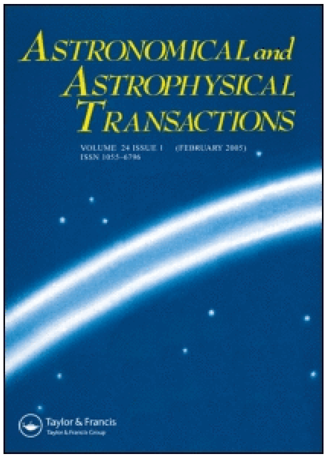

Astronomical & Astrophysical Transactions
The Journal of the Eurasian Astronomical Society
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713453505

Observations and Levy statistics in interstellar scattering

C. R. Gwinn [a]
[a] Department of Physics, University of California, Santa Barbara, CA, USA

Online Publication Date: 01 December 2007
To cite this Article: Gwinn, C. R. (2007) 'Observations and Levy statistics in interstellar scattering', Astronomical & Astrophysical Transactions, 26:6, 525 - 533
To link to this article: DOI: 10.1080/10556790701610506
URL: http://dx.doi.org/10.1080/10556790701610506

Astronomical and Astrophysical Transactions
Vol. 26, No. 6, December 2007, 525–533

Taylor & Francis
Taylor & Francis Group

Observations and Levy statistics in interstellar scattering

C. R. GWINN*

Department of Physics, University of California, Santa Barbara, CA 93106, USA

(Received 19 July 2007)

Levy flights are similar to random walks, but are dominated by rare, large events rather than the accumulation of many steps. Levy flights are not subject to the Central Limit Theorem because the distribution of steps has no second moment; they converge to Levy-stable distributions, which have power-law tails and are stable in form under convolution with themselves. A variety of physical models can produce Levy flights. For such models, the distribution of differences in refractive index has no convergent moments, so that propagation of waves through such a medium is difficult to treat with traditional theoretical methods. Calculations show that Levy flights may help to understand some surprising aspects of radio-wave scattering in the interstellar plasma, including the scaling of broadening time τ with dispersion measure DM, the shape of scatter-broadened images, and the impulse-response function for temporal broadening of narrow pulses.

Keywords: Scattering; Progation; Statistics; Stable distribution; Levy flight

1. Introduction

1.1 *Two games of chance*

Consider two gambling games: in the first, ‘Gauss’, the bettor flips a 2-ruble coin, and he receives one ruble each time the coin falls with the double-headed eagle upward. The bettor is allowed 100 flips. In the second game, ‘Levy’, the bettor is given a 1-ruble coin, and his winnings are *doubled* each time the double-headed eagle falls upward. The bettor must leave with his winnings when the eagle falls downward, or when he reaches 100 flips, whichever comes first.

Interestingly, and perhaps surprisingly, the mean payout for each game is exactly the same: 100 rubles. Of course, for ‘Gauss’ the Central Limit Theorem predicts that the outcome is drawn from a nearly Gaussian distribution, centered at the mean outcome of 100 rubles. The ‘typical’ bettor receives a payout equal to the mean, to within a couple of standard deviations. The mean and standard deviation (or, equivalently, the variance) are related to the first and second moments; higher moments are calculable from these. For the game ‘Levy’, on the other hand, the distribution is approximately power-law, with a most-probable payout of two rubles

*Email: cgwinn@physics.ucsb.edu

Astronomical and Astrophysical Transactions
ISSN 1055-6796 print/ISSN 1476-3540 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/10556790701610506

and a maximum payout of 10^{30}rubles contributing equally to the mean payout. The maximum payout dominates all higher moments, which have only the most tenuous of connections to the experience of the 'typical' bettor. A variant of 'Levy', with no limit on the number of flips, is the subject of the famous 'Saint Petersburg Paradox' of probability and economic theory [1].

1.2 *Stable distributions*

Repeated plays of 'Levy' will quickly converge to a distribution with a power-law tail. Distributions with power-law tails, as in the original Saint Petersburg Paradox, are not subject to the Central Limit theorem because they have no convergent second moment. Rather, such distributions converge to one of the class of 'Levy-stable' or 'alpha-stable' distributions [2]. Of course, the distribution of the sum of random events is the convolution of the distributions of the individual events. Levy distributions are stable in the sense that they yield a scaled copy of the original distribution, when convolved with themselves. Indeed, Levy-stable distributions are attractors under the operation of repeated convolution [3]. Gaussian distributions are but one example of Levy-stable distributions.

