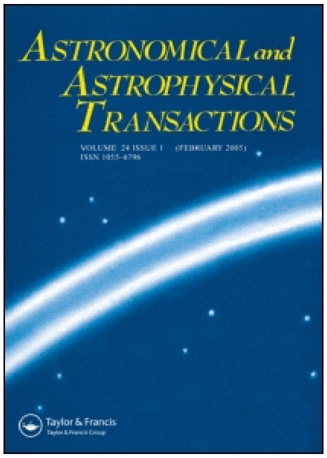

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THE GENERALIZATION OF MAXIMUM ENTROPY METHOD FOR RECONSTRUCTION OF COMPLEX FUNCTIONS

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The Maximum Entropy Method is widely used for reconstruction of real non-negative functions, such as images (intensity distributions) in optics and astronomy. The problem of reconstruction exists not only for real non-negative functions: in radio holography, for example, it is often necessary to reconstruct a coherent source field distribution described by a complex function. In this paper the Generalization of Maximum Entropy Method for reconstruction of functions of different types (real non-negative as well as real with alternating signs and complex ones) is suggested. Though this problem is considered for two-dimensional functions it is evident that the generalization obtained can be applied for functions of different dimensions. Numerical simulation results show high quality of reconstruction of complex functions and stability of the algorithm in the presence of measurement errors.

KEY WORDS Reconstruction of images, maximum entropy method

1. INTRODUCTION

The Maximum Entropy Method is widely used for solving image reconstruction problem, for example, in optics and astronomy [1, 2]. Let us consider the traditional Maximum Entropy Method in discrete form for two-dimensional sequences

$$\min\left(\sum_m \sum_l x_{ml} \ln (x_{ml})\right) \tag{1}$$

$$\sum_m \sum_l x_{ml} a_{ml}^{nk} = A_{nk} \tag{2}$$

$$x_{ml} >= 0 \tag{3}$$

where a_{ml}^{nk} are constants, A_{nk} are measured data (in radio astronomy: samples of visibility function).

The problem (1)–(3) is a nonlinear optimization problem with constraints in the form of equality (2) and inequality (3).

Using the Lagrange method it is easy to obtain a solution for x_{ml}

$$x_{ml} = \exp\left(-\sum_n \sum_k \alpha_{nk} a_{ml}^{nk} - 1\right) \tag{4}$$

where α_{nk}—Lagrange or dual factors which can be found by maximization of the dual functional without supplementary conditions

$$\max\left(\sum_m \sum_l \exp\left(-\sum_n \sum_k \alpha_{nk} a_{ml}^{nk} - 1\right) + \sum_n \sum_k \alpha_{nk} A_{nk}\right). \tag{5}$$

It is clear that x_{ml} in accordance with (4) is always non-negative. But in several cases it is necessary to reconstruct real sequences with alternating signs or complex ones. Such a situation arises, for example, in radio astronomy or in radio holography when it is required to build an image of coherent sources.

In this paper the Generalization of Maximum Entropy Method (1)–(3) is suggested. The possibilities of this method for high quality reconstruction and stability in the presence of errors were investigated by numerical simulation technique.

2. RECONSTRUCTION OF REAL FUNCTION WITH ALTERNATING SIGNS

In this case the minimizing functional cannot be written as (1), because x_{ml} can take negative values and the logarithm of a negative value is not determined in the real domain. Therefore it is proposed to modify functional (1) in the following way

$$\min\left(\sum_m \sum_l |x_{ml}| \ln |x_{ml}|\right) \tag{6}$$

where $|*|$ is the absolute value of $*$.

To avoid the absolute value in (6) and reduce the optimization problem to the traditional problem (1)–(3) let us represent the sequence x_{ml} in the form

$$x_{ml} = y_{ml} - z_{ml} \tag{7}$$

where both y_{ml} and z_{ml} are non-negative.

