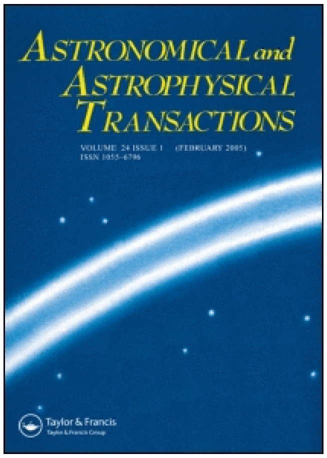


Astronomical & Astrophysical Transactions
The Journal of the Eurasian Astronomical Society
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713453505

# On a generalization of Kepler's third law

V. A. Vujičić [a]
[a] Mathematical Institute, Serbian Academy of Sciences, Beograd, Serbia

Online Publication Date: 01 December 2005
To cite this Article: Vujičić, V. A. (2005) 'On a generalization of Kepler's third law', Astronomical & Astrophysical Transactions, 24:6, 489 - 495
To link to this article: DOI: 10.1080/10556790500528008
URL: http://dx.doi.org/10.1080/10556790500528008

*Astronomical and Astrophysical Transactions*
Vol. 24, No. 6, December 2005, 489–495



# On a generalization of Kepler's third law

V. A. VUJIČIĆ*

Mathematical Institute, Serbian Academy of Sciences, Knez Mihailova 35,
PO Box 367, 11001 Beograd, Serbia

*(Received 5 July 2005)*

In the educational and scientific literature, Kepler's third law is seen as only approximately correct. Therefore, a so-called 'generalized Kepler's third law' has been introduced as correct. Only when the planet masses are considered is the standard Kepler's third law obtained. In this paper, we prove that 'the generalized Kepler's third law' is neither physically nor mathematically based on the axioms of mechanics; thus it is in fact not correct.

*Keywords*: Kepler's law; Constant of gravitation; Force of attraction

## 1. Introduction

Johannes Kepler first published his third law on 15 May 1618 (see p. 55 of [1] and p. 526 of [2]): '... the period of time of only two planets rotating around the Sun is equal one-and-a-half-ton time proportion of their average distances from the Sun'.

That definition completely corresponds to the modern formulation of Kepler's third law [3]: 'The square of the period of revolution of a planet about the Sun is proportional to the cube of the mean distance of the planet from the Sun.' 'The squares of the periodic times are proportional to the cubes of the major axes.' According to Sommerfeld (pp. 43–44 of [4]): 'Kepler greeted this' statement of the cove of this law with the enthusiastic statement: finally I have brought to light and verified beyond all my hopes and expectations that the whole Nature of Harmonies permeates to the fullest extent, and in all its details, the motion of the heavenly bodies; not, it is true, in the manner in which I had earlier thought, but in a totally different, altogether complete way.'

About 50 years later, Isaac Newton (p. 504 of [5]) wrote: 'PHENOMENON IV. The Star periods of the revolution of the five main planets, and also of the Sun around the Earth, the Earth around the Sun, stay in a half-cubic relation to their average distances from the Sun'. 'This relation, which was found, is recognized by all. All astronomers agree about the duration of the revolution; the sizes of the orbits, however, were determined carefully from the observations of Kepler and Bullio.'

*Email: vvujicic@mi.sanu.ac.yu

*Astronomical and Astrophysical Transactions*
ISSN 1055-6796 print/ISSN 1476-3540 online 
http://www.tandf.co.uk/journals
DOI: 10.1080/10556790500528008

According to Goldstein (p. 80 of [6]): 'Actually, Kepler was concerned with specific problem of planetary motion in the gravitational field of the Sun. A more precise statement of his law would therefore be: the squares of the periods of the various planets are proportional to the cube of their major axis'.

Duboshin wrote (p. 526 of [2]): 'In 1619, Kepler deduced the law that connects the whole Sun system in the following way: the squares of the rotation times of planets around the Sun stay in a cubic relation to their major axis.' The following definitions were given by Arnol'd (p. 40 of [7]): 'III Kepler's law: the rotation time along the elliptical orbit depends only on the magnitude of the major axis.' 'The squares of the rotation periods for various elliptical orbits stay in a cubic relation to their major axis.' Those definitions completely correspond to the modern formulation of Kepler's third law.

Based on the above quotations and on numerous other references in the literature (see, for example, [8] and p. 54 of [9]), we can conclude the following.

