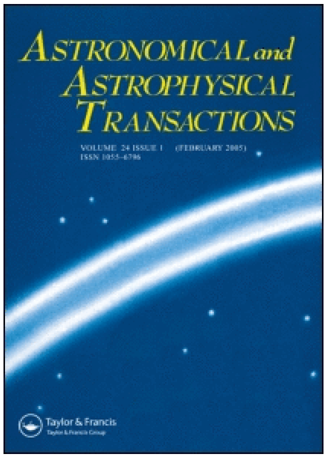

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THE ACTING ANALYTICAL THEORY OF ARTIFICIAL EARTH SATELLITES

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The author considers problems of the practical construction of the analytical theory of the motion of artificial Earth satellites and the relevant computational programme. He reports about the construction of this theory, algorithms and programmes. He also points out their merits, gives assessments of the accuracy of calculations of the coordinates of satellites as well as turning to concrete examples.

KEY WORDS Artificial Earth satellites, theory of motion

1. INTRODUCTION

The analytical theory of the motion of artificial Earth satellites has been worked out by many authors [1, 2, 3, 4, 5]. Until the accuracy of observations of satellites amounted to 2–5 m, the analytical methods of calculating coordinates were applied for solving many problems. With this accuracy, analytical methods call for much less time or calculations than methods of numerical integration.

Over the past 20 years the accuracy of observations of satellites has increased roughly by 100 times. Using modern laser ranging, it is possible to measure the distance to the satellite with an accuracy of 3–5 cm. The attempts to base calculation programmes on the analytical theory of the motion of artificial Earth satellites that enable one to calculate the coordinates of the satellite with such high accuracy have not led to the desired result. Difficulties cropped up due to the considerable increase of the length of the formulae of the theory while striving for the accuracy of 1 cm. Even the use of systems of programming of analytical transformations on the computer [4, 6, 7] caused problems.

However, the necessity of the further development of analytical methods and algorithms of calculating the coordinates of satellites is prompted at least by three aspects. In the first place, the analytical theory can be successfully used for solving applied problems in which the required accuracy is not so high. For instance, the setting of a ground directional aerial to bear on the satellite is among such problems. Secondly, the analytical theory makes it possible, without large expenses of the time of calculations, to assess the influence of various disturbing factors. These estimates can be used, for example, for choosing the needed combination of the terms of the expansion of functions representing the right-hand sides of differential equations of the motion of artificial Earth satellites in case of numerical integration. Discarding of non-essential terms can lead to a great saving of calculation time. Analytical methods enable one to determine the sensitivity of the given combination of observations to variations of different parameters of the expansion of the geopotential. This should be known while determining the parameters from observations. Thirdly, the problem of advantages of numerical and analytical methods with the high accuracy of calculating the coordinates of the satellite cannot be regarded as closed. It is especially important to consider the case when the coordinates of the satellite are calculated with due account for the expansion of the geopotential with the accuracy of up to harmonics of very high orders and degrees, for instance, 180. In such a case, the calculation of the right-hand sides of equations of motion during numerical integration, require a great calculational expense. At the same time, it is known that the magnitude of disturbances from harmonics on the order of 180 is small. Therefore, such disturbances can be determined without high relative accuracy, which is quite accessible in practice while using analytical methods of calculating the coordinates of the satellite.

As can be judged from the latest publications on the problem under consideration, and also from our experience, the compilation of suitable algorithms that effectively realize analytical methods, is the main issue in using analytical methods for calculating the coordinates of satellites with high accuracy.

This paper briefly describes the results obtained by the author in the field under consideration. The main ideas, and partly their implementation, have been described in the author's papers [8–12].

2. AN INTERMEDIATE ORBIT

The construction of an analytical theory of the motion of the satellite is usually based on an intermediate model of motion that includes the effect of the main forces acting on the satellite. The impact of other factors is taken into account by methods of the theory of disturbances. The Keplerian orbit is the simplest case of intermediate motion.