In addition, let the sequences y_{ml} and z_{ml} satisfy the following conditions

$$\begin{aligned} &\text{if} \quad x_{ml} > 0 \quad \text{then} \quad z_{ml} \Rightarrow 0 \quad \text{and} \quad x_{ml} \simeq y_{ml} \\ &\text{if} \quad x_{ml} < 0 \quad \text{then} \quad y_{ml} \Rightarrow 0 \quad \text{and} \quad x_{ml} \simeq -z_{ml}. \end{aligned} \tag{8}$$

Then functional (6) can be rewritten as follows

$$\min\left(\sum_m \sum_l y_{ml} \ln(y_{ml}) + z_{ml} \ln(z_{ml})\right). \tag{9}$$

Let (9) be modified as

$$\min\left(\sum_m \sum_l y_{ml} \ln(a y_{ml}) + z_{ml} \ln(a z_{ml})\right) \tag{10}$$

where a is a parameter which can be chosen so that the conditions (8) are realized. That will be shown below.

The supplementary conditions (2) can be rewritten in the following way

$$\sum_m \sum_l (y_{ml} - z_{ml}) a_{ml}^{nk} = A_{nk} \tag{11}$$

$$y_{ml}, z_{ml} >= 0. \tag{12}$$

Analysis of the solution of the optimization problem (10)–(12) shows that the value of the parameter a influences the realization of the conditions (8) and therefore the reconstruction quality.

Let us show how this problem of condition optimization (10)–(12) can be represented as a simpler dual nonconditional optimization problem.

Let us construct a Lagrange functional using (10) and (11)

$$L = \sum_m \sum_l (y_{ml} \ln (a y_{ml}) + z_{ml} \ln (a z_{ml}) + \sum_n \sum_k \alpha_{nk} \left(\sum_m \sum_l (y_{ml} - z_{ml}) a_{ml}^{nk} - A_{nk} \right)$$

where α_{nk} = Lagrange or dual factors.

Let us find the minimum of L from the extremum existence condition

$$\frac{\mathrm{d}L}{\mathrm{d}y_{ml}} = 0, \qquad \frac{\mathrm{d}L}{\mathrm{d}z_{ml}} = 0:$$

$$\frac{\mathrm{d}L}{\mathrm{d}y_{ml}} = 1 + \ln (a) + \ln (y_{ml}) + \sum_n \sum_k \alpha_{nk} a_{ml}^{nk} = 0$$

$$\frac{\mathrm{d}L}{\mathrm{d}z_{ml}} = 1 + \ln (a) + \ln (z_{ml}) - \sum_n \sum_k \alpha_{nk} a_{ml}^{nk} = 0$$

$$\ln (y_{ml}) = -\sum_n \sum_k \alpha_{nk} a_{ml}^{nk} - 1 - \ln (a), \qquad \ln (z_{ml}) = \sum_n \sum_k \alpha_{nk} a_{ml}^{nk} - 1 - \ln (a)$$

$$y_{ml} = \exp \left(-\sum_n \sum_k \alpha_{nk} a_{ml}^{nk} - 1 - \ln(a) \right) \tag{13}$$

$$z_{ml} = \exp \left(\sum_n \sum_k \alpha_{nk} a_{ml}^{nk} - 1 - \ln (a) \right). \tag{14}$$

As can be seen from (13) and (14), conditions (12) are satisfied.

By substituting (13) and (14) into the Lagrange functional we obtain the dual optimization problem without supplementary conditions

$$\begin{aligned}
&\min \left(\sum_m \sum_l y_{ml} \left(-\sum_n \sum_k \alpha_{nk} a_{ml}^{nk} - 1 - \ln (a) + \ln (a) \right) \right. \\
&+ z_{ml} \left(\sum_n \sum_k \alpha_{nk} a_{ml}^{nk} - 1 - \ln (a) + \ln (a) \right) \\
&\left. + \sum_n \sum_k \alpha_{nk} \sum_m \sum_l (y_{ml} - z_{ml}) a_{ml}^{nk} - \sum_n \sum_k \alpha_{nk} A_{nk} \right) \Rightarrow \min \left(\sum_m \sum_l (-y_{ml} - z_{ml}) \right. \\
&\left. - \sum_n \sum_k \alpha_{nk} A_{nk} \right) \Rightarrow \max \left(\sum_m \sum_l (y_{ml} + z_{ml}) + \sum_n \sum_k \alpha_{nk} A_{nk} \right).
\end{aligned} \tag{15}$$

The dual factors α_{nk} are determined by solving (15). The solution sought for can be found by substituting the value of α_{nk} into (13) and (14).