In principle, because the maximum payout of 'Levy' is finite, the distribution has finite moments, and the Central Limit Theorem predicts that repeated plays will finally converge to a Gaussian distribution. However, to approach this limit requires $\gg 10^{30}$ plays, so that the bettor will win the top prize many times, and fully sample the distribution. A power-law Levy distribution is much more descriptive of the 'typical' experience of a bettor.

In one dimension, two parameters characterize Levy-stable distributions; for symmetric distributions, a single parameter suffices. This parameter is usually denoted α, leading to the equivalent term 'alpha-stable' for Levy-stable. Here I use β (and, later, γ) for the Levy parameter, to avoid confusion with the power-law index characterizing turbulence. Symmetric Levy-stable distributions do not have a general analytic form, but are the Fourier transform of an exponentiated power-law:

$$P_\beta(y) = \int_{-\infty}^{\infty} \mathrm{d}\mu \exp(i\mu y) \exp(-C|\mu|^\beta), \tag{1}$$

where the constant C sets the scale of the distribution. Values of $0 < \beta \leq 2$ are of interest. The value $\beta = 2$ corresponds to a Gaussian distribution, and $\beta = 1$ to a Cauchy or Lorentzian distribution. The function P_β is not positive definite for $\beta > 2$ and is not normalizable for $\beta \leq 0$. Distributions in two or more dimensions possess a much greater variety of Levy-stable forms, with the multi-dimensional analogs of equation 1 representing but one such possibility.

Levy flights are the analogs of random walks, but with steps chosen from Levy-stable rather than Gaussian distributions; the term 'flight' reflects the fact that the process is dominated by a few large steps, rather than the outcome of many small ones (unless, of course, $\beta = 2$). Levy flights are important in many random processes that can be described as the sums of independent, random events. For example, Levy distributions describe short-term fluctuations of financial markets and are important in pricing derivative securities [4].

2. Models for wave propagation

Variations in refractive index bend radio waves propagating through the interstellar medium. In principle, one can imagine that the statistics of deflection might be Gaussian or Levy, resembling either of the two games introduced above; or some more complicated prescription. These mathematical possibilities raise the question of what the statistics of deflection tell about the physics of the density fluctuations, including the turbulence responsible for them; and indeed whether Levy-like deflections are physically possible. They also raise the question of whether the microscopic statistics of deflection are measurable, and if so how. The remainder of this paper outlines ongoing work on answers to these questions.

2.1 *Gaussian statistics*

Theories of wave-propagation in random media usually assume Gaussian statistics. In particular, these theories commonly describe the relationship between the moments of the propagating wave field and the moments of differences in refractive index. In strong scattering, this relationship was found for the second moment by Tatarskii and for the fourth moment by Shishov. Their elegant formulations have dominated studies of wave propagation in random media for a half-century. For interstellar scattering of radio waves, the relevant field is the amplitude and phase of one polarization of the electric field, and the fluctuations in refractive index arise from local fluctuations in electron density.

2.2 *Turbulence*

The Kolmogorov theory of turbulence motivates an elegant prescription for fluctuations in refractive index. This theory predicts a power-law scaling of difference in turbulent velocity Δv with difference in position Δx:

$$\Delta v \propto \Delta x^{1/3}. \tag{2}$$

Strictly speaking, this relation holds for the third moment of the velocity difference projected onto the separation vector [5, 6]; other powers of velocity may depart from this form because of intermittency of turbulence [6, 7]. Note also that the distribution of velocity fluctuations need not be Gaussian; it may reflect scaling of any distribution of turbulent velocity. The observed differences in refractive index are usually ascribed to density and temperature differences at constant pressure (entropy differences), convected by the turbulent motions to all scales of turbulence. Such differences in density should follow the same scaling as velocity differences: $\Delta n \propto \Delta x^{1/3}$. If the statistics of density fluctuations are Gaussian, one can define the the density structure function D_n:

$$D_n(\Delta\vec{x}) \equiv \langle (n(\vec{x} + \Delta\vec{x}) - n(\vec{x}))^2 \rangle \propto |\vec{x}|^{\alpha-2}. \tag{3}$$

Here, the angular brackets $\langle \ldots \rangle$ indicate an average over an ensemble of statistically identical systems. The assumption of Gaussian statistics, and useful convergent moments, allows evaluation of that average in equation 3. Any fractal picture of turbulence, with the additional assumption of Gaussian fluctuations, leads to the power-law form given by the final proportionality. The index $\alpha = 11/3$ is characteristic of the Kolmogorov picture. The density structure function is the starting point for most theoretical discussions of wave scattering.

