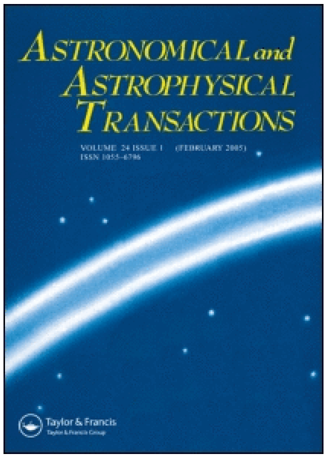


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# ON ZERO-VELOCITY SURFACES INSIDE AND OUTSIDE A HOMOGENEOUS ROTATING AND GRAVITATING ELLIPSOID

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Surfaces of zero velocity and regions of possible motions of a small body in the gravitational field of a homogeneous rotating ellipsoid are examined. The existence of stable motions in Hill's sense is shown. Applications of the results obtained to problems both on a satellite of triaxial planet motion and on a star inside and outside a galaxy are considered.

Keywords: Zero-velocity surfaces; Homogeneous rotating ellipsoid; Gravitational field

Differential equations of a passively gravitating material point M inside and outside a homogeneous rotating ellipsoid T in a proper system of rectangular coordinates x, y, z of the body T have the form

$$\begin{aligned}\frac{d^2x}{dt^2} - 2\Omega\frac{dy}{dt} &= \frac{\partial U}{\partial x},\\ \frac{d^2y}{dt^2} + 2\Omega\frac{dx}{dt} &= \frac{\partial U}{\partial y},\\ \frac{d^2z}{dt^2} &= \frac{\partial U}{\partial z},\end{aligned} \tag{1}$$

Here

$$U = V + \frac{\Omega^2}{2}(x^2 + y^2), \tag{2}$$

where $\Omega$ is a constant angular velocity of body's T rotation and V is its gravity potential.

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The directions of the x, y and z axes are chosen along the grand, intermediate and small axes respectively of the ellipsoid T so that

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad a \geq b \geq c, \tag{3}$$

is the equation of the ellipsoidal surface.

The potential V has different forms inside and outside the homogeneous ellipsoid. As to the outside region the potential of an ellipsoid is slightly different from that of a homogeneous ball of radius R, the volume of the ellipsoid being equal to the volume of the ball. Then

$$a^2 = R^2 + \alpha, \qquad b^2 = R^2 + \beta, \qquad c^2 = R^2 + \gamma, \tag{4}$$

where $\alpha$, $\beta$ and $\gamma$ are constants that are small in comparison with $R^2$ and that satisfy the relation

$$\alpha + \beta + \gamma = 0.$$

The potential of this ellipsoid can be represented in the form

$$V = \frac{GM}{r}\left(1 + \frac{3}{10}\,\frac{\alpha x^2 + \beta y^2 + \gamma z^2}{r^4} + \cdots\right), \tag{5}$$

where only terms of the first order with respect to $\alpha$, $\beta$ and $\gamma$ are included. Here G is the constant of gravity, M is the mass of the ellipsoid T and

$$r = (x^2 + y^2 + z^2)^{1/2}. \tag{6}$$

At the same time, the potential inside of the homogeneous ellipsoid is represented as

$$V = V_0 - \tfrac{1}{2}(V_1 x^2 + V_2 y^2 + V_3 z^2), \tag{7}$$

where $V_0$, $V_1$, $V_2$ and $V_3$ are constant values.

The differential Eqs. (1) have a first integral of the Jacobi type

$$\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = U + h, \tag{8}$$

where h is an arbitrary constant of integration.

The integral (8) for $h < 0$ allows us to determine regions of possible motion of the small body M in the form

$$U \geq C, \qquad C = -h. \tag{9}$$

The surface

$$U = C \tag{10}$$

is a boundary of the region of possible motions.

Singular points of the surface (10) coincide with the liberation points determined by the system of equations

$$\frac{\partial U}{\partial x} = 0, \qquad \frac{\partial U}{\partial y} = 0, \qquad \frac{\partial U}{\partial z} = 0. \tag{11}$$

For the interior of ellipsoid Eqs. (1) the unique libration point $L_1$ is located at the centre of the ellipsoid.

