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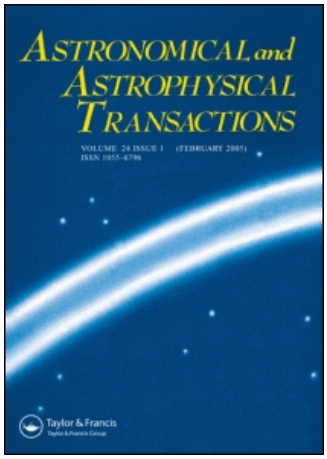

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Gravity anomalies in the Earth's crust and upper mantle

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Gravity anomalies in the Earth's crust and upper mantle

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The contribution to the Earth external and internal gravity field from isostatically equilibrated masses of relief, density jumps on Moho boundary and anomalous masses of crust and upper mantle of the Earth in linear and quadratic approximation is found. Significant gravity anomalies are revealed in isostatically equilibrated crust and mantle, which can lead to gravitational and dynamic instability. The deviation of the external field of the Earth in isostatic equilibrium from observations demonstrates the disturbance of isostatic equilibrium in Earth.

Keywords: Internal gravitational field of the Earth; Crust and upper mantle of the Earth's; Isostatic equilibrium; Gravity anomalies

1. Introduction

The external gravitational field of the earth is one of key parameters characterizing density distribution inside the Earth. The difference between the measured field and the field caused by a rotating spheroid in hydrostatic equilibrium represents the anomalous part of gravity field. However, determination of real distribution of anomalous density is not a well-posed problem. Therefore, it is preferable to separate the contribution of the part of the Earth, the density structure of which can be found with help of other geophysical information and some other, physically and theoretically justified, hypothesis. The crust and the upper mantle of the Earth can be considered as such Earth structure.

The study of statistical connections between the heights of equivalent relief and the depths of mantle surface (M) for various regions of the Earth has revealed non-linearity for both relief load transfer on M, and the return transmission caused by Archimedean forces [1]. It illustrates the absence of isostatic compensation of relief on M, and existence of density and stress anomalies in the crust and the upper mantle of the Earth. The knowledge of direct and inverse regression functions has allowed the separation of equation for total equilibrium of the crust and the upper mantle in two components: (1) partial compensation of crust pressure on surface M; (2) pressure compensation on some depth in the mantle for the parts of the crust, not compensated on M, (similar equations for Archimedean equilibrium).

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Solution of equilibrium equations for the $5^\circ \times 5^\circ$ cells of the whole Earth allowed to determine the optimum depths of compensation, with minimized functional, depending on density and stress anomalies in the crust and the upper mantle. The comparative analysis of conditions of the crust isostatic equilibrium for various variants of compensation for each cell allows to construct the maps of optimum (in sense of potential energy minimum) distribution of anomalous masses and stresses in the crust and the upper mantle (reaching $\pm 7 \times 106\,\mathrm{kg/m^2}$ and ± 15 MPa) [2].

The main purpose of this work is to determine the contribution of isostatically equilibrated masses of crust, mantle and density jump on boundary M to the external and internal gravity field of the Earth.

2. Linear approximation

In linear approximation the laterally distributed anomalous masses can be presented as a simple layer of continuous density distributed on a sphere. In this case, there is a linear relation [3] between the coefficients of the expansion of a simple layer density in terms of spherical harmonics and the Stokes constants caused by the layer's contribution:

$$\begin{Bmatrix} \Delta C_{nm}^{(s)} \\ \Delta D_{nm}^{(s)} \end{Bmatrix} = \frac{3}{2n+1} \left(\frac{R_s}{R_0}\right)^2 \frac{\Delta\sigma_s}{\overline{\sigma} R_0} \left(\frac{R_s}{a}\right)^n \begin{Bmatrix} A_{nm}^{(s)} \\ B_{nm}^{(s)} \end{Bmatrix},$$

here $\Delta C_{nm}^{(s)}$, $\Delta D_{nm}^{(s)}$ are contributions to Stokes constants; R_s is a mean radius of a layer; R_0, $\overline{\sigma}$, a are average radius, density and major semi-axis of Earth ellipsoid, respectively;

$$\Delta\sigma_s H_s(\varphi, \lambda) = \Delta\sigma_s \sum_{n=1}^{N} \sum_{m=0}^{n} \left(\overline{A}_{nm}^{(s)} \cos m\lambda + \overline{B}_{nm}^{(s)} \sin m\lambda\right) \overline{P}_{nm}(\sin\varphi)$$

