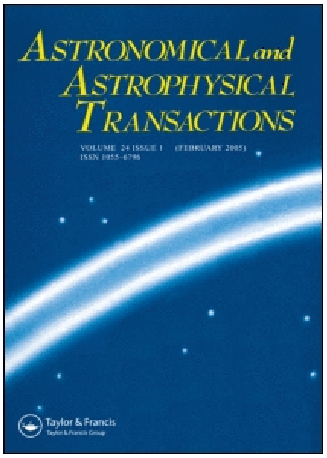

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Gravitational–tidal evolution of planetary subsystems of the Sun

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The results of some of our work and new research in which the motion of celestial bodies is considered to be subject to dissipative interactions is generalized in this article. In a number of problems the interaction of celestial bodies according to the law of universal gravitation is supplemented by a dissipative force. In the framework of this model, the evolution processes are studied, stationary configurations of an n-body systems are found, and their stability is investigated. In other problems the model of the axisymmetric celestial body that consists of a solid core and a viscoelastic mantle is considered. In such a model, the dissipation of energy under deformations is taken into account. The tidal evolution of a planet rotational motion is analysed in detail. Also phase portraits are constructed and an qualitative analysis of the global evolution of the dynamic characteristics of the Sun's planetary subsystems is presented on the basis of averaged evolution equations.

Keywords: Dissipative interactions; Model of viscoelastic celestial body; Energy dissipation under deformations; Dynamic evolution of the Sun's planetary subsystem

1. Introduction

The creation of high-precision theories of the motion of natural celestial bodies is a complicated mathematical problem. It is known that gravitational and tidal moments play a dominating part in the evolution of the dynamic characteristics of planets and their satellites in the Solar System [1, 2]; the orbital motion and rotation around the mass centre change greatly as a consequence of tidal interaction. The first fundamental research in this field was conducted at the end of the nineteenth century by the celestial mechanic and cosmogonist George Darwin [1].

Adopting the Kozlov–Eneev [3] cosmogony theory as the model of formation of the Solar System as a result of the evolution of the protoplanetary cloud that initially had a larger size into protoplanets, it may be shown that at the initial stage of dynamic evolution of the Solar System the tidal evolution of rotational motion of the planets played a particularly important part in their formation. This evolution occurred several orders move quickly than in modern

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times. At present, because of the very weak tidal evolution of the rotations of the planets and their satellites, the values of their angular velocities can remain close to the resonant values for a long time, even if the corresponding resonant rotations are unstable. Dissipation leads the system to evolve to a state of minimum total energy; therefore, for a system with tidal dissipation, the resonant state may become the final state.

In the second part of the twentieth century, the advent of radiolocation astronomy with its continuously improving high-precision methods of measurements led to the necessity to correct and to develop the exist theories and to create qualitatively new theories of tidal evolution of the rotational and translational motion of celestial bodies.

To study the evolution processes, researchers used a new theoretical model which considers a planet as an elastic solid body with energy dissipation under deformations and the orbital evolution of which is caused by the work of internal dissipative forces (without considering the thermodynamic processes); the planet itself executes translational–rotational motion. Researchers have shown and emphasized that it is not correct to consider the evolution of a planet's rotational motion separately from its orbital motion and vice versa. Moreover, the research carried out on the evolution processes in the motion of planetary subsystems show the indissoluble ties between the evolutions of the dynamic characteristics of translational–rotational motion of planets and their satellites.

The present method is characterized by a rigorous mathematical model. It is considered that the layer is described by the linear viscoelasticity theory and the deformation process is carried out quasistatically [4]. The model proposed permits us to obtain results mathematically justified in an asymptotical sense for cosmogonic time intervals.

2. Three-body problem with dissipative forces

Let us consider the three-body problem in a classic statement for planar configurations subject to dissipative forces depending on the rate of change in the distance between gravitating bodies. Let the body with mass m_0 be the attractive centre O; we ignore the effect of dissipative forces from one side of this body with mass m_0 (attractive center) acting on the other two bodies (figure 1).

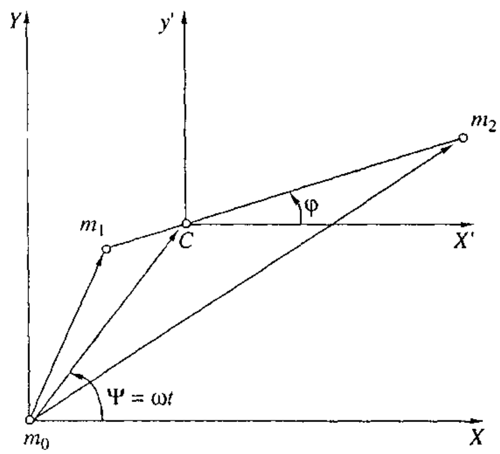

Figure 1. Coordinate systems for the description of motion in the three-body problem with dissipative forces.

Let us set the dissipative force proportional to the rate of change in the distance between the bodies of masses m_1 and m_2 in the form [5]

$$\boldsymbol{F}^{\text{Diss}} = -Q(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{r}_1' - \boldsymbol{r}_2')\boldsymbol{r}^{-10}(\boldsymbol{r}_1 - \boldsymbol{r}_2), \tag{1}$$

where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the radius vectors of the bodies of masses m_1 and m_2, respectively, relative to the inertial coordinate system, $Q > 0$ is a constant coefficient depending on the physical characteristics of the bodies (density, sizes, elastic and dissipative properties of the bodies), and r is the distance between the bodies of masses m_1 and m_2, where $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$.

Let point C be the barycentre of the bodies of masses m_1 and m_2; $\boldsymbol{R}$ is the radius vector of point C. The expression for the system's potential energy up to terms of order $(r/R)^3$ has the form

$$\Pi = -fm_1m_2r^{-1} - fm_0\left(-\left(\frac{m_1m_2}{(m_1+m_2)}\right)r^2R^{-3} - \frac{3}{2}\left(\frac{m_1m_2}{(m_1+m_2)}\right)(\boldsymbol{R},\boldsymbol{r})^2R^{-5}\right), \tag{2}$$

where f is the gravitational constant. Let us introduce the designation $m = m_1m_2/(m_1+m_2)$ and rewrite equation (2) in the form

$$\Pi = -fm_1m_2r^{-1} + \frac{1}{2}fm_0mr^2R^{-3}[1 + 3\cos^2(\varphi - \psi)], \tag{3}$$

where the angles φ and ψ are determined according to figure 1.

The equations of mutual motion of the centres of mass of the planet and satellite in the Delaunay variables L, Λ, ℓ and g are derived from Hamilton's variational principle

$$\int_{t_1}^{t_2}[\delta(L\ell' + Gg') - \delta H + \delta A]\,\mathrm{d}t = 0, \tag{4}$$

where δA is the elementary work of forces of gravitational interaction subject to the work of elastic and dissipative forces on the possible motions of the system.