The effective path of constructing the analytical theory of the motion of artificial Earth satellites based on the non-Keplerian intermediate orbit was shown in papers [5, 13, 14]. The gain is that the intermediate orbit of the satellite based on solving the generalized problem of two fixed centres fully takes into account the influence of the second and third zonal harmonics of the expansion of the potential of the Earth's gravity. The trajectory of the satellite moving along an intermediate orbit is described by the force function [5, 14].

$$W = \frac{fm}{2}\left\{\frac{1+\sqrt{-1}\sigma}{\sqrt{x^2+y^2+[z-c(\sigma+\sqrt{-1})]^2}} + \frac{1-\sqrt{-1}\sigma}{\sqrt{x^2+y^2+[z-c(\sigma-\sqrt{-1})]^2}}\right\},$$

where x, y, z—rectangular coordinates of the satellite, f—gravitational constant, m—mass of the Earth. Constants c and σ are determined from the relationship:

$$c = r_0\left[J_2 - \frac{J_3^2}{(2J_2)^2}\right]^{1/2}, \qquad \sigma = \frac{J_3}{2J_2}\left[J_2 - \frac{J_3^2}{(2J_2)^2}\right]^{-1/2},$$

where r_0—mean equatorial Earth radius, J_2 and J_3—coefficients of the second and third zonal harmonics of the expansion of the potential of the Earth's gravity, respectively.

It is usual practice to estimate the smallness of the disturbing factors with respect to the potential of the Earth's gravity as material points. For the Earth, J_2 is of the order of 10^{-3} while J_3 has the order of 10^{-6}. This is why the first order of smallness is ascribed to the second zonal harmonic. The remaining harmonics will be of the second order of smallness.

The formulae of the intermediate orbit express rectangular coordinates and velocity of the satellite as explicit functions of time and six elements of the orbit. They are similar to formulae of the Keplerian orbit, but contain series with respect to degrees of parameters:

$$\varepsilon_1 = J_2 - \varepsilon_2^2, \qquad \varepsilon_2 = \frac{J_3}{2J_2},$$

which have the first order of smallness.

Formulae derived manually with the accuracy of up to terms of the second order were published in papers [5, 14]. Paper [9] reports the creation of the calculational programme, making it possible to derive sought for formulae, with any necessary accuracy, with the aid of the computer. They were derived with the accuracy up to terms of the fourth order.

Expressions for coordinates contain three linear functions of time similar to the mean anomaly under the signs of sines and cosines. Time coefficients in these functions are also presented by series according to powers of small parameters ε_1, ε_2. Terms of such series are called secular.

Let us now consider several examples of comparing the calculations with the results of numerical integration according to formulae of the intermediate orbit. This will demonstrate the accuracy of the formulae of the intermediate orbit of the satellite, which would fully take into account the influence of the second and third zonal harmonics in the expansion of the potential of the Earth's gravity. All periodical terms were taken into account to an accuracy of up to the second order of magnitude and all secular terms were taken into consideration with the accuracy of up to the third order of magnitude. This means that periodical terms contained factors:

$$J_2, \frac{J_3}{J_2}, J_2^2, J_3, \frac{J_3^2}{J_2^2}$$

and secular terms contained factors:

$$J_2, J_2^2, \frac{J_3^2}{J_2^2}, J_2^3, \frac{J_3^2}{J_2}.$$

Two satellites were taken as examples. One of them is hypothetical (Hypo) with orbital elements: the mean motion is 12.196 revolutions per 24 hours, the eccentricity is 0.1 and the inclination is 49°.82. The other satellite similar to the geodetic satellite Starlet has the elements: the mean motion is 13.82 revolutions per 24 hours, the eccentricity is 0.021, the inclination is 49°.82. The investigations were carried out during the time interval of 150 days. Comparisons were made of the rectangular coordinates obtained for some intermediate time moments according to the formulae, and from the numerical integration of the equations of the satellite's motion. For the Hypo satellite, deviations did not exceed 2.2 cm and for the Starlet satellite 1.8 cm.

3. THE METHOD OF THE THEORY OF DISTURBANCES

The impact of the factors that are not included in the model of intermediate motion is taken into account by methods of the perturbation theory. These methods can be divided into two groups: the methods of the small parameter and methods of canonical transformations. Methods of canonical transformations also use the smallness of some parameters. Some papers stress the merits of the modification of the method of canonical transformations—the Hori-Deprit method [7, 15]. The main advantage is that while calculating the perturbations as functions of osculating elements and perturbations as functions of mean elements, use is made of the same transformation function. However, in the practical application of the theory these advantages are often inessential. The merits of any method can be correctly assessed only while compiling the relevant computational programme and solving concrete problems.