is a representation of a simple layers' density as expansion series of normalized surface harmonics with degrees $n < N$; $s = 1, 2, 3, 4$; $\Delta\sigma_1$, H_1 are mean density and height of equivalent (*i.e.*, reduced to the homogeneous density $\Delta\sigma_1 = 2.67 \cdot 10^3\,\mathrm{kg/m^3}$) relief relatively to the mean radius $R_1 = R_0 - 1.66\,\mathrm{km}$; $\Delta\sigma_2 = 0.31 \cdot 10^3\,\mathrm{kg/m^3}$, H_2 is mean density jump on the boundary M and of the height of mantle surface referred to mean radius $R_2 = R_0 - 22\,\mathrm{km}$; $\Delta m_3 = \Delta\sigma_3 H_3$ are anomalous masses distributed in the crust and conventionally referred to the mean radius $R_3 = R_0 - 11\,\mathrm{km}$; $\Delta m_4 = \Delta\sigma_4 H_4$ are anomalous masses distributed in the upper mantle and conventionally referred to the mean radius $R_4 = R_0 - 40\,\mathrm{km}$ (according to average depth of isostatic compensation found in ref. [2]).

The summary contribution of anomalous structures of the crust and the mantle to the external gravity potential and gravity force is then defined as follows:

$$\Delta V_\mathrm{e}(r, \varphi, \lambda) = \frac{f M_0}{r} \sum_{s=1}^{4} \sum_{n=1}^{N} \left(\frac{a}{r}\right)^n Y_n^{(s)}(\varphi, \lambda) \tag{1}$$

$$\Delta g_\mathrm{e}(r, \varphi, \lambda) = -\frac{\partial V_\mathrm{e}}{\partial r} = \frac{f M_0}{r^2} \sum_{s=1}^{4} \sum_{n=1}^{N} (n+1) \left(\frac{a}{r}\right)^n Y_n^{(s)}(\varphi, \lambda), \tag{2}$$

here M_0 is the mass of the Earth,

$$Y_n^{(s)}(\varphi, \lambda) = \sum_{m=0}^{n} \left(\Delta C_{nm}^{(s)} \cos m\lambda + \Delta D_{nm}^{(s)} \sin m\lambda\right) \overline{P}_{nm}(\sin\varphi),$$

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$\overline{P}_{nm}(\sin\varphi)$ are Kaula – normalized elementary spherical harmonics (associated Legendre functions), according Kaula.

Similarly, one can define the contribution of these structures to an internal gravitational field in upper and lower crust and in upper mantle. Thus computing the gravity anomalies in the upper crust (for $R_3 < r < R_1$), the relief contribution ($s = 1$) is described by formulas:

$$\Delta V_i\,(r, \varphi, \lambda) = \frac{f M_0}{R_s} \sum_{n=1}^{N} \left(\frac{r}{R_s}\right)^n Y_n^{(s)}\,(\varphi, \lambda), \tag{3}$$

$$\Delta g_i\,(r, \varphi, \lambda) = -\frac{f M_0}{R_s^2} \sum_{n=1}^{N} n \left(\frac{r}{R_s}\right)^{n-1} Y_n^{(s)}\,(\varphi, \lambda). \tag{4}$$

Other layers' contribution ($s = 2, 3, 4$) is defined by formulas similar to equations (1) and (2):

$$\Delta V_{\mathrm{e}}\,(r, \varphi, \lambda) = \frac{f M_0}{r} \sum_{n=1}^{N} \left(\frac{R_s}{r}\right)^n Y_n^{(s)}\,(\varphi, \lambda), \tag{5}$$

$$\Delta g_{\mathrm{e}}\,(r, \varphi, \lambda) = \frac{f M_0}{R_s^2} \sum_{n=1}^{N} (n+1) \left(\frac{R_s}{r}\right)^{n+2} Y_n^{(s)}\,(\varphi, \lambda), \tag{6}$$

here

$$\begin{Bmatrix} \Delta C_{nm}^{(s)} \\ \Delta D_{nm}^{(s)} \end{Bmatrix} = \frac{3}{2n+1} \left(\frac{R_s}{R_0}\right)^3 \frac{\Delta\sigma_s}{\overline{\sigma}} \begin{Bmatrix} a_{nm}^{(s)} \\ b_{nm}^{(s)} \end{Bmatrix}, \tag{7}$$

$$\begin{Bmatrix} a_{nm}^{(s)} \\ b_{nm}^{(s)} \end{Bmatrix} = \begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix}_1 = \frac{1}{R_s} \begin{Bmatrix} \overline{A}_{nm}^{(s)} \\ \overline{B}_{nm}^{(s)} \end{Bmatrix} \tag{8}$$

are coefficients of expansion $h_s = H_s(\varphi, \lambda)/R_s$ in terms of normalized surface harmonic.