The canonical equations of motion of the considered system of bodies are represented in the form

$$\begin{aligned} L' &= -\frac{\partial H}{\partial \ell} + Q_\ell, \quad \ell' = \frac{\partial H}{\partial L} - Q_L,\\ \Lambda' &= -\frac{\partial H}{\partial g} + Q_g, \quad g' = \frac{\partial H}{\partial \Lambda} - Q_\Lambda. \end{aligned} \tag{5}$$

Here,

$$H = H_0 + \frac{1}{2}fm_0mr^2R^{-3}[1 + 3\cos^2(\varphi - \psi)], \tag{6}$$

where $H_0 = -(1/2)\mu^2m^3L^{-2}$ is the Hamiltonian of the unperturbed problem; $\mu = f(m_0 + m_1)$; Q_L, Q_Λ, Q_ℓ and Q_g are the generalized forces corresponding to changes in the coordinates L, Λ, ℓ and g, respectively.

The expression for the work of dissipative forces on possible motions of the system is determined by the relation

$$\begin{aligned} \delta A = &-Q(|\boldsymbol{r}_1' - \boldsymbol{r}_2'|)\left(\frac{(\boldsymbol{r}_1 - \boldsymbol{r}_2)}{r}, \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\right)\\ &- Q(|\boldsymbol{r}_2' - \boldsymbol{r}_1'|)\left(\frac{(\boldsymbol{r}_2 - \boldsymbol{r}_1)}{r}, \delta(\boldsymbol{r}_2 - \boldsymbol{r}_1)\right) \end{aligned}$$

$$= -2Q\boldsymbol{r}'\left(\frac{\boldsymbol{r}}{r}, \delta\boldsymbol{r}\right)$$
$$= -2Qr'\,\delta r. \tag{7}$$

Since $r = r(L, \Lambda, \ell)$, then $\partial r/\partial g = 0$ and

$$\delta r = \frac{\partial r}{\partial L}\delta L + \frac{\partial r}{\partial \Lambda}\delta\Lambda + \frac{\partial r}{\partial \ell}\delta\ell. \tag{8}$$

Then system (5) can be rewritten as follows:

$$\begin{aligned} L' &= -\frac{\partial H}{\partial \ell} - \frac{2Qp_r}{m}\frac{\partial r}{\partial \ell}, \quad \ell' = \frac{\partial H}{\partial L} + \frac{2Qp_r}{m}\frac{\partial r}{\partial L}, \\ \Lambda' &= -\frac{\partial H}{\partial g}, \quad g' = \frac{\partial H}{\partial \Lambda} + \frac{2Qp_r}{m}\frac{\partial r}{\partial \Lambda}, \end{aligned} \tag{9}$$

where p_r is the generalized momentum.

Using the relations $r = a(1 - e\cos w)$, $a = (L^2\mu^{-1}m^{-2}$, $e^2 = 1 - \Lambda^2 L^{-2}$, $\ell = w - e\sin w$, $\cos w = (e + \cos v)(1 + e\cos v)^{-1}$ and $\varphi = g + w$ (where w and v are the eccentric and true anomalies, respectively), the Hamiltonian of the perturbed problem can be rewritten as follows:

$$H = -\frac{1}{2}\mu^2 m^3 L^{-2} + \frac{1}{2}fm_0 m R^{-2}(1 - e\cos w)^2[1 + 3\cos^2(w + g - \psi)]. \tag{10}$$

The generalized momentum p_r is written in the form

$$p_r = (2\mu m^3 r^{-1} - \Lambda^2 r^{-2} - \mu^2 m^4 L^{-2})^{1/2} = \mu m^3 e\sin w L^{-1}(1 - e\cos w)^{-1}. \tag{11}$$

Let us average the right-hand sides of the first two equations of system (9) over the fast variables ℓ and ψ. We obtain the averaged equations

$$\Lambda' = 0, \quad L' = -2Qm^{-1}Le^2\left(\frac{1}{2} + \frac{1}{8}e^2 + \cdots\right). \tag{12}$$

At $t \to \infty$, $e \to 0$, i.e. $L \to \Lambda$, the orbits of motion of the points of masses m_1 and m_2 around the barycentre C tend to be circular.

3. Evolution of the motion of *n* gravitating hoops

Let us consider the problem of the evolution of inclinations of the planes of planetary orbits. To simplify the problem, let us make the following assumption: we shall consider all orbits as circular, having a constant radius. It is known that averaging the equations of motion over fast angular variables in some sense is equivalent to 'smearing' a mass point along its orbit. Therefore, let us consider the model problem of motion of n solid gravitating hoops, moving along concentric spheres, instead of the problem of the evolution of inclinations of orbit planes.

Let m_i and $\boldsymbol{R}_i$ be the hoop masses and radii, respectively. Let us take advantage of the variables introduced by Zhuravlev and Klimov [6] and represent the angular moments of the hoops in the form

$$\boldsymbol{\Lambda}_i = \frac{1}{2}C_i\boldsymbol{\Omega}_i + C_i\gamma_i'\boldsymbol{e}_i, \quad \boldsymbol{\Omega}_i = \boldsymbol{e}_i \times \boldsymbol{e}_i'. \tag{13}$$

Here, C_i are the moments of inertia of the hoops relative to the symmetry axis, $\boldsymbol{e}_i$ are the unit vectors of the symmetry axes of the hoops and γ_i are the angles of rotation of the hoops around the symmetry axis. The equations of motion follow from the theorem of change in the angular momentum:

$$\boldsymbol{\Lambda}_i' = \frac{1}{2}C_i(\boldsymbol{e}_i \times \boldsymbol{e}_i'') + C_i\gamma_i''\boldsymbol{e}_i = \boldsymbol{M}_i, \tag{14}$$

where $\boldsymbol{M}_i = \boldsymbol{M}_i^{\mathrm{Grav}} + \boldsymbol{M}_i^{\mathrm{Diss}}$ are the moments of the forces acting on a hoop. The gravitational moment is represented in the form

$$\boldsymbol{M}_i^{\mathrm{Grav}} = \sum_{j=1(j\neq i)}^{n} \left(\iint (\boldsymbol{r}_i \times \boldsymbol{r}_i)|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-3} f\rho_i\rho_j\,\mathrm{d}s_i\,\mathrm{d}s_j \right), \tag{15}$$

where f is the gravitational constant, ρ_i and ρ_j are the linear densities of the ith and jth hoops respectively, and $\mathrm{d}s_i$ and $\mathrm{d}s_j$. are the differentials of arc lengths of the ith and jth hoops, respectively; integrals are taken over the ith and jth hoops. The dissipative moments are represented in the form

$$\boldsymbol{M}_i^{\mathrm{Diss}} = \sum_{j=1(j\neq i)}^{n} \left(\iint (\boldsymbol{r}_i \times \boldsymbol{r}_j) Q_{ij}|\boldsymbol{r}_i - \boldsymbol{r}_j|\,|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-9}\,\mathrm{d}s_i\,\mathrm{d}s_j \right), \tag{16}$$

according to the expression for dissipative force given below; $Q_{ij} = Q_{ij} > 0$.