2.3 *Levy statistics*

For turbulent media with non-Gaussian statistics, the structure function might not exist. However, scalings analogous to it can exist. For any fractal theory of turbulence, the form of the distribution of density fluctuations, $P(\Delta n)$, is expected to be independent of separation Δx. The characteristic scale of $P(\Delta n)$ will vary as a power of separation. In such a case at least two parameters characterize the distribution of density fluctuations: the power-law index of the scaling with separation, α, with $\alpha = 11/3$ for the Kolmogorov theory; and the Levy parameter for the distribution of density differences at a point, β, with $\beta = 2$ for Gaussian fluctuations [8]. In principle this provides a simple prescription for generating a model medium, using the definition of the Levy distribution (equation 1 or its higher-dimensional analog), and the convolution theorem for Fourier transforms. This is perhaps the simplest model for wave propagation by Levy flight. An optical approximation relates the bending angle of a wavefront to the Δn; Levy distributions of Δn then lead to a Levy flight. In practice, Levy flights, and the indices α and β, can arise from a variety of turbulent and non-turbulent physical pictures, as discussed in the remainder of this section.

A ray obliquely incident on an abrupt step in refractive index, such as a thin shock, will experience deflection. Snell's Law shows that if the angle of incidence is uniformly distributed, the angle of deflection will be drawn from a Cauchy distribution [9]. This distribution is Levy-stable, as noted above, and so repeated deflections by a series of randomly-oriented steps yield a Cauchy distribution as well. (This model is to be distinguished from that of Lambert and Rickett [10], which involves steps *tangent* to the line of sight.) Strong scattering, as observed at most radio wavelengths for most interstellar lines of sight, requires interference among multiple rays, which in turn implies that the steps in refractive index must bend, or have limited size, across the 'scattering disk': the region from which the observer receives radiation. Plausibly, if the step is convoluted with fractal index equal to the Kolmogorov index (perhaps because of convection by Kolmogorov turbulence), one might expect $\alpha = 11/3$ to describe spatial decorrelation of the density step. Boldyrev and Köpnigl [11] propose a related physical model for scattering in the interstellar medium including ionization physics; interestingly, this model does *not* involve convection of density differences from much larger spatial scales.

Kolmogorov [7] pointed out that turbulent flows may be expected to be intermittent; in other words, only some modes of the velocity field are active [6]. Moments of the velocity field can describe this intermittency. The higher moments are sensitive to rare, large fluctuations of the velocity field, rendering experimental measurement challenging. Effects of intermittency on the second and fourth moments of the velocity field, and the corresponding moments of density variations important for scattering, are small. Intermittency is conceptually related to the idea of Levy flights in that rare, large fluctuations become important in averages; however, it is distinct in that the moments of those fluctuations remain finite.

Lack of finite moments complicates theoretical description of wave propagation through media with Levy-stable, rather than Gaussian distributions of density differences. To date, our efforts have focused on simple pictures related to observables [8, 9, 12, 13].

3. Observables and observations

3.1 *Scaling of τ with distance: Sutton paradox*

Sutton [14] was the first to point out that temporal broadening of pulses increases with distance more rapidly than Gaussian statistics would predict, for more distant pulsars. Observations show that τ increases with dispersion measure DM as $\tau \propto \mathrm{DM}^4$, for more distant pulsars with $\mathrm{DM} \gtrsim 30\,\mathrm{cm}^{-3}\,\mathrm{pc}^{-1}$. Figure 1 shows this relationship. Sutton adopted dispersion measure as a proxy for distance. As Sutton noted, one expects angular deflection of a wavefront $\Delta\theta$ to increase with distance d as $\langle \Delta\theta^2 \rangle \propto d$ for a random walk with $N_s \propto d$ steps. The geometric time delay from extra path length then scales as $\tau \propto \Delta\theta^2 d \propto d^2$. Sutton proposed, as an explanation, that scattering in the interstellar plasma is highly nonstationary, in that essentially all of the scattering material for a line of sight is concentrated into the single most