For the field in lower crust (for $R_2 < r < R_3$) the formulae (3) and (4) determine the contribution from both the relief ($s = 1$) and the anomalous masses of the crust ($s = 3$). The contribution of remaining structures ($s = 2, 4$) is defined by formulas (5) and (6).

In the top of upper mantle adjoining to the boundary M (for $R_4 < r < R_2$), formulae (3) and (4) define the contribution of anomalous masses of relief and crust, and density jump on M ($s = 1, 2, 3$). Also in the bottom of upper mantle (for $r < R_4$), the field induced by all anomalous layers ($s = 1 \div 4$) is defined by equations (3) and (4).

3. Quadratic approximation

In reality the relief masses and especially the anomalous masses, caused by density jump on M, are not simple spherical layers, but are height distributed by relatively to the reference ellipsoid $r_{\dot{y}}$. In this case, for $s = 1, 2$ the external field can be described by equations (5) and (6), and expansion coefficients can be found by integration over relief masses:

$$\begin{Bmatrix} \Delta C_{nm}^{(s)} \\ \Delta D_{nm}^{(s)} \end{Bmatrix} = \frac{\Delta\sigma_s}{(2n+1)\,M_0} \iiint r'^{n+2} \overline{P}_{nm}(\sin\varphi') \cos m\lambda'\, \mathrm{d}r'\, \mathrm{d}\lambda'\, \mathrm{d}\sin\varphi',$$

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here

$$\int r'^{n+2}\,\mathrm{d}r' = \frac{r_{\acute{y}}^{n+3}}{n+3}\left[\left(1+\frac{H_s}{r_{\acute{y}}}\right)^{n+3}-1\right]$$
$$\approx R_s^{n+3}\left[\frac{H_s}{R_s}+\frac{n+2}{2}\left(\frac{H_s}{R_s}\right)^2+\alpha\,(n+2)\,\frac{H_s}{R_s}\overline{P}_2\,(\sin\varphi)\right]$$

in quadratic approximation, $r_{\acute{y}} = R_s\left(1-\alpha\overline{P}_2\,(\sin\varphi)\right)$, $\alpha = (2/3)e \approx 10^{-3}$, e is oblateness of reference ellipsoid.

Therefore, if one takes into account quadratic terms and ellipsoidal structure of reference surface some additional terms arise in formula (8), namely

$$\begin{Bmatrix} a_{nm}^{(s)} \\ b_{nm}^{(s)} \end{Bmatrix} = \begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix}_1 + \frac{n+2}{2}\begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix}_2 + \alpha\,(n+2)\begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix}_3. \tag{9}$$

Here the first term corresponds to coefficients of expansion of function h_s (*i.e.*, to linear approximation). The term $\{\}_2$ with the index 2 corresponds to coefficients of expansion of a function $(h_s)^2$, and the term $\{\}_3$ to coefficients of expansion of function $h_s\overline{P}_2\,(\sin\varphi)$. Computing gravity forces on an ellipsoid similar to reference ellipsoid and similarly oriented with respect to it, the terms taking into account the ellipsoidal structure (with index 3), are absent in quadratic approximation.

One can derive similar formulas for the inner field. In this case, it is necessary to use equation (10) instead of equation (8) in expressions (3), (4) and (7):

$$\begin{Bmatrix} a_{nm}^{(s)} \\ b_{nm}^{(s)} \end{Bmatrix} = \begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix}_1 - \frac{n-1}{2}\begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix}_2 + \alpha\,(n-1)\begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix}_3. \tag{10}$$

Comparing equations (6) and (9) with equations (4) and (10), one can see that quadratic term contributions to both external and internal gravity forces have the same sign. They differ only by factors $(n+1)(n+2)$ and $n(n-1)$ for external and internal fields, respectively. Contributions from both linear terms and terms induced by ellipsoidal form, have opposite sign for exterior and interior gravity forces.