Let us transform the expression for the moments of the gravitational force (15) and dissipative force (16) to a form convenient for further consideration. Consider separately each integral (15) that is under the sign of summation. Let us use a specially introduced coordinate system (figure 2). Then

$$\iint (\boldsymbol{r}_i \times \boldsymbol{r}_j)|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-3} f\rho_i\rho_j\,\mathrm{d}s_i\,\mathrm{d}s_j = \frac{\boldsymbol{e}_i \times \boldsymbol{e}_j}{|(\boldsymbol{e}_i \times \boldsymbol{e}_j)|} \iint \left(\frac{\boldsymbol{e}_i \times \boldsymbol{e}_j}{|(\boldsymbol{e}_i \times \boldsymbol{e}_j)|}, (\boldsymbol{r}_i \times \boldsymbol{r}_j) \right)$$
$$\times\, f\rho_i\rho_j|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-3}\,\mathrm{d}s\,\mathrm{d}s_j, \tag{17}$$

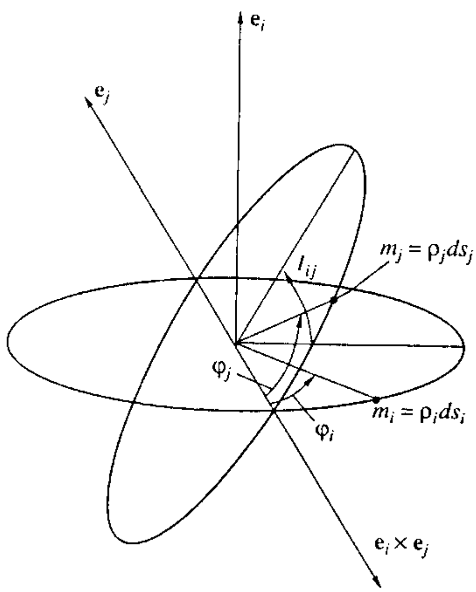

Figure 2. Coordinate system for the calculation of the moment of forces for the interaction of hoops.

i.e. the vector of the gravitational moment of the interaction of a hoop pair is parallel to the line of intersection of planes of these hoops. Using expressions for $\boldsymbol{r}_i = R_i(\cos\psi_i, \sin\psi_i, 0)$, $\boldsymbol{r}_j = R_j(\cos\psi_j, \sin\psi_j\cos\vartheta_{ij}, \sin\psi_j\sin\vartheta_{ij})$, where $\cos\vartheta_{ij} = (\boldsymbol{e}_i, \boldsymbol{e}_j)$, let us transform equation (17) to the form [7]

$$\iint (\boldsymbol{r}_i \times \boldsymbol{r}_j)|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-3} f\rho_i\rho_j \, \mathrm{d}s_i \, \mathrm{d}s_j = (\boldsymbol{e}_i \times \boldsymbol{e}_j)\frac{\partial}{\partial(\boldsymbol{e}_i, \boldsymbol{e}_j)}\left(\iint f\rho_i\rho_j|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-3}\,\mathrm{d}s_i\,\mathrm{d}s_j\right), \tag{18}$$

and, consequently,

$$\boldsymbol{M}_i^{\mathrm{Grav}} = \sum_{j=1(j\neq i)}^{n}\left[(\boldsymbol{e}_i \times \boldsymbol{e}_j)\frac{\partial}{\partial(\boldsymbol{e}_i, \boldsymbol{e}_j)}\left(\iint f\rho_i\rho_j|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-3}\,\mathrm{d}s_i\,\mathrm{d}s_j\right)\right]. \tag{19}$$

The dissipative moment, after some transformations, is represented as

$$\boldsymbol{M}_i^{\mathrm{Diss}} = \sum_{j=1(j\neq i)}^{n} Q_{ij}\Big(\iint (\boldsymbol{r}_i \times \boldsymbol{r}_j)|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-10}([(\boldsymbol{\Omega}_i - \boldsymbol{\Omega}_j) \times \boldsymbol{r}_i], \boldsymbol{r}_j) + ([(\gamma_i'\boldsymbol{e}_i - \gamma_j'\boldsymbol{e}_j) \times \boldsymbol{r}_i], \boldsymbol{r}_j)\,\mathrm{d}s_i\,\mathrm{d}s_j\Big). \tag{20}$$

From equations (14), (19) and (20) it follows that the system will be in equilibrium in the following two cases.

(i) All $\boldsymbol{e}_1 = \boldsymbol{e}_2 = \cdots = \boldsymbol{e}_0$, $\gamma_1' = \gamma_2' = \cdots = \gamma_0'$ (in this case, $\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_2 = \cdots = \boldsymbol{\Omega}_n =$ constant), i.e. all hoops are in one plane and rotate with the same angular velocity and this plane also rotates with a constant angular velocity.

(ii) $\boldsymbol{e}_1 = \boldsymbol{e}_2 = \cdots = \boldsymbol{e}_m = \boldsymbol{e}_0$, $\boldsymbol{e}_{m+1} = \boldsymbol{e}_{m+2} = \cdots = \boldsymbol{e}_l = \boldsymbol{e}_{00}$, $\boldsymbol{e}_{l+1} = \boldsymbol{e}_{l+2} = \cdots = \boldsymbol{e}_n == \boldsymbol{e}_{000}$, the vectors $\boldsymbol{e}_0, \boldsymbol{e}_{00}$ and $\boldsymbol{e}_{000}$ are orthogonal, $\gamma_1' = \gamma_2' = \cdots = \gamma_n' = 0$ and $\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_2 = \cdots \boldsymbol{\Omega}_n =$ constant. In other words, the hoops are separated into three groups so that the hoops of each group are in one plane and stationary relative to it. All three planes are mutually orthogonal and rotate with the same constant angular velocity.

Let us prove that, in case (i), the equilibrium is asymptotically stable.

Let $A_i = (1/2)C_i$; let us multiply vectorially each ith equation (14) from the left by $\boldsymbol{e}_i'$ and, taking the formula of the triple vector product into account, we obtain

$$A_i\boldsymbol{e}_i(\boldsymbol{e}_i'', \boldsymbol{e}_i') + C_i\gamma_i''[\boldsymbol{e}_i' \times \boldsymbol{e}_i] = \sum_{j=1(j\neq i)}^{n}\left[\boldsymbol{e}_i(\boldsymbol{e}_i', \boldsymbol{e}_j)\frac{\partial}{\partial(\boldsymbol{e}_i, \boldsymbol{e}_j)}\left(\iint f\rho_i\rho_j|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-1}\,\mathrm{d}s_i\,\mathrm{d}s_j\right)\right] + (\boldsymbol{e}_i' \times \boldsymbol{M}_i^{\mathrm{Diss}}). \tag{21}$$

