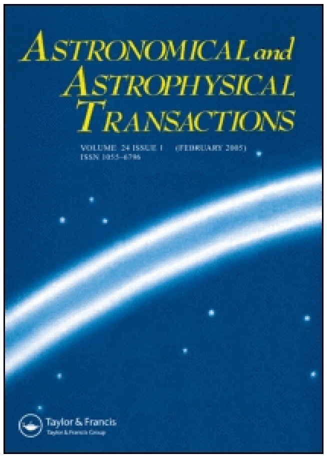

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STABILITY OF MOTION IN HILL'S SENSE IN THE PROBLEM OF MANY BODIES

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A criterion on the stability of motion due to Hill in the problem of many absolutely rigid bodies possessing an arbitrary form and structure of the density distribution is established. A graphical illustration of the results obtained is given for the case of three bodies. The instability in Hill's sense for the classical problem of n material points is shown. Conditions of possible disintegration of the n-body system when removing even one body to an infinitely large distance from another body are considered.

Keywords: Many-body problem; Hill stability; Mutual collisions

Let $M_1, M_2, \ldots, M_n$ be a system of n absolutely rigid bodies with an arbitrary form and size. Also, let $m_1, m_2, \ldots, m_n$ be their respective masses.

The force function of such a system of actively gravitating bodies can be represented in the form

$$U = \sum_{j>i} U_{ij}. \tag{1}$$

Here U_{ij}, are force functions of mutual attraction between every pair of bodies M_i and M_j. The latter functions can be expanded in ascending powers of inverse mutual distances having the form (Duboshin, 1975)

$$U_{ij} = f\,\frac{m_i m_j}{r_{ij}} + \cdots, \tag{2}$$

where f is the universal constant of gravitation and r_{ij} is the mutual distance between the mass centres of M_i and M_j.

Suppose that r_{ij} satisfy the inequalities

$$r_{ij} > a_i + a_j, \tag{3}$$

where a_i is the maximal distance from the mass centre of the M_i body to its surface. Note that the absence of binary collisions between bodies M_i and M_j are guaranteed owing to the

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inequality (3). It is proposed that the existence of the initial system is completed if at least one pair of bodies collides.

As a consequence of the inequality (3) the force function U of the above system is limited:

$$U \leqslant V, \tag{4}$$

where V is a constant corresponding to some compact optimal packing of all the n bodies when every body touches at least two other bodies. No other compact packing of the n bodies can give for the force function U a value larger than V.

This compact packing of all the bodies can be represented as an analogue of the full collision in the classical problem of n mass points.

It is to be noted that, where there is full collision of all n bodies, pairs of these bodies can exist with a mutual distance that exceeds the sum $a_i + a_j$. However, the latter does not influence subsequent results.

Suppose now that one of these bodies, for example M_k, is removed to an infinite distance and let $U^{(k)}$ be the mutual attraction force function of the remaining (n − 1)-body system. This function is evidently restricted:

$$U^{(k)} \leqslant V^{(k)}, \tag{5}$$

where $V^{(k)}$ is a maximal value of the $U^{(k)}$ function, $V^{(k)}$ corresponding to collisions of the $n-1$ bodies $M_1, \ldots, M_{k-1}, M_{k+1}, \ldots, M_n$.

We suppose in addition that the bodies are numbered so that the following inequalities are satisfied:

$$V^{(1)} \leqslant V^{(2)} \leqslant \cdots \leqslant V^{(n)} \leqslant V, \tag{6}$$

where V is the constant mentioned above.

The latter conditions are fulfilled if the following inequalities are satisfied:

$$\frac{m_1}{a_1} \geqslant \frac{m_2}{a_2} \geqslant \cdots \geqslant \frac{m_n}{a_n} \tag{7}$$

and

$$m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n. \tag{8}$$

It is well known that differential equations of the many-body problem have an energy integral. In the barycentric coordinate system this integral is written in the form

$$T - U = h, \tag{9}$$

where T is the kinetic energy of the n-body system and h is the barycentric constant of energy.

It is obvious that for real motions the kinetic energy T of the above-mentioned system cannot be negative. Consequently, for $h < 0$ the regions of possible motions are determined by the inequality

$$U \geqslant C. \tag{10}$$

Here $C = -h$ is a new arbitrary constant of integration such that $0 < C \leqslant V$. Surfaces of zero kinetic energy determined by the equality

$$U = C \tag{11}$$

are the boundaries of the regions of possible motions.