The formulae expressing the coefficients $\{a_{nm}, b_{nm}\}_2$, $\{a_{nm}, b_{nm}\}_3$ via $\{a_{nm}, b_{nm}\}_1$, were derived by mathematical modeling of symbol computation using mathematical software packages for computer algebra. In order to solve this problem, two approaches have been elaborated and implemented. The first method is based on direct expansion of product of finite series of elementary surface harmonics by carrying out the corresponding integration in symbolic form. The second method demands significantly less execution time. It is based on Clebsh–Gordan series [4] for the product of two Legendre associated functions P_{nm} with different index sets, and consists of transformation the expression for series product for h_s and selection in this expression the terms containing function P_{nm} with some fixed index set. The time of execution for numerical computation and hugeness of derived expressions rapidly increase with growth of degree N. We have found corresponding analytical expressions up to $N = 5$ inclusively and have developed a computation algorithm for arbitrary degree N. The final formulae can

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be presented in general form:

$$\{a_0\}_2 = \sum_{n=1}^{N}\sum_{m=0}^{n}(a_{nm}^2 + b_{nm}^2) = \iint (h_s)^2 \,\mathrm{d}\lambda' \,\mathrm{d}\sin\varphi',$$

$$\{a_{nm}\}_2 = \sum_{k}\sum_{s}\alpha_{kslq}\left(a_{ks}\cdot a_{lq} \pm b_{ks}\cdot b_{lq}\right)_1,$$

$$\{b_{nm}\}_2 = \sum_{k}\sum_{s}\alpha_{kslq}\left(a_{ks}\cdot b_{lq} \mp b_{ks}\cdot a_{lq}\right)_1,$$

here summation is carried out on coefficients $k + l \geq n$, $|s \pm q| = m$, and $\alpha^{k+l\geq n|s\pm q|=m}$ are numerical coefficients of first order of smallness: $\left(0.1 \leq \left|\alpha_{kslq}\right| \leq 1.8\right)$.

Notice that considering the quadratic members of relief expansion of degree N, introduces an additional contribution to potentials' harmonics of degree $n = 0 \ldots 2N$. Besides, the value of this contribution increases with the growth of n. The contribution to zero harmonics indicates the difference between the mean radius of an isometric sphere and the mean radius of relief reference surface.

4. Numerical results and conclusions

The maps of gravity anomalies induced by contribution of isostatically equilibrated structures of crust and mantle, are presented in figures 1 to 3. They are calculated in quadratic approximation. For this calculus we used the expansion of degree $N = 36$ of relief heights, of depths M and of anomalous masses of crust and mantle, obtained in our previous works [1, 2]. The quadratic terms contribution is most significant in regions where M boundary is situated in depth and reaches 40 mGal in Tibet region. It is even greater than the contribution from linear approximation in free air (28 mGal). The contribution to zero harmonics of external field is 4 mGal.

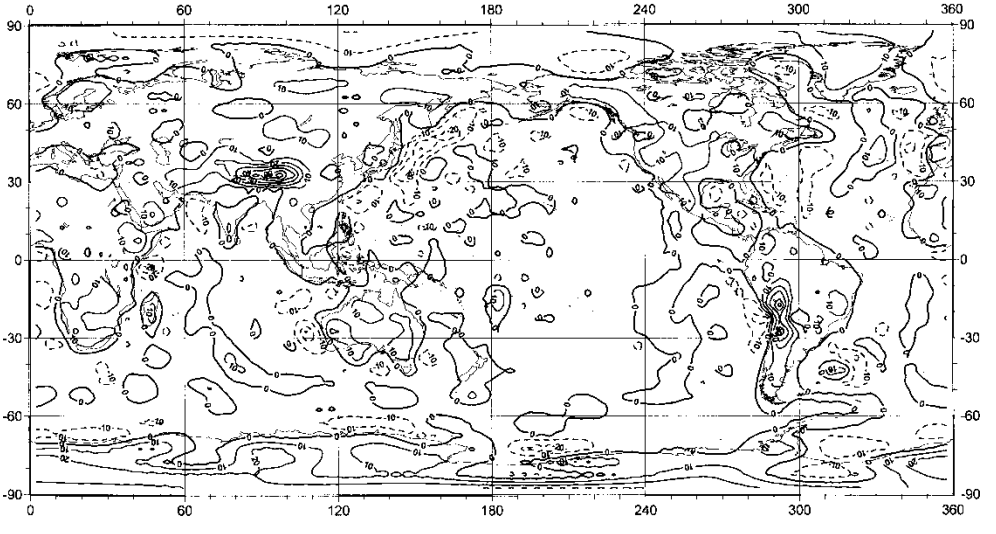

Figure 1. Summary contribution from isostatically equilibrated anomalous Earth structures of crust and mantle to gravity force anomaly in free air (in mGal). Section of isolines is 10 mGal, range of variation is (−34/68) mGal.