The solution of (15) may be found by using any gradient method.

The solutions (13) and (14) have a peculiarity : the product of y_{ml} and z_{ml} depends on the parameter a

$$y_{ml}z_{ml} = \exp(-2 - 2\ln(a)) = K(a). \tag{16}$$

From the last expression it is clear that $K(a)$ influences the accuracy of the realization of conditions (8) and, consequently, the reconstruction quality. An increase of a results in a decrease of $K(a)$ and an improvement of the reconstruction quality, which will be shown by numerical simulation.

3. RECONSTRUCTION OF COMPLEX FUNCTIONS

Since the real and imaginary parts of the complex function are real functions with alternating signs, let us represent the complex sequence in the following way

$$r_{ml} + jq_{ml} = (x_{ml} - y_{ml}) + j(z_{ml} - v_{ml}) \tag{17}$$

where

$$x_{ml},\ y_{ml},\ z_{ml},\ v_{ml} >= 0. \tag{18}$$

By analogy with (8) let the sequences x_{ml}, y_{ml}, z_{ml} and v_{ml} satisfy the following conditions

$$\begin{array}{llllll} \text{if} & r_{ml} > 0 & \text{then} & y_{ml} \Rightarrow 0 & \text{and} & r_{ml} \approx x_{ml} \\ \text{if} & r_{ml} < 0 & \text{then} & x_{ml} \Rightarrow 0 & \text{and} & r_{ml} \approx -y_{ml} \\ \text{if} & q_{ml} > 0 & \text{then} & v_{ml} \Rightarrow 0 & \text{and} & q_{ml} \approx z_{ml} \\ \text{if} & q_{ml} < 0 & \text{then} & z_{ml} \Rightarrow 0 & \text{and} & q_{ml} \approx -v_{ml}. \end{array} \tag{19}$$

Because r_{ml} and q_{ml} are independent of one another it is proposed to minimize the following functional

$$\min\left(\sum_m \sum_l |r_{ml}| \ln |r_{ml}| + |q_{ml}| \ln |q_{ml}|\right). \tag{20}$$

If conditions (19) are satisfied, the optimization problem can be rewritten by analogy with (10)–(12) in the following way

$$\min\left(\sum_m \sum_l x_{ml} \ln(ax_{ml}) + y_{ml} \ln(ay_{ml}) + \sum_m \sum_l z_{ml} \ln(az_{ml}) + v_{ml} \ln(av_{ml})\right) \tag{21}$$

$$\begin{aligned} &\sum_m \sum_l (x_{ml} - y_{ml})a_{ml}^{nk} - (z_{ml} - v_{ml})b_{ml}^{nk} = A_{nk} \\ &\sum_m \sum_l (x_{ml} - y_{ml})b_{ml}^{nk} + (z_{ml} - v_{ml})a_{ml}^{nk} = B_{nk} \end{aligned} \tag{22}$$

$$x_{ml},\ y_{ml},\ z_{ml},\ v_{ml} >= 0 \tag{23}$$

where a_{ml}^{nk}, b_{ml}^{nk} are constants, A_{nk}, B_{nk} are the real and imaginary parts of the measured samples respectively.