In this paper the Poisson small parameter method is used [16, 17]. Elements of the adopted intermediate orbit of the satellite are similar to the elements of the Keplerian orbit. Let us denote through α_1, α_2, α_3 the elements similar to the semi-major axis, to the eccentricity and to the inclination of the satellite's orbit and through β_1, β_2, β_3 the elements similar to the mean anomaly, to the angular distance of the perigee from the ascending node and to the longitude of the ascending node of the intermediate orbit, respectively.

Differential equations of the satellite's perturbed motion will be written in chosen variables in the form:

$$\frac{d\alpha_i}{dt} = A_i, \qquad \frac{d\beta_i}{dt} = n_i + B_i \qquad (i = 1, 2, 3),$$

where:

$$A_i = \sum_{j=1}^{3} a_{ij} \frac{\partial R}{\partial \beta_j}, \qquad B_i = -\sum_{j=1}^{3} a_{ji} \frac{\partial R}{\partial \alpha_j} \qquad (i = 1, 2, 3).$$

Here t—time, R—perturbing function, while a_{ij}, n_i—functions depend only on α_1, α_2, α_3. The perturbing function is regarded as small since it describes the impact of factors that are not taken into account in the intermediate orbit. Its order of smallness depends on the choice of the satellite's intermediate orbit. If the intermediate orbit is Keplerian, the main term of the perturbing function is due to the Earth's oblateness and for satellites with the height interval of 1000 to 20,000 km it can be considered the small value of the first order. In the case of the intermediate orbit that takes into account the second zonal harmonic in the expansion of the potential of the Earth's gravity, the perturbing function for the above-mentioned class of satellites will have the second order of smallness. To solve equations by the small parameter method it is necessary to expand right-hand sides into series by powers of small parameters and to search for the solution also in the form of the series. Let us denote the order of smallness by the upper index in brackets.

For the Keplerian intermediate orbit of the satellite the expansion of the perturbing function into powers of small parameters under consideration consists just of two terms:

$$R = R^{(1)} + R^{(2)}.$$

The coordinates of the satellite in the non-Keplerian intermediate orbit depend on small parameters ε_1, ε_2. In this case the perturbing function has the second order of smallness and its expansion will be written in the form:

$$\dot{R} = R^{(2)} + R^{(3)} + R^{(4)} + \cdots.$$

The procedure of the small parameter method is described in the textbook [16]. For the given case the procedure is considered in detail in [17]. Therefore, it is not given here. It consists in the successive fulfilment of analytical operations over the perturbing function. These are operations of three types: 1) differentiation (∂) with respect to arguments α_1, α_2, α_3, β_1, β_2, β_3, 2) *integration* ($\int$) with respect to time, 3) multiplication ($\times$) of derivatives from the perturbing function by the expression that is obtained after the integration. Since the perturbing function represents a series containing up to several tens of thousands of terms, the multiplication operation ($\times$) is very time-consuming even for computers.

As it follows from [8, 10, 11], the disturbing function for all perturbing factors of a gravitational nature is represented in the form of a trigonometric series with coefficients depending only on α_1, α_2, α_3, and with arguments that are linear in time. The periods of harmonics of such expansion are determined by combinations of frequencies of rotational processes: the satellite's orbital motion, the revolution of the node and the perigee of the orbit of the satellite, similar processes for the Moon and the Sun, and the Earth's rotation. The disturbances in the motion of the satellite of the class under consideration have the following typical periods: short-periodical—the period of the satellite's revolution, diurnal—the period of the Earth's revolution, mean-periodical—half-period of the Moon's revolution and long-periodical—periods of the revolution of the node and the perigee of the satellite's orbit, the half-period of the Sun's revolution. After the integration of each trigonometric term of the expansion of the perturbing function with respect to time ($\int$), the frequency of one harmonic is written in the denominator. This is why for long-periodical terms each process of integration with respect to time lowers the order of the smallness of the perturbation by unity. Table 1 shows the minimal orders of the smallness of perturbations obtained at each stage of applying the small parameter method for Keplerian and non-Keplerian intermediate orbits.