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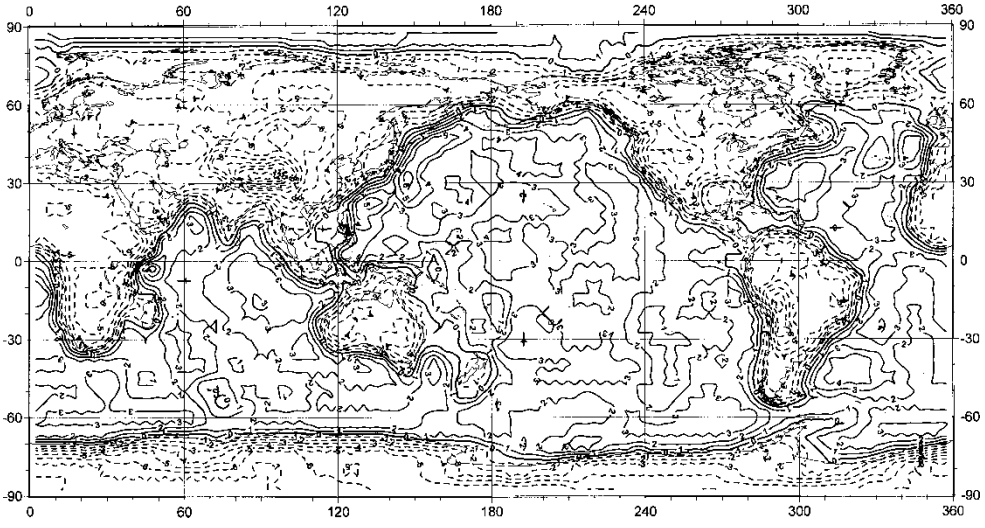

Figure 2. Summary contribution from isostatically equilibrated anomalous Earth structures to gravity force anomaly in medial crust (in 100 mGal). Section of isolines is 100 mGal, range of variation is (−1300/700) mGal.

The crust of continents (figure 2) is characterized by large negative gravity anomalies. It stimulates permanent uprise of relief surface and lowering of bottom boundary *M* from the very beginning of continents' origin. Positive anomalies in oceans' crust induce the inverse process: lowering of the ocean's bottom and rise of the *M* boundary. It is possible that these processes have caused such a great difference in thickness of continents and oceans' crust. The field of anomalies at highest levels of the upper mantle (figure 3) is more inhomogeneous. Thus, the MidOcean Ridges are characterized by large negative anomalies, that also stimulate the rise of *M* surface. Also large positive anomalies in deep basins cause the inverse process. The field in bottoms of the upper mantle (figure 4) is quieter: the anomalies do not exceed 110 mGal. It is mainly characterized by positive anomalies located under the continents (except for the