Now we shall give the following definition. Let $r_{ij}^{(0)} = r_{ij}(t_0)$ be any initial value of mutual distance between M_i and M_j and let $C^{(0)}$ be the corresponding value of constant C such that $C^{(0)} = U(r_{ij}^{(0)})$. Then the motion of the n-body system in mutual distances chosen as Cartesian coordinates is called stable in Hill's sense if there exists a constant C* such that $C^{(0)} > C^*$ for which regions of possible motions $U(r_{ij}^{(0)}) \geqslant C^{(0)}$ have a final extent. On the contrary, if there is no C*, then we shall say that, in the coordinate system $Or_{12}, \ldots, r_{n-1,n}$ considered, all motions of the n-body system are unstable in Hill's sense.

It is easy to establish the following criterion of Hill's stability in the problem of many bodies:

$$V^{(n)} < C \leqslant V. \tag{12}$$

Really, if $C > V^{(n)}$, then with the help of Eqs. (4), (6) and (10) we obtain

$$V^{(n)} < C \leqslant U < V.$$

Also, from the definition of $V^{(n)}$ it is obvious that, under a time change in infinite limits $(-\infty < t < +\infty)$, all bodies remain at final mutual distances from each other. None of these bodies can be removed to an infinitely large distance from another body because this is only possible when $C \leqslant V^{(n)}$. Therefore the n-body motion under $C > V^{(n)}$ in the space of mutual distances is always limited by a closed surface of zero kinetic energy $U = V^{(n)} + \epsilon$, where $\epsilon > 0$ is a sufficiently small constant. This completes the proof of stability in Hill's sense of the n-body motions.

If the constant C does not satisfy the criterion (12), then at least one body can be removed from the others at an infinitely large distance. In fact, if

$$V^{(n-1)} < C \leqslant V^{(n)},$$

then the body M_n can leave the system. This is affirmed both by the inequalities

$$V^{(n-1)} < C \leqslant U^{(n)} \leqslant V^{(n)}$$

and the $V^{(n-1)}$ and $V^{(n)}$ definitions. On the contrary, the departure to infinity of both M_{n-1} and any other body is impossible because this gives rise to the contradictory inequalities $V^{(n-1)} < C \leqslant U^{(n-1)} \leqslant V^{(n-1)}$.

If the constant C takes values satisfying the inequalities

$$V^{(n-2)} < C \leqslant V^{(n-1)},$$

then with the help of analogous considerations it can easily be shown that either M_n or M_{n-1} can leave the n-body system.

Finally, provided that

$$0 < C \leqslant V^{(1)},$$

any of the n bodies can be removed to an infinitely large distance from the system. Moreover, in the latter case, two or more bodies can leave the system at once. However, to find out this possibility (by determination of the appropriate value of C) it is necessary to investigate with the help of an analogous method the (n − 1)-body system, the (n − 2)-body system and so on.

Let us remark that, if the criterion of Hill's stability is satisfied, it does not exclude collisions between some bodies of the system.

One can confirm that when the criterion (12) is fulfilled, either the n-body system always exists (as $t \to \pm\infty$) or, at some moment of time t, two or more bodies collide and consequently the existence of the original n-body system is completed.

It is easy to prove that, when the criterion (12) is fulfilled, the value of constant C is so large that the majority of mutual distances between the bodies (and possibly all of them) in agreement with the inequality (10) are comparable with the sizes of the bodies themselves.

For simplicity, suppose that all bodies are spheres of radii a_i and their mutual distances are equal each other ($r_{ij} = r$). Also, under the $V^{(n)}$ definition, consider approximately that, in process of the (n − 1)-body collision, every pair of these bodies touches each other. Then from the inequalities

$$U \geqslant C > V^{(n)} \tag{13}$$

we obtain an estimation for the size of the region where the criterion (12) is satisfied. This estimation is given by the following inequality:

$$r < \frac{n}{n-2}(a_1 + a_2). \tag{14}$$

The latter means that the fulfilment of the criterion for Hill's stability requires such a narrow region for at least some of the bodies that the probability of collisions becomes sufficiently large.

Because of the propositions suggested, the estimation (14) slightly reduces the size of the stability region and the above-mentioned bodies can occupy a somewhat greater space. However, even when the value of r is achieved over a longer time, the probability of collisions between bodies of the system becomes significant.