By multiplying each equation in a scalar manner by $\boldsymbol{e}_i$ and taking equation (14) into account, we derive

$$\sum_i \frac{1}{2}A_i(\boldsymbol{e}_i')^2 = \sum_{i,j(i<j)}\iint f\rho_i\rho_j|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-1}\,\mathrm{d}s_i\,\mathrm{d}s_j - \sum_{i,j(i<j)} Q_{ij}(\boldsymbol{\Omega}_i - \boldsymbol{\Omega}_j), \iint (\boldsymbol{r}_i \times \boldsymbol{r}_j)|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-10}\{([(\boldsymbol{\Omega}_i - \boldsymbol{\Omega}_j) \times \boldsymbol{r}_i], \boldsymbol{r}_j) + ([(\gamma_i'\boldsymbol{e}_i - \gamma_j'\boldsymbol{e}_j) \times \boldsymbol{r}_i], \boldsymbol{r}_j)\}, \mathrm{d}s_i\,\mathrm{d}s_j). \tag{22}$$

As the equations

$$C_i \gamma_i' \gamma_i'' = \gamma_i'(\boldsymbol{e}_i, \boldsymbol{M}_i^{\mathrm{Diss}}) \tag{23}$$

hold, then, adding equation (22) and summing equation (23), we obtain

$$\begin{aligned}
&\sum_i \frac{1}{2} A_i (\boldsymbol{e}_i')^2 - \sum_{i,j\,(i<j)} \iint f\rho_i\rho_j |\boldsymbol{r}_i - \boldsymbol{r}_j|^{-1}\,\mathrm{d}s_i\,\mathrm{d}s_j + \sum_i \frac{1}{2} C_i (\gamma_i')^2 \\
&\quad = \sum_{i,j\,(i<j)} Q_{ij}((((\boldsymbol{\Omega}_i - \boldsymbol{\Omega}_j + \gamma_i'\boldsymbol{e}_i - \gamma_j'\boldsymbol{e}_j, \iint (\boldsymbol{r}_i \times \boldsymbol{r}_j)|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-10} \\
&\qquad \times ([(\boldsymbol{\Omega}_i - \boldsymbol{\Omega}_j) \times \boldsymbol{r}_i], \boldsymbol{r}_j) + ([(\gamma_i'\boldsymbol{e}_i - \gamma_j'\boldsymbol{e}_j) \times \boldsymbol{r}_i], \boldsymbol{r}_j)\,\mathrm{d}s_i\,\mathrm{d}s_j) \\
&\quad = \sum_{i,j\,(i<j)} Q_{ij} \iint (\boldsymbol{\Omega}_i - \boldsymbol{\Omega}_j + \gamma_i'\boldsymbol{e}_i - \gamma_j'\boldsymbol{e}_j, (\boldsymbol{r}_i \times \boldsymbol{r}_j))^2 |\boldsymbol{r}_i - \boldsymbol{r}_j|^{-10}\,\mathrm{d}s_i\ \mathrm{d}s_j \\
&\quad \le 0.
\end{aligned} \tag{24}$$

In this case, equality to zero is reached only in the equilibrium position.

For equilibrium of type (i), the energy of the system will be its Lyapunov function

$$\begin{aligned}
V = &\sum_i \frac{1}{2} A_i (\boldsymbol{e}_i')^2 - \sum_{i,j\,(i<j)} \iint f\rho_i\rho_j |\boldsymbol{r}_i - \boldsymbol{r}_j|^{-1}\,\mathrm{d}s_i\,\mathrm{d}s_j + \sum_i \frac{1}{2} C_i (\gamma_i')^2 \\
&+ \sum_{i,j\,(i<j)} \iint f\rho_i\rho_j |\boldsymbol{r}_i - \boldsymbol{r}_j|^{-1}\,\mathrm{d}s_i\,\mathrm{d}s_j \big|_{(\boldsymbol{e}_i' = \boldsymbol{e}_0')} - \sum_i \frac{1}{2} A_i (\boldsymbol{e}_0')^2 - \sum_i \frac{1}{2} C_i (\gamma_0')^2.
\end{aligned} \tag{25}$$

From equation (25), it can be seen that $V = 0$ in the equilibrium position, $V' < 0$ in a very small neighbourhood of the equilibrium position, $V > 0$ in the neighbourhood of the equilibrium position. Thus, all conditions of the Lyapunov theorem are met and the asymptotic stability of the equilibrium position position of type (i) is established.

For proof of the instability of the equilibrium positions of type (ii), let us draw attention to the fact that, when the hoop planes deviate slightly from strict orthogonality, a moment arises that tends to disturb the given equilibrium position and to put the hoops in one plane, and this means instability.

The equilibrium position and its stability analysis obtained upon solution of the model problem of the motion of n solid gravitating hoops moving along concentric spheres can be used to study the evolution of the natural hoops of gigantic planets of the Solar System.

4. Regular features of tidal evolution in the rotational motion of a viscoelastic planet

4.1 *Model of a deformable planet*

Let us consider the model of a planet as an elastic solid body with the dissipation of energy under deformations. A planet consists of a rigid part (a core) and an isotropic elastic layer. In its undeformed state, the planet is dynamically compressed. The axis of symmetry of the elastic part coincides with the axis of dynamic symmetry of the whole planet. Displacements of particles of the elastic medium at the boundary with the rigid part are equal to zero, and the rest of the boundary is free. The motion of the planet allows us to describe it in the framework of classic dynamics, and its deformable states are considered without thermodynamic processes.

The description of the deformable state of a celestial body is based on a modal approach, small deformations being considered to be linearly elastic [8]. In the case of an axisymmetric elastic part with axisymmetric boundary conditions, the vector of elastic displacement may be written in the following form:

$$\boldsymbol{u}(\boldsymbol{r}, t) = \sum_{k,m=0}^{\infty} [q_{km}(t)\boldsymbol{V}_{km}(\boldsymbol{r}) + p_{km}(t)\boldsymbol{W}_{km}(\boldsymbol{r})], \tag{26}$$

where $q_{km}(t)$ and $p_{km}(t)$ are the generalized normal coordinates (modal variables) describing the body motion corresponding to the inner degrees of freedom, and $\boldsymbol{V}_{km}(\boldsymbol{r})$ and $\boldsymbol{W}_{km}(\boldsymbol{r})$ are the fundamental functions (natural models) of free oscillations of the elastic part, corresponding to the fundamental (natural) frequency ν_{km} and satisfying the conditions of orthonormalization.

The reference frame $Cx_1x_2x_3$ is embedded in the rigid part. Its axes are directed along the principal central axes of inertia of the undeformed planet, and Cx_3 is the axis of symmetry. The modal approach requires calculation of the following coefficients of the fundamental functions:

$$b_{kmij} = \int V_{kmi}x_j\,\mathrm{d}x, \quad c_{kmij} = \int W_{kmi}x_j\,\mathrm{d}x,$$
$$\mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3, \tag{27}$$

where V_{kmi} and W_{kmi} are the projections of the vectors $\boldsymbol{V}_{km}$ and $\boldsymbol{W}_{km}$ on the axis Cx_i.

The potential energy of linearly elastic deformations is given by the quadratic functional $E_2[\boldsymbol{u}]$. The rheological properties of the planetary material are described by the Kelvin–Voigt model with the use of the dissipative functional $D[\boldsymbol{u}'] = \chi b E_2[\boldsymbol{u}']$, where χ is the factor accounting for energy dissipation and where b is a positive constant.