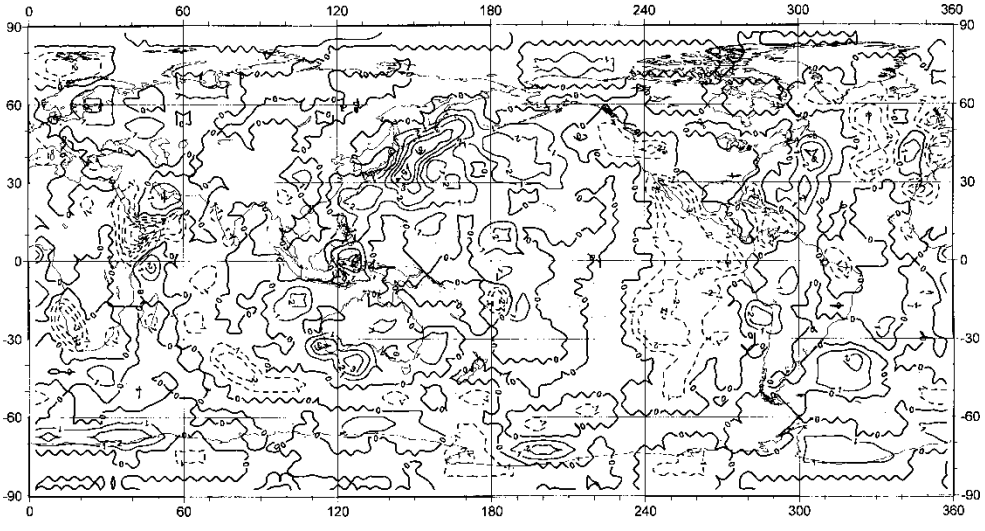

Figure 3. Summary contribution from isostatically equilibrated anomalous structures to gravity force anomaly at highest levels of upper mantle (in 100 mGal). Section of isolines 100 mGal, range of variation is (−700/700) mGal.

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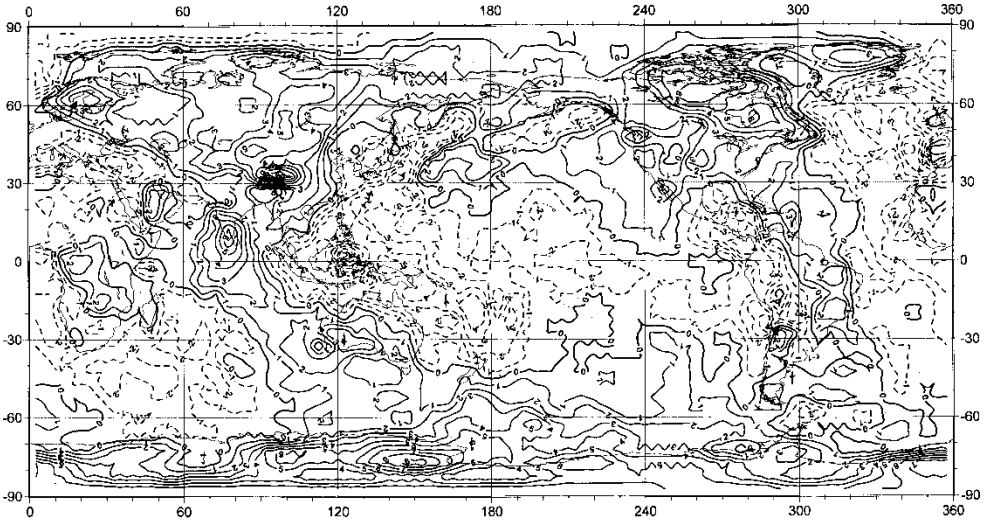

Figure 4. Summary contribution from all anomalous crust and mantle structures to gravity force anomaly in bottoms of the upper mantle (in 10 mGal). Section of isolines 10 mGal, range of variation is (−110/110) mGal.

Western Europe and the Andes), and negative anomalies located under the oceans (except for some zones of ridges and trenches).

Also, it is necessary to underline a large range of gravity force anomaly changes in crust and in upper mantle figures 2 and 3, both lateral and radial. For example, in Tibet region anomalies in upper, mean, and lower crust reach −1.5, −1.3, −1.2 Gal, respectively, and at the highest levels of the mantle it decrease to 0. The maximum of anomalies in the crust and tops of the mantle of the Northwest Pacific basin is of order 0.7 Gal, and of Red Sea riftzone is of order −0.6 Gal. It illustrates the gravitational non-equilibrium of the crust and the upper mantle for the model of the Earth in isostatic equilibrium.

The comparison of anomalies map in free air (figure 1) with gravity force anomalies map for satellite-derived model of potential (figure 5) shows a value of deviation of the upper shells of the Earth from isostatic equilibrium. This value reaches ±110 mGal in anomalies. The contribution of isostaticly non-equilibrated structures of crust and mantle practically does not change anomalies distribution in crust and upper mantle (figures 2 and 3). However, it has essential influence on anomalies pattern in bottoms of upper mantle located below the maximum depths of compensation (*i.e.*, below the depths of the mantle anomalous masses location). Therefore, total contribution to anomalies field of the upper mantle bottoms of all anomalous structures of the crust and mantle is presented in figure 4. We consider here both isostatically equilibrated anomalies and anomalies induced by deviations of crust and tops of the mantle from the state of isostatic equilibrium. The anomalies are conventionally referred to mean maximum compensation depth of 70 km. In great depth, the field decreases for harmonics of degree n as $(r/(R_0 - 70))^{n-1}$, and this decrease is essential for higher harmonics. For example the anomalies of the extent of the order of magnitude 10° ($n = 18$) decrease by twice on depths of 250 km, and for $n < 5$ the decrease on those depths does not exceed than 10%.