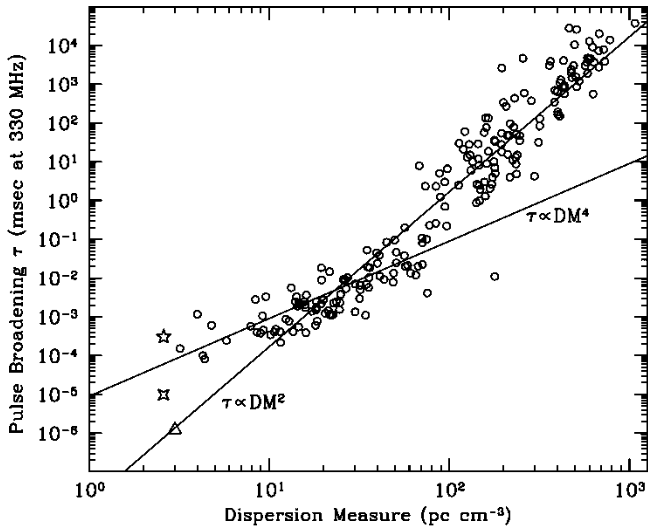

Figure 1. Pulse broadening τ plotted with dispersion measure DM for pulsars for which both are measured, after Pynzar' and Shishov [15]. Lines show expected scaling [15] for $\alpha = 2$, $\beta = 2$ of $\tau \propto \mathrm{DM}^2$, and the $\tau \propto \mathrm{DM}^4$ scaling observed for distant pulsars. Circles are data from Cordes and Lazio [16]. Triangle shows values [17] for pulsar B0950 + 08. Stars show pulsar J0437−4715, which shows two scales of scintillation [18].

strongly-scattering screen. The statistics of scattering are thus reminiscent of the game 'Levy' above, in that the single largest outcome dominates any series of samples. Although τ and DM have been measured for many additional pulsars since Sutton's original work, and the relationship between DM and d has been extensively refined, Sutton's original suggestion has long remained the only viable explanation. Kolmogorov scaling modifies this picture only slightly; it predicts $\tau \propto d^{2.2}$, because of the greater Δn available for large separations.

Boldyrev and Gwinn [8] pointed out that Levy statistics can provide a statistically stationary explanation for the observed scaling. The sum y of N_s Levy-distributed random variables has the distribution $P_{N_s}(y) = P_\beta(y N_s^{-1/\beta}) N_s^{-1/\beta}$, where P_β is the distribution of the elements of the sum. For $\beta = 2$ one recovers the random-walk result. One finds that various combinations of α and β satisfy both the observed scaling of τ with DM for more distant pulsars, and the observed scaling of τ with observing frequency, with uniform scattering along the line of sight. In particular, for the Kolmogorov value of $\alpha = 11/3$, the constraints are met for $\beta \approx 4/5$. Thus, Levy statistics present an alternative explanation for the scaling of τ with DM, but with stationary rather than non-stationary statistics.

3.2 *Angular broadening: Desai paradox*

The observed image of a broadened point source is simply the distribution of the latest deflection angles, for paths that reach the observer. If the scattering medium is stationary, this image will relax to a Levy form [13]. For Gaussian density fluctuations with a Kolmogorov structure function [19], the form is in fact a Levy distribution with Levy parameter $\gamma = 5/3$. In general, the form is a Levy distribution with parameter $\gamma = (\alpha - 2)\beta/2$. Thus, interferometric observations of scattered pointlike sources can determine the microscopic physics of scattering. For Levy parameter $\gamma \neq 2$, the scattered images should show 'cusps' at small scales, and 'halos' at large scales, relative to Gaussian distributions of flux density. Conveniently, interferometry measures the Fourier transform of the source intensity distribution, facilitating direct comparison between observations and equation 1 or its two-dimensional analog.