The equations of motion consist of the equations governing the rotation of the planet as a whole and the equations determining the deformation. The main assumption, which has a solid physical basis, is as follows: the attenuation time of free oscillations of the elastic part at the lowest fundamental frequency ν is much less than the characteristic time of motion of the system as a whole. The equations for the normal coordinates q_{km} and p_{km} are singularly perturbed; so investigation of such equations is usually carried out by the boundary layer method (using equations with a small parameter and a higher derivative with respect to time). The motion attained after attenuation of the oscillations with high fundamental frequencies is a long-period motion. It corresponds to the regular part of the solution of equations for the normal coordinates, which describes the forced oscillations of planet deformation. This particular solution is applicable for determining the deformations in an asymptotic sense from a certain time instant. It satisfies quasistatic equations for the modal variables. These equations define the generalized coordinates q_{km} and p_{km} as functions of the variables that describe the motion of the planet as a whole.

4.2 *Evolution equations of motion and their analysis*

If the mass centre of a planet is rotating along a circular orbit in a central gravitational field of forces while the planet is subjected to tides, then the evolution of its motion is described by the following equations [9]:

$$\begin{aligned} I_2 = {} & -k\{I_2[\lambda_1(1 + 2x^2 - 3x^4) + 4\lambda_2(3 + 2x^2 + 3x^4)] \\ & - 4C\omega_0 x[4\lambda_2(1 + x^2) + \lambda_1(1 - x^2)]\}, \end{aligned}$$

$$
\begin{aligned}
I_3 &= 4k\{I_2 x[(4\lambda_2-\lambda_1)(1-x^2)-8\lambda_2] \\
&\quad + C\omega_0[(\lambda_0+\lambda_2-\lambda_1)(1-x^2)^2+2(4\lambda_2-\lambda_1)x^2+2\lambda_1]\} \qquad (28)\\
k &= \frac{9}{8}\varepsilon^3\kappa\omega_0 C^{-1}, \quad \varepsilon=\frac{\omega_0}{\nu}, \quad \kappa=\chi b\nu, \quad x=\frac{I_3}{I_2},\\
\lambda_0 &= \rho_2\sum_{m=0}^{\infty}(c_{0m11}-c_{0m33})^2\sigma_{0m}^{-2}, \quad \lambda_1=\rho_2\sum_{m=0}^{\infty}(b_{1m23}-b_{1m32})^2\sigma_{1m}^{-2},\\
\lambda_2 &= \rho_2\sum_{m=0}^{\infty}b_{2m12}^2\sigma_{2m}^{-2}, \quad \sigma_{im}=\frac{\nu_{im}}{\nu} \quad (i=0,1,2), \quad k_i,\lambda_i>0.
\end{aligned}
$$

Here, I_2 is the magnitude of the proper angular momentum vector $\boldsymbol{G}$ of the planet; I_3 is the projection of $\boldsymbol{G}$ on to the normal $\boldsymbol{n}$ to the orbital plane; $x=\cos\delta$, where δ is the angle between the vectors $\boldsymbol{G}$ and $\boldsymbol{n}$; ω_0 is the angular velocity of the mean orbital motion of the mass centre of the planet; $\rho_2=$ constant is the density of its elastic part.

For further investigation of the evolution of the system considered, it is reasonable to proceed to the variables $\Psi=C^{-1}I_2$ (Ψ is the angular velocity of proper rotation of the planet) and x. According to equation (28), we have

$$
\begin{aligned}
\Psi &= -k\Psi_{12}(\Psi-\Psi_1), \qquad \Psi_1=\frac{\Psi_{11}\omega_0}{\Psi_{12}},\\
x' &= -k\Psi^{-1}\Psi_{22}(\Psi-\Psi_2), \qquad \Psi_2=\frac{\Psi_{21}\omega_0}{\Psi_{22}},\\
\Psi_{11}(x) &= 4x[\lambda_1(1-x^2)+4\lambda_2(1+x^2)],\\
\Psi_{12}(x) &= \lambda_1(1-x^2)(1+3x^2)+4\lambda_2(3+2x^2+3x^4), \qquad (29)\\
\Psi_{21}(x) &= 4[\lambda_0(1-x^2)^2+\lambda_1(1-x^2)+\lambda_2(1+2x^2-3x^4)],\\
\Psi_{22}(x) &= x[3\lambda_1(1-x^2)^2+4\lambda_2(1+2x^2-3x^4)],\\
\operatorname{sgn}\Psi_{11} &= \operatorname{sgn}\Psi_{22}=\operatorname{sgn}x, \quad \Psi_{12},\Psi_{21}>0.
\end{aligned}
$$

Let us construct a phase picture of the system motion (figure 3). It follows from the first of equations (29) that, for $\Psi>\Psi_1(x)$, the derivative $\Psi'<0$, and the rotation of the planet decelerates; if $\Psi<\Psi_1(x)$, then $\Psi'>0$, and the rotation of the planet accelerates. Using the expressions

$$
\begin{aligned}
&\frac{\mathrm{d}\Psi_{12}}{\mathrm{d}x}=x\frac{\mathrm{d}\Psi_{11}}{\mathrm{d}x}, \qquad \frac{\mathrm{d}\Psi_{11}}{\mathrm{d}x}=4[\lambda_1(1-3x^2)+4\lambda_2(1+3x^2)],\\
&\frac{\mathrm{d}\Psi_1}{\mathrm{d}x}=\omega_0\left(\frac{\mathrm{d}\Psi_{11}}{\mathrm{d}x}\right)(1-x^2)[\lambda_1(1-x^2)+4\lambda_2(3+x^2)],
\end{aligned} \qquad (30)
$$

it may be shown that Ψ_1 is an odd and monotonically increasing (if $\lambda_1<8\lambda_2$, as is assumed) function of x with the asymptotes $\Psi=\pm\omega_0$. It follows from the second of equations (29) that, for $x>0$, the value of x increases or decreases provided that the spin rate Ψ of the planet satisfies the relations $\Psi<\Psi_2$ or $\Psi>\Psi_2$, respectively. For $x<0$, the inequality $\Psi>\Psi_2$ always holds (since $\Psi_2<0$); so the value of x increases monotonically, while $|x|$ decreases. Stationary tilts δ of the rotational axis correspond to the values $x_1=1$ and $x_2=-1$ when the axis is directed upwards and downwards respectively along the normal to the orbital plane ($\delta_1=0$ and $\delta_2=\pi$).

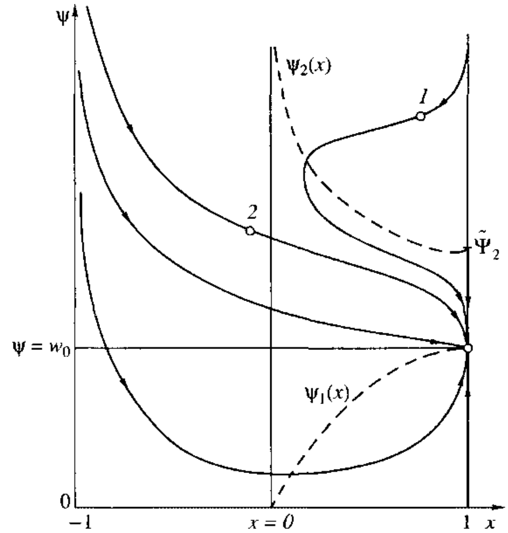

Figure 3. Phase portraits of the evolution of the rotational motions of planets: 1, Mars; 2, Uranus.

