rm cr}-(2-n)}{m_{\rm cr}-1} - 1\right)\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 0, \tag{11}$$

where we have denoted

$$m_{\rm cr} = 1 + \frac{n-1}{nq+n-1}. \tag{12}$$

Note that $m_{\rm cr} = 2$ for dust matter. We shall see that this notation is not accidental, the quantity (12) has some crucial meaning. Eq. (11) is not solvable generally in elementary functions. We give here solutions for some interesting cases.

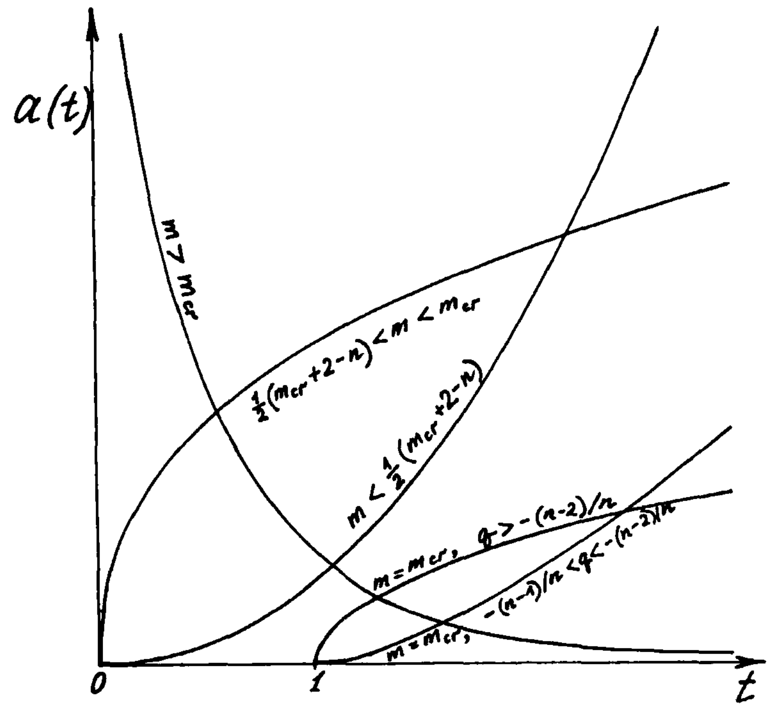

Figure 1 $q > -(n-1)/n$. The cases $m = [m_{\text{cr}} + (2-n)]/2$ and $m = m_{\text{cr}}$, $q = -(n-2)/n$ are not indicated because they give trivial linear and logarithmic solutions, respectively.

(1) *A flat n-dimensional space* ($k = 0$): Eq. (11) can be reduced to an equation in full differentials. Let also $q \neq -1$ as yet.

(1a) $m \neq m_{\text{cr}}$ and $q \neq -1$. The solution of Eq. (11) is

$$a = Ct^{\gamma_{1a}}, \quad \gamma_{1a} = \frac{2(m_{\text{cr}} - m)}{m_{\text{cr}} - (2-n)}, \tag{13}$$

with C in arbitrary constant. For $n > 1$ and any $q > -(n-1)/n$, we have $m_{\text{cr}} > 2-n$ and therefore the behavior of the solution (13) is mainly determined by the sign of the term $m_{\text{cr}} - m$. For $m < m_{\text{cr}}$, the scale factor increases from zero to infinity. As for the derivative $\dot{a}$ at the point $t = 0$, it is zero for $m < [m_{\text{cr}} + (2-n)]/2$ and it is equal to infinity for $[m_{\text{cr}} + (2-n)]/2 < m < m_{\text{cr}}$. If $m > m_{\text{cr}}$, the scale factor collapses from infinity to zero for infinite t. The case $m = 1$ gives solutions already obtained in [1]. The dependence on time of the matter parameters is as follows:

$$\varepsilon, p', p \propto t^{-2}. \tag{14}$$

(1b) $m = m_{\text{cr}}$ and $q \neq -1$. The solution is of a logarithmic nature and starts at the point $t = 1$:

$$a = C(\ln t)^{\gamma_{1b}}, \quad \gamma_{1b} = \frac{2}{n(q+1)},$$

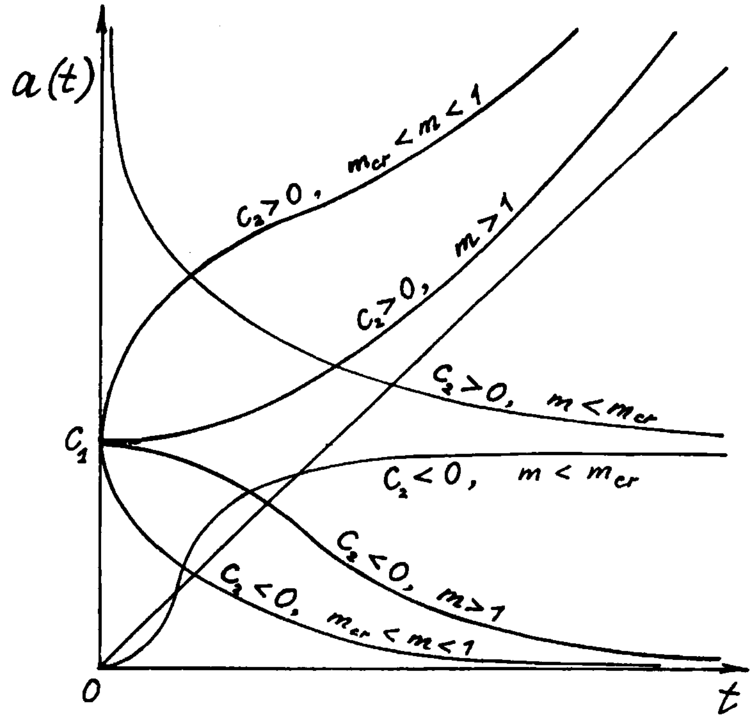

Figure 2 $q = -1$ ($m_{\rm cr} = 2 - n$). Here $m_{\rm cr} < 1$ because we have $n > 1$. The case $m < m_{\rm cr}$ with our restriction $m > 0$ is possible for $1 < n < 2$ only.

for $q < -(n-2)/n$, the solution has an inflexion (it is not indicated at Figure 1), and asymptotically for large t

$$\varepsilon, p', p \propto t^{-2}(\ln t)^{-1}. \tag{15}$$

Figure 1 shows the qualitative behavior of the scale factor when $q > -(n-1)/n$ for all possible values of m. If $n \geq 2$, then $\varepsilon > 0$ for all the increasing solutions (including $m = m_{\rm cr}$) and $\varepsilon < 0$ for the decreasing one; if $n < 2$, then $\varepsilon < 0$ for the increasing solutions when $0 < m < (2-n) < m_{\rm cr}$.

Consider now the case $q = -1$. It is interesting not only by its physical content but also by that it is an isolated case for Eq. (11). The solutions below do not contain an explicit dependence on n but only via $m_{\rm cr}$: $m_{\rm cr} = 2 - n$.

(1c) $q = -1$ and $m \neq m_{\rm cr}$:

$$a = C_1 \exp(C_2 t^{\gamma_{1c}}), \quad \gamma_{1c} = \frac{m_{\rm cr} - m}{m_{\rm cr} - 1}.$$

The behavior of the solution here is determined by the sign of $m_{\rm cr} - m$ and the coefficient C_2 (see Figure 2). Asymptotically, for large t, if $m > m_{\rm cr}$, then $\varepsilon, p', p \propto$

$t^{2(\gamma_{1c}-1)}$, and if $m < m_{cr}$, then $\varepsilon, p', p \propto t^{\gamma_{1c}-2}$. All the solutions without inflexions have a constant positive sign of ε. For those with the inflexions the sign of ε changes: asymptotically, $\varepsilon > 0$ when $C_2 > 0$ and $\varepsilon < 0$ when $C_2 < 0$.

(1d) $q = -1$ and $m = m_{cr}$. The solution is:

$$a = Ct^{\gamma_{1d}}$$

with arbitrary γ_{1d}, and the time dependence of ε, p' and p is again (14), however $\varepsilon < 0$ for $0 < \gamma_{1d} < 2$.

Thus, we have considered all the possibilities in the case (1).

(2) *The stationary case.* It follows from the set of equations (8) that if a solution is stationary ($a = const$) then any dependence on m disappears and

$$\varepsilon = \frac{n(n-1)}{2}\frac{k}{a^2}, \quad p' = -\varepsilon = -\frac{n(n-1)}{2}\frac{k}{a^2}, \quad p = -\frac{(n-1)(n-2)}{2}\frac{k}{a^2}.$$

It means that the pure temporal part of the stress-energy tensor is isotropic: $T_{AB} = \varepsilon\delta_{AB}$, and that $q = -(n-2)/n$. For $n > 1$ a stationary Universe with $k = -1$ might exist if ε is negative.

(3) *Two-dimensional* ($n = 2$) *dust-filled* ($q = 0$) *space.* In this case we do not restrict ourselves to $k = 0$, although it unifies some subcases of the cases (1) and (2). For any k,

$$a = \begin{cases} Ct^{2-m}, & \mathrm{m}\neq 2, \\ C(\ln t), & \mathrm{m}=2. \end{cases}$$

Here $m_{cr} = 2$. The behavior of solutions is the same as in Figure 1. Time dependence of ε and p' ($p = 0$) is described by (14) and asymptotically by (15) again for $m \neq 2$ and $m = 2$, respectively. For $m \leq 2$ and $k = 0, +1, \varepsilon$ is positive, and for $m > 2$ and $k = -1, 0, \varepsilon$ is negative; in the remaining cases, ε changes its sign.

The stationary solution for $k = +1$ and any m is

$$\varepsilon = \frac{1}{a^2}, \quad p' = -\varepsilon = -\frac{1}{a^2}.$$

Such the solution for $m = 1$ was obtained in [1]. We stress that it is of great importance because the dimensionality of space may tend to two on large cosmological scales [1].

Our analysis is as yet preliminary. It is worth nothing that there exists a tendency consider solutions with $\varepsilon < 0$ as physical ones because they may be provided by a quantum matter and lead to new physics. For example, the famous "time-machine" solution [4] requires violation of the weak energy condition.

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