Outside the ellipsoid there are four liberation points $L_2$, $L_3$, $L_4$ and $L_5$ placed on the continuation of the major and minor axes in the x–y plane symmetrically with respect to the ellipsoidal centre (Batracov, 1957). It should be noted that there are two infinitely removed liberation points $L_{\pm\infty}$ (Lukyanov, 1989). The total collection of libration points located both inside and outside the ellipsoid can be represented in the form

$$L_1 = L_1(0, 0, 0), \tag{12}$$
$$L_2 = L_2(a_0 + \alpha' a_0, 0, 0), \qquad L_3 = L_3(-a_0 - \alpha' a_0, 0, 0), \tag{13}$$
$$L_4 = L_4(0, a_0 + \beta' a_0, 0), \qquad L_5 = L_5(0, -a_0 - \beta' a_0, 0), \tag{14}$$
$$L_{\pm\infty} = L_{\pm\infty}(0, 0, \pm\infty), \tag{15}$$

where

$$a_0^3 = \frac{GM}{\Omega^2}, \qquad \alpha' = \frac{3}{10}\frac{\alpha}{a_0^2}, \qquad \beta' = \frac{3}{10}\frac{\beta}{a_0^2}. \tag{16}$$

The libration points $L_2$, $L_3$, $L_4$ and $L_5$ are conical singular points and $L_1$ and $L_{\pm\infty}$ are isolated singular points of the family of surfaces (10).

Values of the constant C corresponding to libration points satisfy the inequalities

$$C_0 > C_{23} > C_{45} > C_{\pm\infty}, \tag{17}$$

where the following notation is chosen:

$$C(L_1) = V_0 = C_0, \qquad C(L_2) = C(L_3) = C_{23},$$
$$C(L_4) = C(L_5) = C_{45}, \qquad C(L_{\pm\infty}) = C_{\pm\infty} = 0.$$

The forms of zero-velocity surfaces and regions of possible motions inside and outside the gravitating ellipsoid are represented in Figures 1 and 2 with the help of sections x–y and y–z in the coordinate planes respectively.

The character of the change in these surfaces and the regions of possible motions is considered during the process of constant decrease in C, starting from $C = \infty$. First of all, our considerations begin at surfaces close to a cylinder of very large radius having the z axis as the axis of symmetry (see curve $C_1$ in Figs. 1 and 2).

Provided that $C = C_0$ the $L_1$ point appears at the centre of the ellipsoid. Then, if $C < C_0$, some oval-like region of possible motions around the point $L_1$ occurs (see curves $C_2$ and $C_3$ in Figs. 1 and 2). At some value C depending on the dimensions of the ellipsoid T, oval-like curves occur outside the ellipsoid (see curve $C_4$). If the dimensions of the gravitating ellipsoid are infinitely small, such oval curves appear at very large C.