Table 1 S—secular disturbances, Sh—short-periodical, L—long-periodical

		Keplerian orbit			*Non-Keplerian orbit*		
No. of stage	*Operations*	*S*	*Sh*	*L*	*S*	*Sh*	*L*
1	∂, $\int$	1	1	—	2	2	1
2	$\times$, ∂, $\int$	2	2	1	3	3	2

As evident from the table, to obtain all disturbances of the prescribed order of smallness for the non-Keplerian intermediate orbit in the small parameter method it is necessary to fulfil by one step less. It is of special importance that it is necessary to carry out a one multiplication of series less. The payment for such simplification is the necessity to obtain expressions for $R^{(3)}, R^{(4)}, \ldots$. The expressions for $R^{(3)}$ in cases of perturbations due to the non-sphericity of the Earth and from the Moon and the Sun are derived manually in [10, 11]. As a result it turns out that the problem of the advantages of an intermediate orbit can be solved only through the realization of the theory in the form of the computational programme. At least it is obvious that to obtain all secular and short-periodical perturbations up to the second order of smallness and all long-periodical perturbations of the first order it is more advantageous to use the intermediate orbit based on the solution of a generalized problem of two fixed centres. The implementation of such theory shows that in time intervals of up to five days, the accuracy of about 2 m in the satellite's coordinates is obtained.

4. THE EXPANSION OF THE PERTURBING FUNCTION

The initial expression for the geopotential U as a function of the spherical coordinates of the satellite is used by us in the form:

$$U = \frac{fm}{r} \sum_{n=0}^{N} \sum_{k=0}^{n} \frac{r_0^n}{r^n} P_{nk}(\sin\varphi)(C_{nk}\cos k\lambda + S_{nk}\sin k\lambda),$$

where r—geocentric distance, φ—latitude, λ—longitude of the satellite with respect to the Greenwich meridian, P_{nk} $(\sin\varphi)$—Legendre functions (with $k=0$—Legendre polynomials), r_0, C_{nk}, S_{nk}—numerical expansion coefficients. The limit of summing N is chosen depending on the required accuracy of calculating the coordinates of the satellite and on the height of the satellite above the Earth. Practically with the modern accuracy of observations N can assume values of up to 180.

The initial expression for the perturbing function conditioned by the attraction of the external perturbing body (the Moon or the Sun) is of the form:

$$R = fm'\left(\frac{1}{\Delta} - \frac{xx' + yy' + zz'}{r'^2}\right)$$

where m'—mass of the perturbing body, Δ—distance from the satellite to the perturbing body, x', y', z', r'—geocentric coordinates and the distance of the perturbing body.

To fulfil the analytical solution of differential equations for elements of the intermediate orbit, the perturbing function must be expressed through elements. Working out the algorithm of solving the problem, we expressed the perturbing function in the general form for all perturbing factors of the gravitational nature:

$$R = \sum_{j_1 \cdots j_9} R^{(k)}_{j_1 \cdots j_9} \binom{\sin}{\cos} (j_1\beta_1 + j_2\beta_2 + j_3\beta_3 + j_4 S + j_5\lambda_1 + j_6\lambda_2 + \cdots j_9\lambda_5),$$

where S—sideral time, $\lambda_1, \ldots, \lambda_5$—fundamental arguments and the mean longitude of the Moon. The index k determines the choice of the trigonometric function: $k=0$ for sin, $k=1$ for cos. The values $R^{(k)}_{j_1\cdots j_9}$ depend on elements α_1, α_2, α_3. Such a form of expansion was first adopted in [8]. It turns out that the perturbing function conditioned by tides of the absolutely elastic Earth can also be expressed in this form [18]. Concrete expressions of values $R^{(k)}_{j_1\cdots j_9}$ for various types of perturbing factors are obtained in [8, 10, 11, 18, 19].