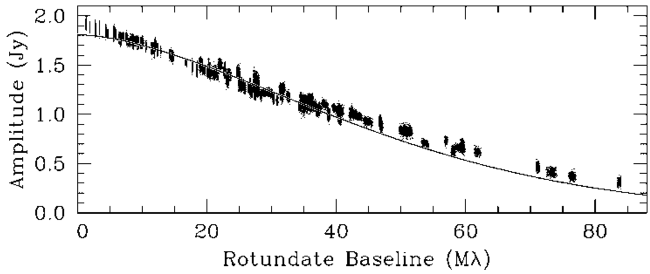

Figure 2. Distribution of correlated flux density with baseline length for a strongly scattered source, from Desai and Fey [20]. Line shows the best-fitting Gaussian model, which is nearly indistinguishable from a Kolmogorov model with Gaussian density fluctuations in this case. The source shows additional flux density at short baselines, indicating a 'halo' around the source relative to the model, and at long baselines, indicating a 'cusp'. The source is 2021 + 317, an extragalatic source in the Cygnus region, observed at 6 cm wavelength. The 'rotundate' baseline corrects for anisotropic scattering of the source. Reprinted with permission from Desai and Fey [20].

High-quality images of scattered sources are uncommon, and are naturally limited to strongly scattered sources with high flux density. Desai and Fey [20] observed four extragalactic sources seen through the Cygnus region. They observe significant angular anisotropy for all sources: this must be taken into account in their analysis. When compared with Gaussian models, some sources show excess flux correlated density both at short baselines (corresponding to large-angular-scale structures) and at long baselines (corresponding to small angular scales). Figure 2 shows an example. Intrinsic source structure might be responsible for large-scale structures, but small-scale intrinsic structure should be blurred by scattering. Thus cusps, or equivalently elevated visibility on long baselines, are difficult to mimic with intrinsic structure. The form expected for a Gaussian–Kolmogorov model ($\alpha = 11/3$, $\beta = 2$) does not differ enough from a pure Gaussian to explain the discrepancy: a fit to the data demands smaller values of α or β, or both.

Interestingly, shapes are not the same among the sources observed, or even among different-frequency observations for the same source. This variation might reflect variations in α or β in space or with bending angle: the latter possibility is similar to the 'truncated' Levy flights seen in the financial markets, where statistics return to Gaussian form for the largest excursions [21].

Halos also appear around heavily-scattered masers, where they have been interpreted as scattering local to the masing material [22]. Some observations suggest cuspy structures as well, from detections on baselines where scatter-broadening should make the source completely resolved [23]. The large variability of these shapes for individual maser spots within a cluster suggests large variations in the scattering process, and possibly an origin near the highly-turbulent regions where masers lie.

3.3 *Pulse broadening: Williamson paradox*

The shape of a heavily scattered pulse is the convolution of the intrinsic pulse shape with the impulse-response function of the scattering medium. This 'pulse-broadening' function carries information on the distribution of scattering material along the line of sight, as well was on the distribution of scattering angles. The evolution of the pulse shape along the line of sight cannot be expressed as a convolution. Analytic forms are known only for a uniform medium with $\alpha = 4$, $\beta = 2$, and for a thin screen. Only typical broadening time τ can be determined for more complicated forms or distributions [13, 25–27]. Most measurements of widths of scattered pulses match the observed pulses to approximations for the observed forms [28, 29], which however are not correct in detail.