By comparing figures 5, 2 and 3 one can see that there is significant anomalous gravity field gradient for some regions. These gradients are located in places of transition, both from an external field to the crust field, and from the crust field to the field in upper mantle. In continental area this gradient direction is opposite to the direction of normal gradient at transition from crust to the external field. In some mountain regions for example, the Tibet and the Andes, its absolute value can even be greater than the normal gradient, the latter being of order of

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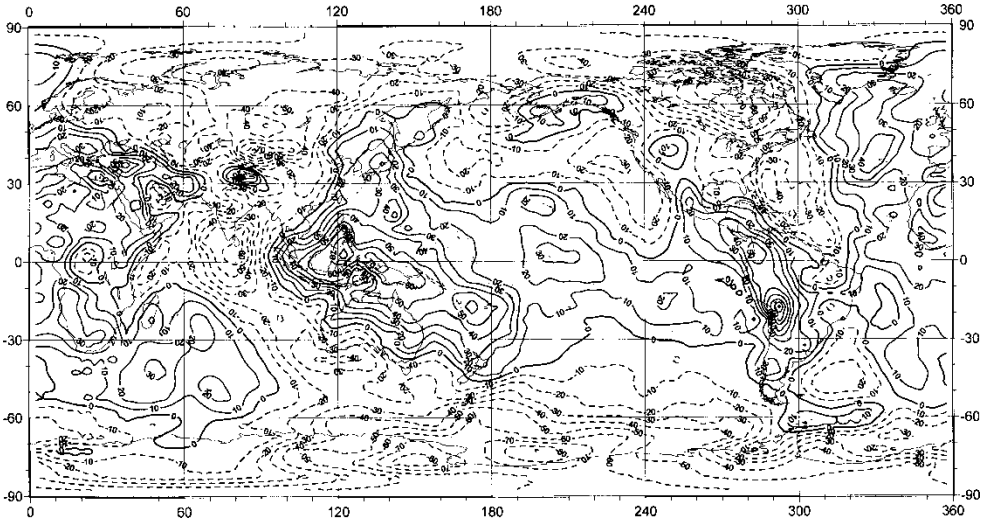

Figure 5. Anomalies of gravity force in the free air in satellite model of potential ($N = 36$) referring to hydrostatic equilibrium of the Earth (in mGal). Section of isolines 10 mGal, range of variation is (−91/93) mGal.

80 mGal/km in crust and upper mantle. Thus resulting gravity force increases at a movement upwards. Such field variations promote decompression of crust substances and can result both in ascending and in descending motions (according gradient direction), and to the rise of continents surface and to lowering of M boundary under them. However, in oceanic regions the picture should be opposite because of gradient variations: compression of the crust and lowering of ocean bottom.

It is interesting to note also comparing the maps, that in some regions the field character does not vary with transition from crust into mantle (the Western Europe, the Red Sea region, a stable area of the Pacific Ocean). However, in Mid-Ocean Ridges regions, in trenches and mountain regions of continents the anomalous field has inverse character both at transition through M boundary, and at transition from up of upper mantle (above the depth of compensation) into a bottom. The backward field inversion is especially characteristic for MOR region at mantle-crust transition. It promotes the uprise of heavy substance from mantle into crust, the upper mantle decompression and crust compression. The backward inversion is also characteristic for the Northwest Pacific area adjoining trenches. In this area there are possible growth of crust and mantle substance density, up to maximum depth of compensation (about 200 km). This effect is in perfect coherence with anomalous field, but after that compression can be replaced by decompression. Remark that the gravity anomalies in adjoining land areas remain positive even in deep regions of a mantle. It can induce the effect of inclined (under the continent) position plates of heavier substance.

The conclusions from the work are next:

(1) Evaluating relief and density jump on boundary M contribution to external gravitational field it is necessary to take into account quadratic terms.
(2) In isostatically equilibrated crust and mantle significant anomalies of the gravity field do arise leading to gravitational and dynamic instability.
(3) The deviation of the external gravity field of isostatically equilibrated Earth from the empirical observation testifies the disturbance of isostatic equilibrium of the Earth.

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