The solution of the optimization problem (21)–(23) is expressed as

$$
\begin{aligned}
x_{ml} &= \exp\left(-\sum_n \sum_k (\alpha_{nk} a_{ml}^{nk} + \beta_{nk} b_{ml}^{nk}) - 1 - \ln(a)\right)\\
y_{ml} &= \exp\left(\sum_n \sum_k (\alpha_{nk} a_{ml}^{nk} + \beta_{nk} b_{ml}^{nk}) - 1 - \ln(a)\right)\\
v_{ml} &= \exp\left(-\sum_n \sum_k (\alpha_{nk} b_{ml}^{nk} - \beta_{nk} a_{ml}^{nk}) - 1 - \ln(a)\right)\\
z_{ml} &= \exp\left(\sum_n \sum_k (\alpha_{nk} b_{ml}^{nk} - \beta_{nk} a_{ml}^{nk}) - 1 - \ln(a)\right).
\end{aligned}
\tag{24}
$$

The dual factors α_{nk} and β_{nk} can be found by maximization of the dual functional

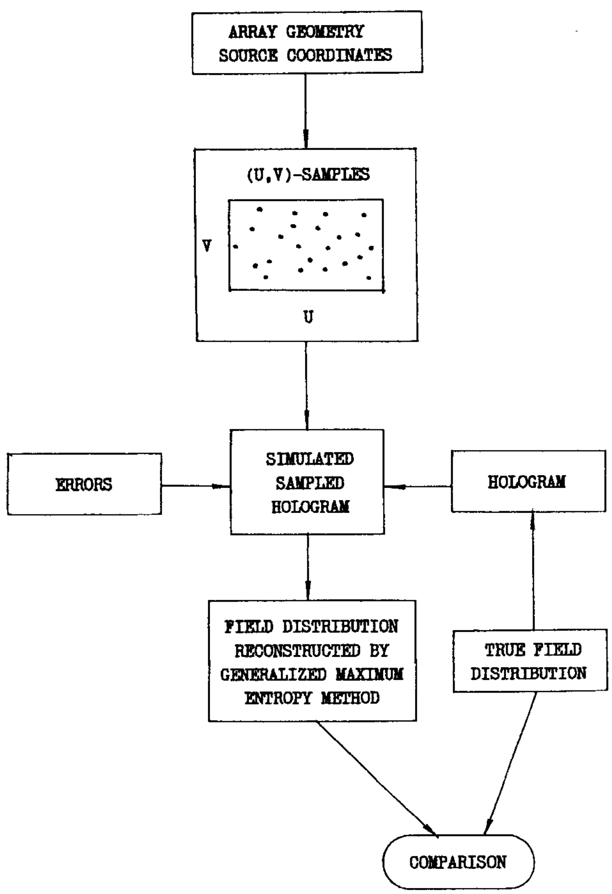

Figure 1 Scheme of the simulation algorithm for demonstrating the utility of the Generalized Maximum Entropy Method.