The variational equation for the first steady-state motion is written as

$$\Delta' = 8k\Psi^{-1}[4\lambda_2\Psi - \omega_0(4\lambda_2 + \lambda_1)]\Delta, \quad x = 1 - \Delta. \tag{31}$$

It follows from equation (31) that the motion $x = x_1$ is stable for $\Psi < \tilde{\Psi}$, where $\tilde{\Psi} = \omega_0(1 + \lambda_1/4\lambda_2)$, and unstable otherwise. The second steady-state motion $x = x_2$ is unstable for arbitrary values of the angular velocity of the rotation of the planet. The function Ψ_2 is odd with respect to x and has the asymptote $x = 0$; for $x \to \pm 1$, the relation $\Psi_2 \to \tilde{\Psi}$ holds. Calculating the expression for $\partial\Psi_2/\partial x$, which for brevity is not presented here, it can be demonstrated that, for $x > 0$ $(0 < \delta < \pi/2)$, the function Ψ_2 has a minimum Ψ_*, $\Psi_* > \omega_0$ while, for $x \to 1$, $\partial\Psi_2/\partial x \to +\infty$.

The differential equation for the phase trajectories of the system (29) is written in the following form:

$$\frac{d\Psi}{dx} = \Psi\Psi_{12}(\Psi - \Psi_1)(\Psi - \Psi_2)^{-1}\Psi_{22}^{-1}. \tag{32}$$

It is seen from equation (32) that, for the values $|x| = 1$, i.e. when the rotational axis of the planet is situated close to the orbital plane normal, $|d\Psi/dx| \to \infty$. With this orientation of the rotational axis, the relative change in the spin rate of the planet proceeds much more quickly than the evolution of the axial tilt δ. For the values of the angle $\delta \in [\pi, \pi/2](x \leq 0)$ corresponding to the proper rotation of the planet, reversed with respect to the orbital motion of the mass centre, the inequality $d\Psi/dx < 0$ always holds, and the spin rate of the planet decreases monotonically.

The above analysis allows us to reveal some regular features of the tidal evolution of the rotational motion of a deformable planet.

(i) All the final motions approach the same limiting pattern, which is forward rotation with the spin rate equal to the angular velocity of the mean orbital motion of the mass centre.
(ii) In the end, the rotational axis of the planet tends to be aligned with the normal to the orbital plane.

The evolution process depends on the initial conditions, which correspond to one of three sets of phase trajectories. In the case of motion along the trajectories of the first set, initially the rotational axis twins towards the orbital plane (the value $x = \cos\delta > 0$ decreases). On intersection with the curve $\Psi_2(x)$, the value of δ' equals zero. Then a decrease in the angle δ is observed, and the rotational axis restores its direction along the normal to the orbital plane. In the motion along the trajectories of this set, the angular velocity of the proper rotation decreases monotonically to the value ω_0.

The phase trajectories of the second and third sets correspond to evolution with a turnover of the rotational axis of the planet, when the rotation changes its direction from reverse to forward. Here, the tilt angle δ monotonically decreases to zero; however, the spin rate Ψ may evolve in various ways. In the case of the second set of phase trajectories, the monotonically decreasing function Ψ approaches asymptotically the limiting value ω_0 (as well as for the first set of trajectories), whereas the angle δ approximately equals zero. It is noted that, for $x < 0$, the magnitude of the inclination of the phase trajectories $|\mathrm{d}\Psi/\mathrm{d}x|$ at a fixed x increases with increase in the actual value of Ψ. Thus, the prevalence of the effect of the evolution of rotational axis tilt is not observed. For the third set of the phase trajectories, the angular velocity Ψ initially decreases to values smaller than ω_0. On intersection with the curve $\Psi_1(x)$ the condition $\Psi' = 0$ occurs; then Ψ increases monotonically and approaches asymptotically the value ω_0, when the angle δ is approximately equal to zero. In the phase portrait the position of the planets Uranus (the second set) and Mars (the first set) in the modern era are indicated.

Let us consider Uranus as an example. The inclination of the proper rotational axis of this planet with respect to the orbital plane is $\delta = 98°$. The period of Uranus in orbit (having a small eccentricity of 0.046) around the Sun equals 84 years and 8 days, while the period of the proper rotation equals 10 h 49 min. Also, the following values are taken: $\Psi(0) = 1.61 \times 10^{-4}\ \mathrm{rad\,s^{-1}}$ and $\omega_0 = 2.37 \times 10^{-9}\ \mathrm{rad\,s^{-1}}$; therefore the condition $\Psi \gg \omega_0$ holds.

It should be noted that Uranus at present performs a reversed rotation. It follows from the foregoing analysis that this planet performed a reversed rotation in the past as well. While the fast spinning of the planet continues, the rotational axis will be situated close to the orbital plane [9, 10].

5. The three-dimensional modification of the deformable planet–satellite problem in the field of an attractive centre

5.1 *Model problem: the motion of a natural satellite in the field of a viscoelastic planet*

Let us describe the natural motion of the mass centre m_2 of the planet and the mass centre m_1 of the satellite relative to the barycentre in the Delaunay variables L, Λ, H, ℓ, g and h; $\Lambda = |\Lambda|$ is the orbital angular momentum of the mass centres of the satelite and the planet, ℓ is the mean anomaly, h is the longitude of the rising node, g is the angular distance of pericentre and $\cos i = H\Lambda^{-1}$. The rotations of the planet are characterized by the angular momentum vector $\boldsymbol{G}$ and the vector $\boldsymbol{K}$, where $\boldsymbol{K} = \boldsymbol{G} + \boldsymbol{\Lambda}$ is the general angular momentum of the system; $\delta > 0$ is the angle between the vectors $\boldsymbol{G}$ and $\boldsymbol{K}$. The assumption regarding purely axial rotation means that the vector $\boldsymbol{G}$ is guided along the symmetry axis of the planet.

In the unperturbed motion (when tidal deformations are absent) the planet uniformly rotates around the symmetry axis with an angular velocity $\Psi = C^{-1}G$. The rotation axis of the planet and the orbit plane precess around the vector $\boldsymbol{K}$ which is motionless in inertia space; the angular velocities of the precessions are equal. It follows from analysis of the unperturbed motion of the system that when the vectors $\boldsymbol{G}$ and $\boldsymbol{\Lambda}$ occur with a nutational motion relative to the inertia axis, the following expressions are of the same order of magnitude:

$$(\cos\delta)' \approx \mu_1 R^{-3}(A-C)G^{-1}, \quad (\cos i)' \approx \mu_1 R^{-3}(A-C)\Lambda^{-1}, \tag{33}$$

where A and C are the equatorial and centroidal moments of intertia respectively of the planet; $\mu_1 = fm_1$, where f is the gravitational constant. The mass centres of the planet and the satellite rotate along unperturbed elliptic orbits around the barycentre. The pericentres precess in the plane of orbits with an angular velocity of the order of $g' \approx (\cos i)'$.