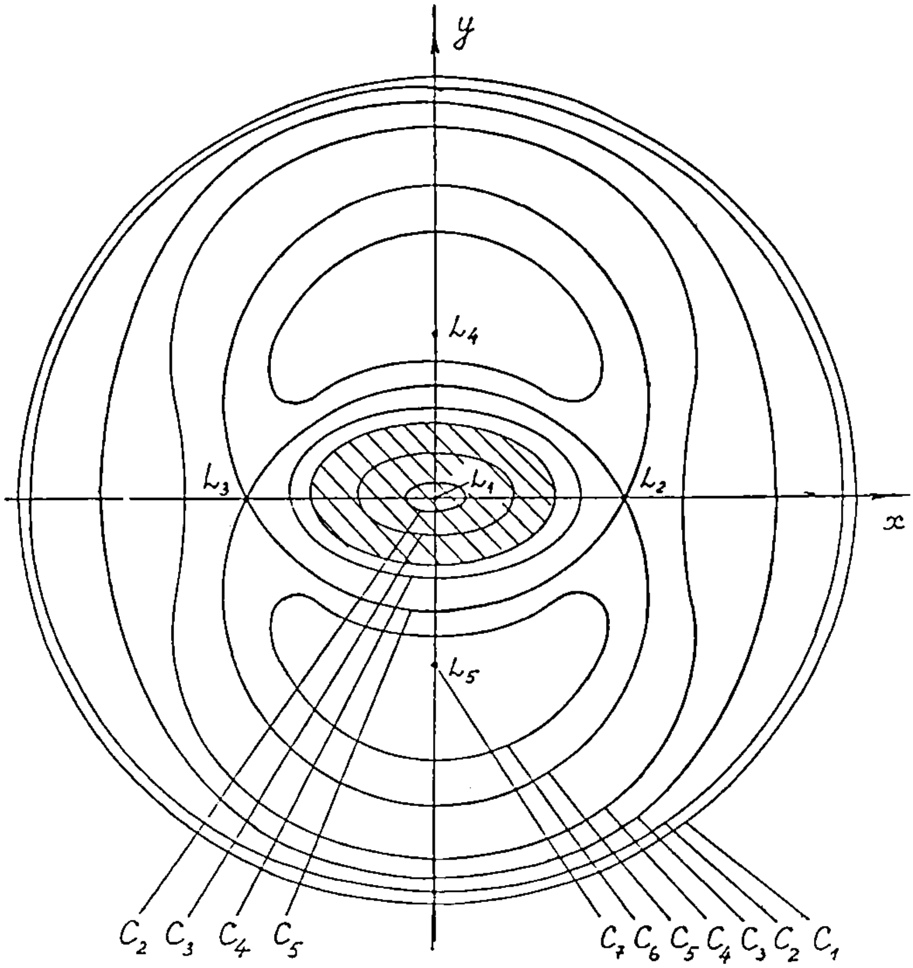


FIGURE 1 A section of zero-velocity surfaces in the x–y plane, showing a section of the gravitating ellipsoid (galaxy). $C_i > C_j$ for $i < j$, $C_5 = C_{23}$, $C_7 = C_{45}$, $C_{11} = V_0$, $C_1 > C_{11} > C_2$.

A further decrease in C to $C_{23}$ gives rise to intersection of the inner oval with the outer cylinder and then a transparent passage occurs in the cylinder. Thus permits the small body M to be removed to infinitely large distance from the ellipsoid (see curves $C_5$ and $C_6$ in Figs. 1 and 2).

Moreover both ‘trunks’ of the cylinder are narrowed to zero in the x–y plane at $C_{45}$ (see curve $C_7$ in Figs. 1 and 2). Then the surface is separated into two parts and does not intersect with the x–y plane (curve $C_8$ in Fig. 2). Each part is attached to one of the infinitely removed points of libration (curve $C_9$ in Fig. 2).

If instead of the triaxial ellipsoid an oblate spheroid is considered ($a = b > c$), then instead of four singular points $L_k$ ($k = 2, 3, 4, 5$) a singular curve emerges, namely the circumference with a centre at the origin of coordinates. The time surfaces of zero velocity for decreasing C intersect the inner oval once along the hole singular circumference for $C = C_{23} = C_{45}$ and afterwards are attached to infinitely removed points. The principal form of the surfaces can be represented if the curves in Figure 2 are rotated around the z axis.