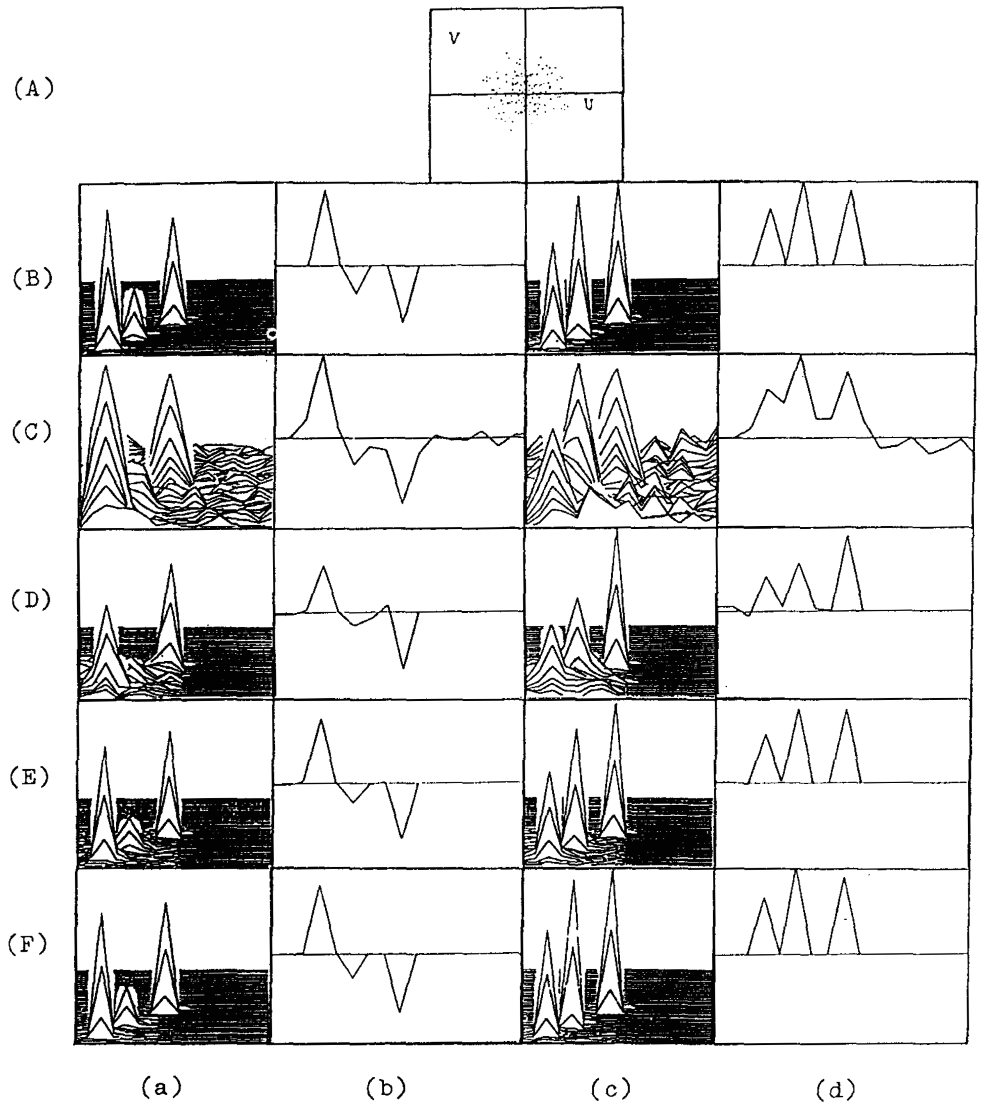

Figure 2 Numerical simulation results for a complex distribution with three point-like components $(0.8+j0.6)$, $(-0.3+j0.9)$, $(-0.6+j0.8)$. (A) (U, V)-samples. (B) True distribution. (C) Distribution obtained without reconstruction algorithm. (D) Reconstructed distribution obtained by using the Generalized Maximum Entropy Method with $a = \exp(-1)$. (E) Reconstructed distribution obtained with $a = \exp(2)$. (F) Reconstructed distribution obtained with $a = \exp(5)$. (a) Absolute value of the real part of a two-dimensional distribution. (b) Diagonal cross-section of the real part. (c) Absolute value of the imaginary part of the two-dimensional distribution. (d) Diagonal cross-section of the imaginary part.

without supplementary conditions

$$\max\left(\sum_m \sum_l (x_{ml} + y_{ml} + z_{ml} + v_{ml}) + \sum_n \sum_k \alpha_{nk} A_{nk} + \beta_{nk} B_{nk}\right)$$

by substituting expression (24) for x_{ml}, y_{ml}, z_{ml}, v_{ml}.

As can be seen from (24), solutions for x_{ml}, y_{ml}, z_{ml}, v_{ml} are related by

$$x_{ml} y_{ml} = z_{ml} v_{ml} = \exp(-2 - 2\ln(a)) = K(a)$$

similarly to (16).

By changing a it is possible to control the reconstruction quality. An increase of a leads to a decrease of $K(a)$ and to closer realization of conditions (19), and therefore to improving the reconstruction quality.