In the perturbed motion the forced deformations of the planet are considered and the regular features of the tidal evolution of the viscoelastic planet–satellite system are studied.

Let us now consider the fast axial rotation of the planet ($\Psi \gg n$). The averaged equations of orbital–rotational motion keeping the terms dependent Ψ have the forms [11]

$$\begin{aligned}
&G' = -k\Psi[\lambda_1\alpha^2(1+3z^2) + 4\lambda_2(3+2z^2+3z^4)]\Phi_2(e),\\
&\Lambda' = k\Psi z[\lambda_1\alpha^2 + 4\lambda_2(1-z^2)]\Phi_2(e), \quad L' = \frac{\Lambda'\Phi_1(e)}{\Phi_2(e)},\\
&x' = -k\Psi\alpha z(1-x^2)^{1/2}G^{-1}[3\lambda_1\alpha^2 + 4\lambda_2(1+3z^2)]\Phi_2(e),\\
&y' = k\Psi(1-y^2)^{1/2}\Lambda^{-1}[\lambda_1\alpha^2 + 4\lambda_2(3+z^2)]\Phi_2(e),\\
&z' = k\Psi\alpha^2\{\Lambda^{-1}[\lambda_1\alpha^2 + 4\lambda_2(3+z^2)] - zG^{-1}[3\lambda_1\alpha^2 + 4\lambda_2(1+3z^2)]\}\Phi_2(e),\\
&(e^2)' = 2(1-e^2)\Phi\Phi_2^{-1}L^{-1}\Lambda'.
\end{aligned} \tag{34}$$

We introduce the following notation here.

$$\begin{gathered}
x = \cos\delta, \quad y = \cos i, \quad z = \cos(\delta + i),\\
\alpha = \sin(\delta + i),\\
\Phi_1(e) = 1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6,\\
\Phi_2(e) = (1-e^2)^{1/2}\left(1 + 2e^2 - \frac{21}{8}e^4 - \frac{3}{8}e^6\right),\\
x, y, \Phi_1(e), \Phi_2(e) > 0, \quad \Phi = \Phi_1(e) - \Phi_2(e)(1-e^2)^{-1/2} > 0.
\end{gathered} \tag{35}$$

The coefficients $k\lambda_1$ and $k\lambda_2$ characterize the moments of tidal forces and must be extended and estimated subsequently on the basis of astrometrical and geophysical measurements. They may be determined approximately.

The obtained averaged equations permit us to study numerically the evolution of natural planetary subsystems of the Solar System which are very far from the Sun; this does not permit us to take into account the influence of the gravitational tides from the Sun on the evolutionary process.

Based on the example of the Neptune–Triton system let us carry out a qualitative analysis of the evolution in the framework of the two body problem. The dynamic characteristics of the system are such that we may assume that there is fast axial rotation of the planet Neptune and that the orbit of Triton is a circle as its eccentricity is small. The analysis of evolutionary equations of the system obtained from equation (34) for $e = 0$ permits us to make the following quality deductions: the magnitude of the angular momentum vector $\boldsymbol{G}$ of the planet decreases monotonically (the rotation of the planet slows down) for any inclinations of Triton's orbit with respect to the equatorial plane of the planet; the reversed orbit of Triton changes to a forward orbit in the course of time; the magnitude of the orbital momentum vector $\boldsymbol{\Lambda}$ of the system decreases monotonically, and the semimajor axis of the orbit decreases correspondingly (figure 4).

The global picture of tidal evolution of the planet–satellite system in the plane of the parameters n, $\beta = e^2$ is presented in figure 5. The phase trajectories separate into two sets (the direction of the motion of a representative point is shown by the arrows). In the case of motion along the trajectories of the first set, initially (when $n > n_2$) the satellite approaches the planet (the value of n increases and the eccentricity of the orbit decreases). It follows from the equation for phase trajectories that an intersection of phase trajectory with the curve $n_2(\beta)$ the value of the inclination of a line tangent to the trajectory equals zero. For $n_1 < n < n_2$, the phase trajectories turn so that, for $n \to n_1$, $|\mathrm{d}n/\mathrm{d}\beta| \to \infty$. In the end, when $n > n_1$, a monotonic increase in the eccentricity e is observed, and the satellite moves away from the planet (n decreases). There is a set of initial conditions, e.g. representative points belonging to a separatrix. The separatrix intersects the curve $n_2(\beta)$ at the point ($e = 0$, $n = \Psi$) which is only one stationary point of the system. All phase trajectories that are below the separatrix are the second set. In the case of motion along these trajectories the semimajor axis and

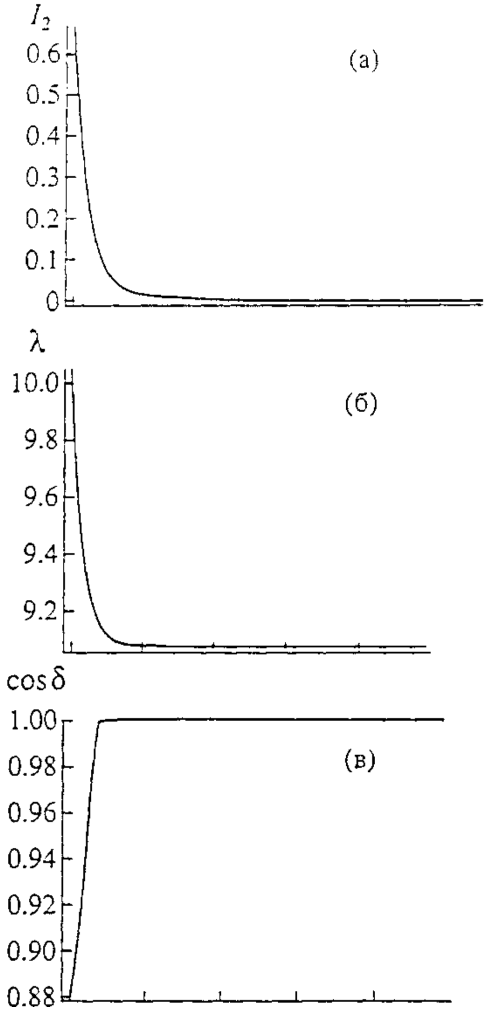

Figure 4. Plots of the relations of evolutionary variables with time in the Neptune–Triton system.