In the particular case of perturbations due to the Earth's non-sphericity we have:

$$j_4 = -j_3, \qquad j_5 = j_6 = \cdots j_9 = 0.$$

Coefficients in the expansion of the perturbing function R can be expressed more briefly through: $R^{(k)}_{j_1 j_2 j_3}$ while expressions for them have the form [19]:

$$R^{(k)}_{j_1 j_2 j_3} = fm \sum_{p=p'}^{p''} \alpha_1^{-q-1} F_{q,j_3,p}(\alpha_3) X_{j_1}^{-q-1,j_2}(\alpha_2) C^{(k)}_{q,j_3},$$

where:

$$q = j_2 + 2p, \qquad C^{(0)}_{q,j_3} = r_0^q \bar{S}_{q,j_3}, \qquad C^{(1)}_{q,j_3} = r_0^q \bar{C}_{q,j_3},$$

functions of the inclination and functions of eccentricity are denoted through $F\ldots(\alpha_3)$, $X^{\cdots}_{\cdots}(\alpha_2)$. The summation limits are determined this way:

$$p' = E\left(\frac{N-j_2}{2}\right), \qquad p'' = -E(-\tfrac{1}{2}\max\{0, -2j_2, 2-j_2, j_3-j_2\}),$$

and:

$$\begin{aligned} &\bar{S}_{q,j} = S_{q,j}, \ \bar{C}_{q,j} = C_{q,j} \quad \text{for} \quad q-j-\text{even}, \\ &\bar{S}_{q,j} = C_{q,j}, \qquad \bar{C}_{q,j} = -S_{q,j}, \ q-j-\text{odd}. \end{aligned}$$

The expression given here for the disturbing function were derived in [8, 10, 11, 19]. They are convenient in compiling the algorithms of the calculation of disturbances. Subsequently such form of expressions was used in [20, 21].

5. THE REALIZATION OF THE THEORY OF THE MOTION OF ARTIFICIAL EARTH SATELLITES

There is a distance between the publication of convenient formulae of the analytical theory of motion and its implementation in the form of the computational programme that makes it possible to determine the coordinates of satellites at the prescribed time moments. Here one has to overcome difficulties associated with restrictions of means of calculation and means of programming.

In [12] I reported the construction of the analytical theory of the motion of the artificial Earth satellite by the Poisson small parameter method on the basis of the above-described non-Keplerian intermediate orbit. The theory is implemented in the form of the computational programme. I obtained all secular and short-periodical terms with the accuracy of up to the third order and all long-period terms up to the second order, inclusive. Two stages of the small parameter method listed in Table 1 are fulfilled. In constructing the theory series containing several thousands of terms are multiplied. These operations are carried out with

Table 2

Satellite orbit			*Deviations*	
Semi-major axis	*Eccentr.*	*Inclin.*	*rms*	*max*
7,969 km	0.01144	47°.227	5.1 cm	15.3 cm
12,275 km	0.00385	110°.005	4.7 cm	14.5 cm

the help of computers without using systems of programming analytical operations in the letter form. Due to the lack of space I do not describe the algorithm of such construction of the theory. Let us consider only performed estimates of the accuracy of the created theory.

The accuracy of the analytical theory of the satellite's motion was estimated through comparison of calculations under the constructed computation programme with the results of the numerical integration of the equations of motion in rectangular coordinates. The accuracy was controlled separately for disturbances due to the Earth's non-sphericity and disturbances due to the impact of the Moon and the Sun.

Let us consider the results of the comparison in the case of disturbances due to the Earth's non-sphericity. The comparison was made at the time interval of 2 days. In the expansion of the potential of terrestrial attraction I took into account all terms up to the 20th order and degree, inclusively. Two model satellites were considered. One of them is similar to the satellite Lageos. Table 2 shows the root-mean-square and maximum deviations of coordinates at the interval of the comparison for two satellites.

The accuracy of the theory in the case of disturbances due to the Moon was estimated for the Lageos satellite in two time intervals. The root-mean-square value of deviations at the 2-day interval made up 34 cm and at the 30-day interval about 2 m. The analysis of the deviations has shown that they are of a periodic nature depending on time with the period of about 14 days. The lowering of the accuracy of the analytical theory of the satellite's motion in the case of disturbances due to the Moon is conditioned by the fact that disturbances with periods of about 14 days which must be obtained while performing the next stages of the small parameter method are not taken into account. These disturbances have a considerable magnitude since in the course of the triple integration with respect to time, the small frequency of the harmonic (1/14 of the revolution per 24 hours) enters three times the denominator of the expression for disturbances.

The merit of the constructed algorithm of the analytical theory is that it is feasible to a first approximation to determine disturbances separately from any harmonic of the expansion of the geopotential. Periodical perturbations due to factors of a gravitational nature can be easily analysed as regards periods and amplitudes.

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