4. NUMERICAL SIMULATION

The possibilities of the proposed Generalized Maximum Entropy Method have been investigated by using the "HOLOGRAPHY" Program Package on an IBM-PC-AT in TURBO-PASCAL created in the Institute of Applied Astronomy of the USSR Academy of Sciences for the simulation of the Radio Holography Aperture Synthesis System for mapping artificial cosmic bodies illuminated with coherent radiation by terrestrial stations. The scheme of the simulation algorithm is given in Figure 1. In Figure 2, simulation results are shown for a point-like

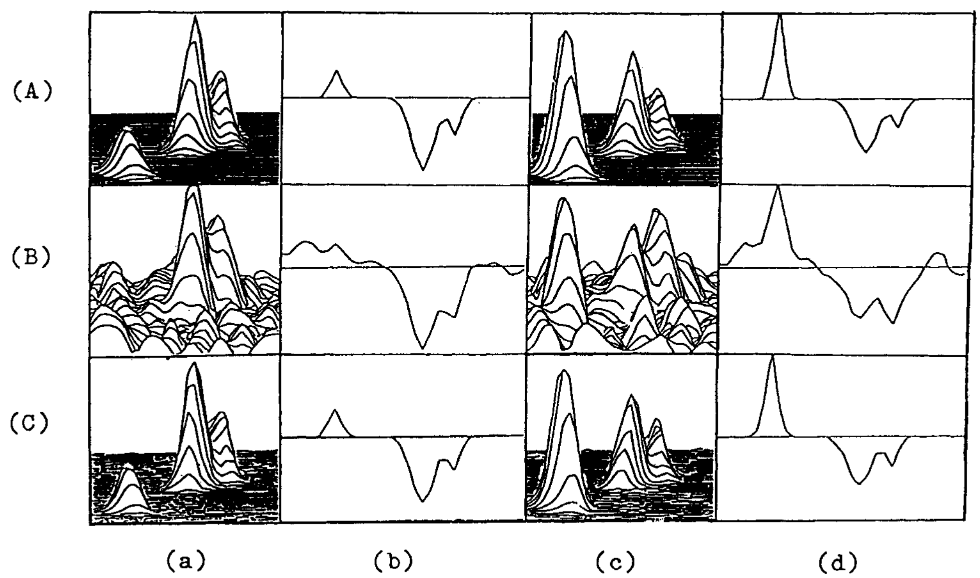

Figure 3 Numerical simulation results for a complex distribution with three gaussian-like components. (A) True distribution. (B) Distribution obtained without using reconstruction algorithm. (C) Reconstructed distribution obtained by using the Generalized Maximum Entropy Method with $a = \exp(7)$. (a) Absolute value of the real part of a two-dimensional distribution. (b) Diagonal cross-section of the real part. (c) Absolute value of the imaginary part of the two-dimensional distribution. (d) Diagonal cross-section of the imaginary part.

complex distribution. Different reconstructed distributions are obtained by using different values of the parameter a. It can be seen that reconstruction quality depends on the value of a, with increasing a leading to improvement of quality. In Figure 3, simulation results are shown for a more complicated complex distribution.

In the above examples measurement errors were taken into account. Random errors uniformly distributed in the range of $[-0.1A, 0.1A]$, where A is a maximum of the modulus of complex distribution, were added to the samples. Computer simulation shows that random errors uniformly distributed in the range of $[-0.25A, 0.25A]$ are tolerable. Thus, simulation results show sufficiently high stability of the Generalized Maximum Entropy Method in the presence of measurement errors.

5. CONCLUSIONS

In this paper a generalization of the well-known Maximum Entropy Method is proposed. This technique is suitable for the reconstruction of functions of different types not only of real non-negative functions, but real functions with alternating signs and complex ones as well. Computer simulation of the generalized algorithm proved its possibilities for high-quality reconstruction of complex distributions and its high stability in the presence of measurement errors. The Generalized Maximum Entropy Method suggested can be effectively used for imaging coherent sources in radio astronomy and holography.

References

1. Image Formation from Coherence Functions in Astronomy. Proceedings of IAU colloquium no. 49 on the formation of images from spatial coherence functions in astronomy, held at Groningen, the Netherlands, 10–12 Aug., 1979, edited by Cornelis Van Schooneveld, D. Reidel Publishing Company Dordrecht: Holland/ Boston: USA, London: England.
2. Picture Processing and Digital Filtering. Edited by T. S. Huang, Springer-Verlag, Berlin, Heidelberg, New-York, 1975.