For the case of the problem of n material points there are no regions of stability in Hill's sense; so V and all $V^{(k)}$ become infinitely large. In other words, under any value of C, all the bodies can be removed to infinity and the existence of the initial n-body system is interrupted.

The total sum for the case in which Hill's stability in the n-body problem is fulfilled can be formulated in the following way.

In the problem of n rigid bodies having an arbitrary form and structure of the density distribution the motions of all bodies are stable in Hill's sense if the criterion (12) is satisfied, but the extents of the regions of stability are very small and comparable with the sizes of the bodies themselves. Hence the fulfilment of criterion (12) indicates that there is a great possibility of the collisions of bodies.

In the problem of n material points, then all motions are unstable in Hill's sense in that, under any initial conditions, each body can leave the system and be removed to an infinitely large distance from the other bodies.

The results obtained were presented graphically in the paper by Lukyanov and Shirmin (2002) for the case of three spheres with a spherical distribution of density. For the case of $n = 3$, regions of possible motions (10) can be represented in the rectangular coordinate system $Or_{12}r_{23}r_{31}$. Sections of these regions defined by the plane $r_{31} = r_{23}$ are hyperbolae in the figure applied where parts of hyperbolae having no physical sense are represented by the dotted area. The analogous form have sections of regions of possible motions defined by the planes $r_{12} = r_{31}$ and $r_{23} = r_{12}$. The corresponding region of stability in Hill's sense is shaded in the same figure. As can be seen in the latter, for the case of material points when $a_1 = a_2 = a_3 = 0$, all hyperbolae are continued along the axis of abscissae up to infinity. However, below this region, there exists a 'split' for possible departure to infinity for at least one of the bodies, that is there is instability in Hill's sense. Note that the latter result was obtained by Lukyanov and Shirmin (2001) (Fig. 1).

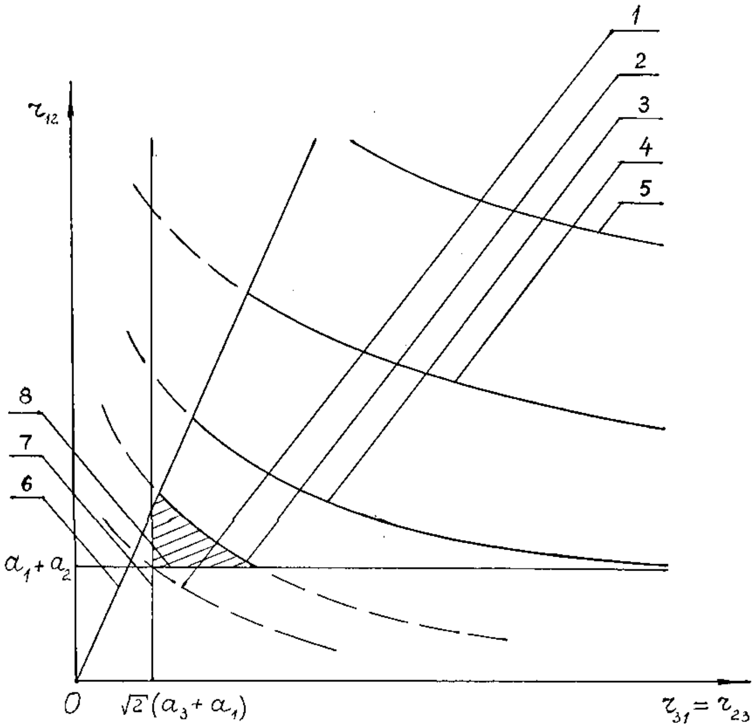

FIGURE 1 Sections of regions of possible motions by the plane $r_{31} = r_{23}$: curve 1, hyperbola under $C = V$ (only one point of it has a physical sense); curve 2, hyperbola under $V^{(3)} < C < V$ (the shaded region illustrates the region of stability in Hill's sense); curve 3, hyperbola under $C = V^{(3)}$, illustrating a limited possibility of stability in Hill's sense; curve 4, hyperbola under $C < V^{(3)}$, illustrating instability of motion in Hill's sense; curve 5, hyperbola under $C \to 0$ which illustrates the total disintegration of the many-body system; curve 6, section defined by the plane $r_{12} = r_{23}+r_{31}$ corresponding to location of the bodies on the same straight line; curve 7, section defined by the plane $r_{31} = a_3+a_1$ corresponding to the binary collision of M_3 and M_1; curve 8, line corresponding to collisions of M_1 and M_2.

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