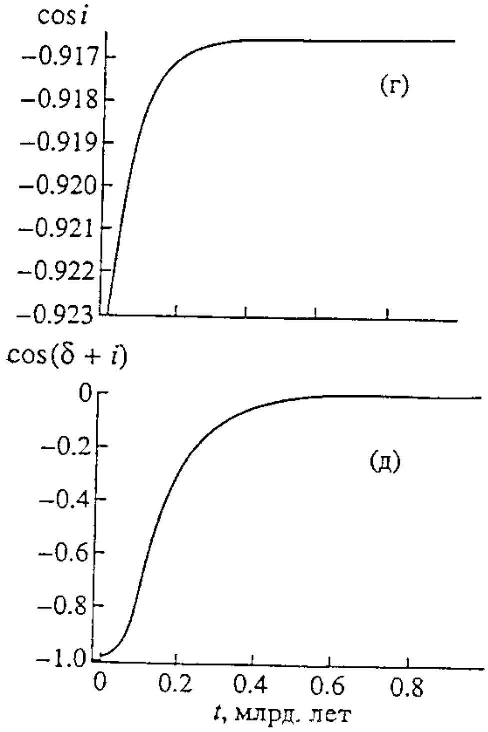

Figure 4. Continued.

the eccentricity of the orbit decrease monotonically; moreover, when $e \approx 1$, the velocity of evolution of n increases (the value $|\mathrm{d}n/\mathrm{d}\beta| \to \infty$ for $\beta \to 1$).

Let us note that the assumption that the angular velocity of the planet rotation is constant is physically well founded if the condition $\boldsymbol{G}(0) \gg \boldsymbol{\Lambda}(0)$ holds. However, for trajectories from the first set for a sufficiently small value of n (i.e. for a sufficiently distance between the satellite and the planet) the values $\boldsymbol{G}$ and $\boldsymbol{\Lambda}$ may be of the same order. In this case the results obtained in this formulation of the two-body problem are not acceptable and it is not correct in particular to make the deduction regarding a further monotonic decrease in n. In the phase portraits (figure 5) the representative points corresponding to Phoebus and Demos are indicated. At the present time the radius of Phoebus's orbit is decreasing monotonically, and Demos is moving away from Mars; moreover the eccentricity of its orbit continues to decrease. A further monotonic decrease in the mean motion of Demos will occur on increase in the eccentricity.

5.2 *Motion of the planet–satellite system in the field of an attractive centre*

The qualitative character of the evolutionary processes of the planet–satellite system in the field of an attractive centre is presented in figure 6. A phase portrait of the evolutionary processes follows from the differential equation

$$\frac{\mathrm{d}\Psi}{\mathrm{d}n} = \eta n(1 - \Psi\Psi_*^{-1})(n - \Psi)^{-1}, \tag{36}$$

with

$$\beta, \eta, n_* > 0, \quad \Psi_* = n(1+\beta)^{-1}, \quad \beta \approx \mathrm{O}(1),$$

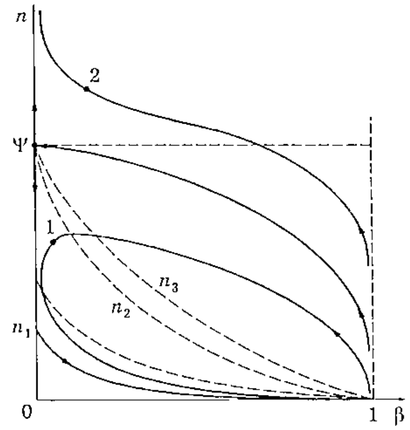

Figure 5. Phase portraits of the evolution of the motions of satellites in the field of a massive planet: 1, Demos; 2, Phoebus.

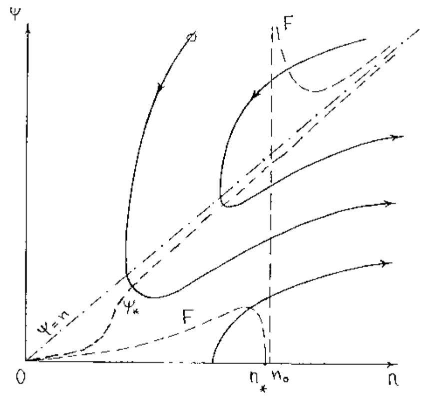

Figure 6. Phase portrait of the evolution of a planet-satellite system in the field of an attractive centre: ∘, the Earth–Moon system.

where n_* represents the mean motion of the barycentre along the orbit. As $\beta > 0$, then, according to equation (36), $\Psi_*(n) < \Psi = n$; moreover, for an increase in n, the function Ψ_* is asymptotic to the straight line $\Psi = n$. The inflection point of Ψ_* is the point $\beta = 0.6$. In the cases when $\Psi > n$ or $\Psi < \Psi_*$, $\mathrm{d}\Psi/\mathrm{d}n > 0$; for $\Psi_* < \Psi < n$, $\mathrm{d}\Psi/\mathrm{d}n < 0$. When $\Psi \to n$, then $|\mathrm{d}\Psi/\mathrm{d}n| \to \infty$; for $\Psi = \Psi_*$, $\mathrm{d}\Psi/\mathrm{d}n = 0$. Let us consider the function

$$F = n^5(y - y_*)N^{-1}. \tag{37}$$

The polynomial $N(y)$ has only one root y_0, namely $N(y_0) = 0$; moreover, $y_0 > y_* = n_*$. It may be shown that the straight lines $\Psi = n$ and $n = n_0$, where $n_0 = y_0$, are asymptotes for F. The stationary point of the function $F = F(y)$ for $y < y_*$ is in the interval $(15/19)y_* < y < y_*$. It follows from equation (37) that, in the cases when $\Psi > F$ (for $y < y_*$), $n < \Psi$ (for $y \leq y_0$) and $n < \Psi < F$ (for $y > y_0$), the value $\mathrm{d}\Psi/\mathrm{d}n > 1$ and the variable Ψ changes more quickly than n. For $\Psi > F(y, y_0)$, $\Psi < \Psi_*(y \geqslant y_*)$, the value $\mathrm{d}\Psi/\mathrm{d}n > 1$ and Ψ changes more slowly than n. It should be noted that for small values of n the plots of the functions F and Ψ_* are nearly the same. Therefore, if the condition $n \ll n_*$ holds, then during the tidal evolution process a representative point of the system arrives in the neighbourhood $\Psi_*(1 - \delta)$, where $0 < \delta = \mathrm{O}(nn_*^{-1}) \ll 1$ and will move along the curve $\Psi = \Psi_*$ while the condition $n \ll n_*$ holds.

Let us consider the Sun–Earth–Moon system and determine the place of the representative point of the Earth–Moon system in the phase portrait (figure 6). The important relation $\boldsymbol{\Lambda} \approx 4.94G$, where G is the value of rotational angular momentum of the Earth and $\boldsymbol{\Lambda}$ is the orbital momentum of the system, should be noted. Therefore the values of $\boldsymbol{G}$ and $\boldsymbol{\Lambda}$ may be considered to be of the same order of magnitude and it is necessary to consider the evolution of the variables Ψ and n jointly. As can be seen in the phase portrait at the present time the Moon is moving away from the Earth and the rotation of the Earth is decelerating. Here, the angular velocity Ψ of the Earth's rotation decreases more quickly than the mean motion n of the Moon: $\Psi' = 55.4n'$. In this connection the Sun's tidal exposure of the Moon to the Earth decreases relatively. The phase portraits that were obtained on the basis of the averaged equation of two model problems may have a basic role in studying the evolutionary processes in the Solar System under the action of gravitational tides.

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