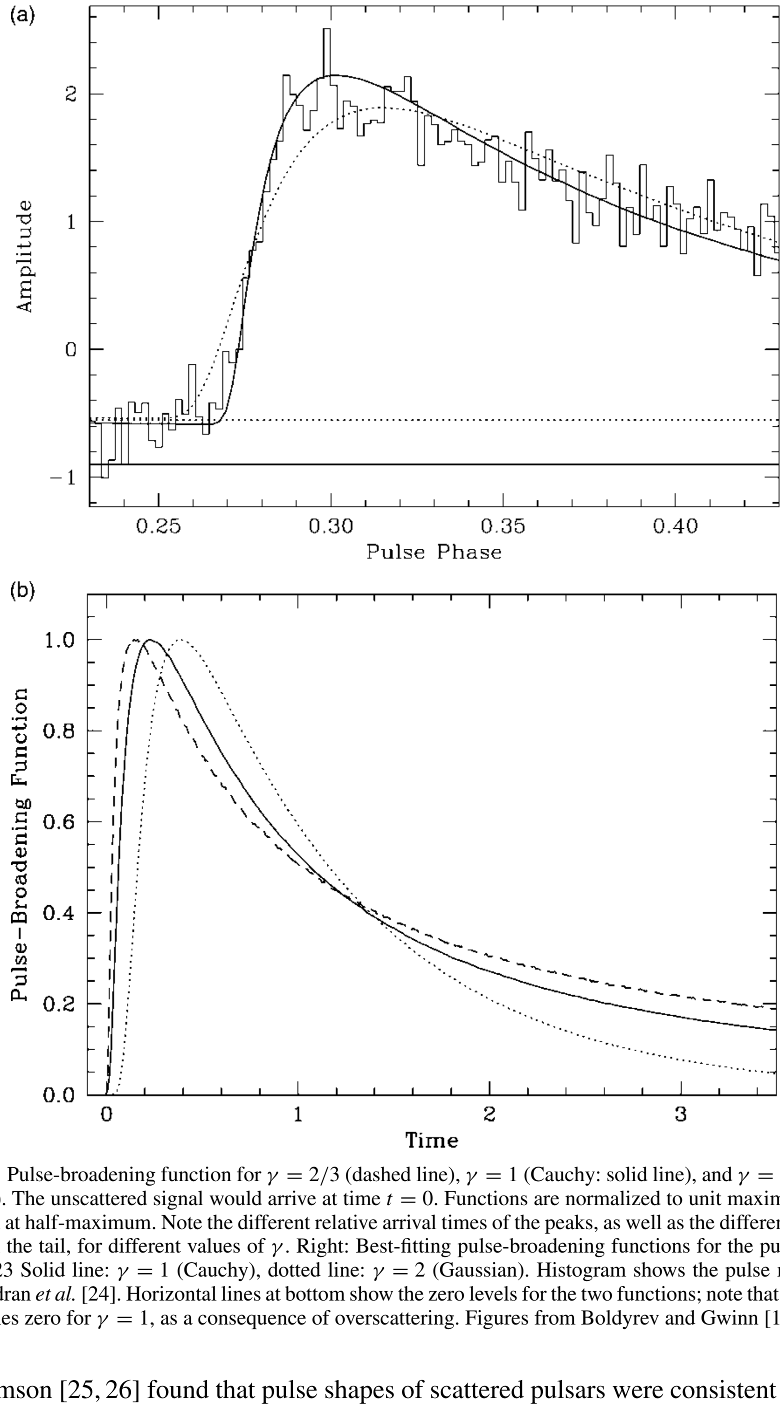

Figure 3. Pulse-broadening function for $\gamma = 2/3$ (dashed line), $\gamma = 1$ (Cauchy: solid line), and $\gamma = 2$ (Gaussian: dotted line). The unscattered signal would arrive at time $t = 0$. Functions are normalized to unit maximum, and the same width at half-maximum. Note the different relative arrival times of the peaks, as well as the different power-law behavior in the tail, for different values of γ. Right: Best-fitting pulse-broadening functions for the pulse of pulsar J1848−0123 Solid line: $\gamma = 1$ (Cauchy), dotted line: $\gamma = 2$ (Gaussian). Histogram shows the pulse measured by Ramachandran *et al.* [24]. Horizontal lines at bottom show the zero levels for the two functions; note that the intensity never reaches zero for $\gamma = 1$, as a consequence of overscattering. Figures from Boldyrev and Gwinn [13].

Williamson [25, 26] found that pulse shapes of scattered pulsars were consistent with scattering by material concentrated into a thin screen, but not with a uniform medium. In our notation, he assumed $\alpha = 4$, $\beta = 2$. The rapid rise and more gradual fall of the impulse-response function, as inferred from observations, best matched a thin-screen model using those parameters.

Levy models predict qualitatively different forms for scatter-broadened pulses [13]. The pulse-broadening function, like angular broadening, is sensitive to the product

$\gamma = (\alpha - 2)\beta/2$. For $\gamma < 2$, the pulse-broadening function has a rapid rise and slow fall, similar to a thin screen. Figure 3 shows sample pulse-broadening functions for three values of γ, and fits of two of these to the pulse shape for J1848 − 0123 as observed by Ramachandran *et al.* [24]. Interestingly, the tail of the Levy model is so long that it leads to 'overscattering', so that the scattered pulse wraps into the following pulse. This leads to an offset of the profile, relative to that of Gaussian models, which may serve to distinguish Levy models from thin-screen models.