(i) Kepler's third law is referring to the motion of planets of the Solar System.
(ii) It establishes a relation between the time $T$ of revolution of planets around the Sun and the large semiaxis of their elliptical orbits:

$$\frac{T_1^2}{a_1^3} = \frac{T_2^2}{a_2^3} = \cdots = \frac{T_n^2}{a_n^3} = K. \tag{1}$$

(iii) The constant $K$ does not depend on the mass of planets nor on the mass of the sun.
(iv) Kepler published his third law (1) about 50 years before Newton's theorems on gravitation.

With respect to Kepler's laws, in the later development of classic and celestial mechanics, several researchers combined some of Newton's theorems on gravitation (book III, p. 510, theorems I, II and III, and p. 519, theorem VII, of [5]) into one law: The force of attraction of the Sun for a planet can be written as

$$F = f\frac{Mm}{\rho^2}, \tag{2}$$

where $M$ is the Sun's mass, $m$ is the planet's mass and $f$ is a coefficient, equal for all planets (see, for example, p. 341 of [10]).

On p. 410 of [11], Appell says: '**Le coefficient** $f$ **étant connu**, relation

$$f(M+m) = \frac{4\pi^2 a^3}{T^2}, \tag{PA}$$

à laquell nous sommes arrivés, donne une valeur approchée de $M + m''$. In Russian, on p. 351 of [10], Appell say: '**Ehsli koefficient** $f$ **izvesten**, to iz sootnosheniya (PA), poluchennogo Appelem, mozhno poluchit' priblizhnoe znachenie dlya $M + m$, *i.e.*, **if the coefficient** $f$ **is known**, the following is obtained:

$$\frac{a^3}{T^2} : \frac{a_1^3}{T_1^2} = \frac{1 + (m/M)}{1 + (m_1/M)}. \tag{3}$$

In a significant number of studies on classical and celestial mechanics, the above-mentioned conditions given by Appell, **'if the coefficient** $f$ **is known'**, has been overlooked; thus equation (3) or (PA) (see, for example, p. 80 of [6], equation 6. 14, p. 45 of [6] and p. 54 of [12]) is called the 'generalized (real, modified, improved, corrected, complete or correct) Kepler's third law in the form

$$\frac{a_i^3}{a_j^3} = \frac{T_i^2(M+m_i)}{T_j^2(M+m_j)}, \quad i \neq j. \tag{4}$$

What differs from Kepler's law is that, in equation (1), masses are not present as well as Kepler's constant $K$; $\dim K = L^3T^{-2}$. So, let us show how equation (4) is obtained and let us analyse its validity.

## 2. Deduction of the 'generalized Kepler's third law'

We observe the motion of two bodies of masses $m_1$ and $m_2$ and position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Newton's second and third laws can be written in the equation forms

$$m_1\ddot{\boldsymbol{r}}_1 = \boldsymbol{F}_1, \tag{5}$$

$$m_2\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}_2, \tag{6}$$

and the mutual attraction theorem in the form

$$F = f\frac{Mm}{\rho^2}, \tag{7}$$

where $\rho = |\boldsymbol{r}_2 - \boldsymbol{r}_1|$ is the distance between the centres of inertia of bodies and $f$ is a factor of proportionality, called the *universal constant of gravitation*.

Differential equations of motion of the observed bodies according to equations (5)–(7), can be written in the forms

$$\ddot{\boldsymbol{r}}_1 = m_1^{-1}\boldsymbol{F}_1, \tag{8}$$

$$\ddot{\boldsymbol{r}}_2 = m_2^{-1}\boldsymbol{F}_2, \tag{9}$$

or

$$\boldsymbol{r}_2 - \boldsymbol{r}_1 = \boldsymbol{\rho} \longrightarrow \ddot{\boldsymbol{r}}_2 - \ddot{\boldsymbol{r}}_1 = \ddot{\boldsymbol{\rho}}. \tag{10}$$

As

$$\ddot{\boldsymbol{\rho}} \times \boldsymbol{\rho} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{\rho}} \times \boldsymbol{\rho}) = \boldsymbol{0}$$

is a space integral

$$\dot{\boldsymbol{\rho}} \times \boldsymbol{\rho} = \boldsymbol{C}, \tag{11}$$

so that the task can be solved with regard to the plane polar coordinate system $\rho, \theta; \boldsymbol{g}_\rho, \boldsymbol{g}_\theta$.