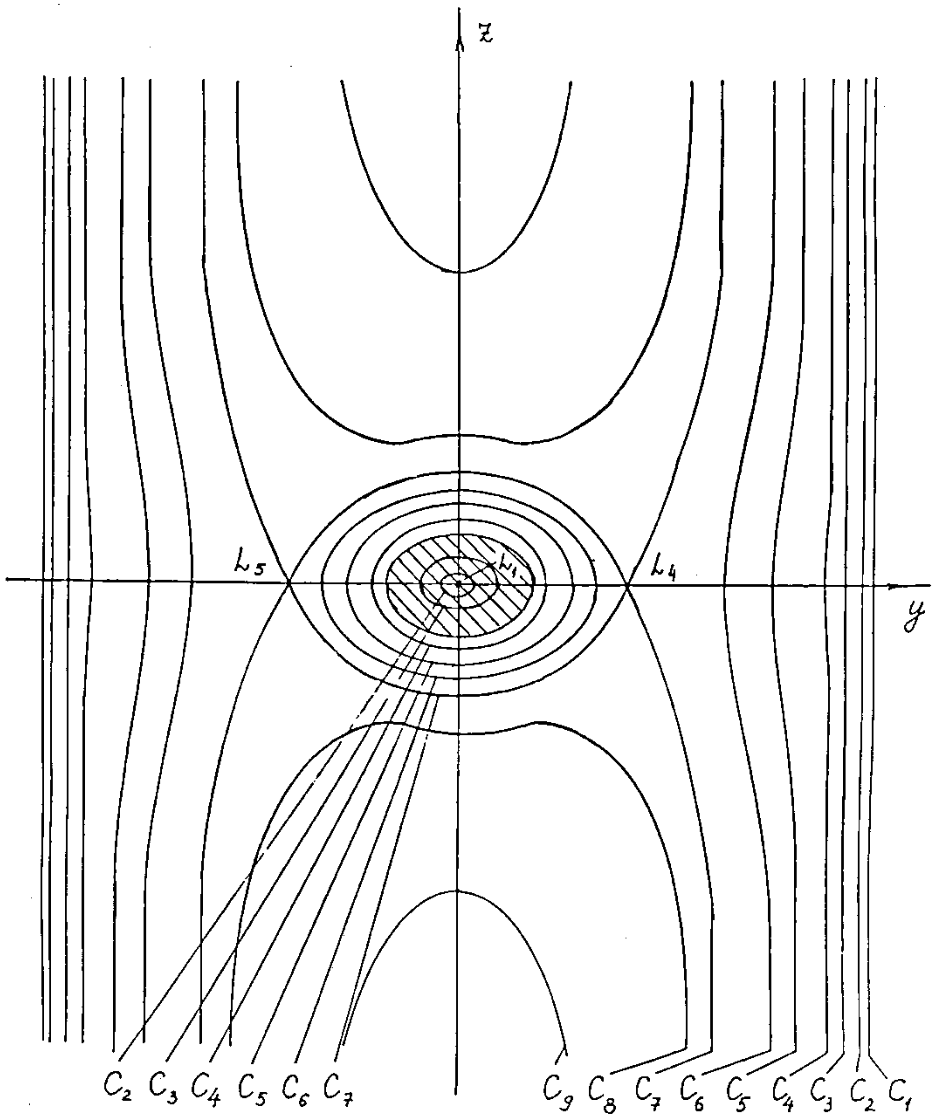


FIGURE 2 A section of zero-velocity surfaces in the y–z plane. The notations is the same as in Figure 1.

The analysis given shows that, in the outer region, motion of the body M of infinitesimal mass is stable in Hill's sense provided that $C > C_{23}$. In other words, if the latter inequality is satisfied, a satellite of a triaxial planet will always move in the vicinity of the planet inside the surface of zero velocity, $C = C_{23}$ (in Figs. 1 and 2 see the final region bounded by curve $C_5$). That is to say, all motions have a bounded (satellite) character. The motions inside the ellipsoid are always stable in Hill's sense with C between $C_0$ and $C_{23}$. Such motions apply to a star inside the galaxy.

Provided that $C < C_{23}$, two 'funnels' appear in the zero velocity surfaces near $L_2$ and $L_3$, and a star can be removed to an infinite distance from the galaxy.

The surface $C = C_{23}$ can be used as a boundary in the Roche model.

With respect to the two 'funnels' in the vicinity of the liberation points $L_2$ and $L_3$ the appearance of two arms in spiral galaxies can be explained with their help. If the galaxy is an oblate ellipsoid of rotation, then instead of arms the whole disc appears around the galaxy when the circumference is a 'funnel'. In this case the gravitating galaxy fills in Roche's cavity (in Figs. 1 and 2 see the region inside the closed parts of curves $C_5$), and the angular velocity of the galaxy's rotation reaches limiting values.

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