4. Summary

Levy flights provide a description of random phenomena where rare, large events are important. Such statistics might arise naturally from various physical pictures of turbulent generation of density fluctuations. Although theoretical calculations are complicated by the fact that such distributions do not have finite moments, we have calculated several observables for comparison with measurements. We find that Levy flights can explain the observed scaling of temporal broadening τ with dispersion measure DM for distant pulsars of $\tau \propto \mathrm{DM}^4$, with stationary statistics for the scattering material rather than the non-stationary statistics traditionally invoked to explain this relation. We find that Levy flights can explain the 'cusps' and 'halos' seen in the scatter-broadened images of pointlike sources. And we find that Levy flights can produce the rapid rise and slow falloff of the impulse-response function inferred for the temporal broadening of pulses from distant pulsars, with a uniform scattering medium rather than a thin screen. Further theoretical work, and comparisons with observations, are in progress.

Acknowledgement

The US National Science Foundation provided partial financial support for this work.

References

[1] D. Bernoulli, Commentarii Academiae Scientiarum Imperialis Petropolitanae. **5** 175 (1738)
[2] B.V. Gnedenko and A.N. Kolmogorov, in *Limit Distributions for Sums of Independent Random Variables, translation*, edited by K.L. Chung (Addison-Wesley, Cambridge, 1954).
[3] J. Klafter, G. Zumofen and M.S. Shlesinger, *Lévy Flights and Related Topics in Physics* (Springer, Berlin, 1995).
[4] J. Voit, *The Statistical Mechanics of Financial Markets* (Springer, Berlin, 2005).
[5] A.N. Kolmogorov, Dokl. Akad. Nauk. SSSR **31** 538 (1941).
[6] U. Frisch, *Turbulence: the Legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[7] A.N. Kolmogorov, J. Fluid Mech. **13** 82 (1962).
[8] S. Boldyrev and C.R. Gwinn, Astrophys. J. **584** 791 (2003).
[9] S. Boldyrev and C.R. Gwinn, Astrophys. Space Sci. **292** 147 (2004).
[10] H.C. Lambert and B.J. Rickett, Astrophys. J. **531** 883 (2000).
[11] S. Boldyrev and A.Königl, Astrophys. J. **640** 344 (2006).
[12] S. Boldyrev and C.R. Gwinn, PRL **91** 131101 (2003).
[13] S. Boldyrev and C.R. Gwinn, Astrophys. J. **624** 213 (2005).
[14] J.M. Sutton, Mon. Not. Roy. Astron. Soc. **155** 51 (1971).
[15] A.V. Pynzar' and V.I. Shishov, Astron. Rep. **41** 586 (1997).
[16] J.M. Cordes and T.J.W. Lazio, astro-ph/0207156 (2002).
[17] J.A. Phillips and A.W. Clegg, Nature **360** 137 (1992).
[18] C.R. Gwinn, C. Hirano and S. Boldyrev, aap. **453** 595 (2006).
[19] B.J. Rickett, araa. **28** 561 (1990).
[20] K.M. Desai and A.L. Fey, Astrophys. J. Supl. **133** 395 (2001).
[21] R.N. Mantegna and H.E. Stanley, Nature **376** 46 (1995).
[22] C.R. Gwinn, Astrophys. J. Lett. **431** L123 (1994).
[23] K.M. Desai, PhD thesis, University of California, Santa Barbara (1995).
[24] R. Ramachandran, D. Mitra, A.A. Deshpande *et al.*, Mon. Not. Roy. Astron. Soc. **290** 260 (1997).

[25] I.P. Williamson, Mon. Not. Roy. Astron. Soc. **157** 55 (1972).
[26] I.P. Williamson, Mon. Not. Roy. Astron. Soc. **163** 345 (1973).
[27] R. Blandford and R. Narayan, Mon. Not. Roy. Astron. Soc. **213** 591 (1985).
[28] O. Löhmer, M. Kramer, D. Mitra *et al.*, Astrophys. J. Lett. **562** L157 (2001).
[29] N.D.R. Bhat, J.M. Cordes, F. Camilo *et al.*, Astrophys. J. **605** 759 (2004).