With regard to that system, the coordinate is

$$\ddot{\boldsymbol{\rho}} = \frac{\mathrm{D}\dot{\rho}}{\mathrm{d}t}\boldsymbol{g}_{\rho} + \frac{\mathrm{D}\dot{\theta}}{\mathrm{d}t}\boldsymbol{g}_{\theta}, \tag{12}$$

where the corresponding coordinates of the acceleration vector are

$$\frac{\mathrm{D}\dot{\rho}}{\mathrm{d}t} = \ddot{\rho} - \rho\dot{\theta}^2, \quad \frac{\mathrm{D}\dot{\theta}}{\mathrm{d}t} = \ddot{\theta} + \frac{2\dot{\rho}\dot{\theta}}{\rho}. \tag{13}$$

From a scalar multiplication of equation (10) by the coordinate vectors $\boldsymbol{g}_{\rho}$ and $\boldsymbol{g}_{\theta}$, according to equation (13) it follows that

$$\ddot{\rho} - \rho\dot{\theta}^2 = \frac{f(M+m)}{\rho^2}, \tag{14}$$

$$\frac{1}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}t}(\rho^2\dot{\theta}) = 0, \tag{15}$$

and from here further that

$$\rho^2\dot{\theta} = C = \frac{2\pi ab}{T}. \tag{16}$$

According to Kepler's first law

$$\rho = \frac{p}{1 + e\cos\ \theta} \tag{17}$$

and Kepler's second law

$$\rho^2\dot{\theta} = C = \text{constant}, \tag{18}$$

where $p$ is a parameter and $e$ is the elliptical eccentricity, it is easy to calculate

$$\dot{\theta} = \frac{2\pi ab}{\rho^2}, \quad \ddot{\rho} = \frac{p-\rho}{p\rho^3}. \tag{19}$$

By substitution into equation (16), we obtain

$$\frac{4\pi^2 a^3}{T^2} = f(M+m). \tag{20}$$

This is Appell's equation (p. 351 of [10] and p. 410 of [11]) on the basis of which the 'generalized Kepler's third law' (2) is deduced. We should thus analyse in detail this relation and its consequences.

### 2.1 *First consequence*

If we estimate a priori that $f$ has just one and the same numerical value from the set of real numbers, *e.g.* 6.62, equation (20) can be written

$$6.62(M + m_i) = \frac{4\pi^2 a_i^3}{T_i^2}, \tag{21}$$

where the subscript $i$ refers to the $i$th planet. By dividing this equation by another equation for the $j$th planet, namely

$$6.62(M + m_j) = \frac{4\pi^2 a_j^3}{T_j^2},$$

the so-called 'generalized Kepler's third law' is obtained:

$$\frac{a_i^3}{a_j^3} = \frac{T_i^2(M + m_i)}{T_j^2(M + m_j)} \tag{22a}$$

or

$$\frac{T_i^2}{a_i^3} = \frac{T_j^2(M + m_j)}{a_j^3(M + m_i)}. \tag{22b}$$

However, it is not known by whom and when it was proved that the coefficient of proportion $f$ has only one numerical value. By carefully reading Newton's *Philosophiæ Naturalis Principia Mathematica* (book I, theorems IV, V, book II, theorems VII, XXII and book III, theorems I, II, III, etc., of [5]), it can be concluded that Newton's coefficients of proportion are not identical.

The various researchers that we quote, use different numerical values for the gravitation constant, *e.g.* the experiment: by Pierre Bouguer (1740), Henry Cavenish (1798), Eötvös (1896), Heyl (l930), Zachradnices (1932), Heyl and Chrzanowski (1934); Chertov (p. 268 of [12]): $f = (6.6720 \pm 0.0041) \times 10^{-11}$ [13]; $G = (6.673 \pm 0.003) \times 10^{-11}$. The list of astronomical constants in the preface to the *Astonomicheskij Ezhegodnik na 1999* (*Astronomical Yearbook for 1999*), (p. 650 of [14]) includes no constant called the universal gravitational constant. The constant $G$ is called the Cavendish gravitational constant. But the basic constant is the Gauss one $k = 0.017\,202\,098\,95$.

Thus, for the different numerical values, a 'generalized Kepler's third law' cannot be obtained.

### 2.2 *Second consequence*

In equation (20), different measurable constants of mass $m_1$ and $m_2$ are present, with a moderate distance $a_{12}$ between two material points and a time $T$ of rotation. The only unknown is $f$, which was determined early (equation 544, p. 536 of [15]) in the form

$$f = \frac{4\pi^2 a^3}{(M + m)T^2}. \tag{23}$$

As $m_1$, $a_1$ and $T_1$ are the measurable and determined constants for some bodies, then the values

$$f_i = \frac{4\pi^2 a_i^3}{(M + m_i)T_i^2} \tag{24}$$

are constant for the $i$th planet. If it is taken into consideration see, for example, (equation (17), p. 388 of [16] or equation (f), p. 415 of [17]) that for the Gauss constant there is only one value 0.017 202 098 95 (p. 650 of [14]), then

$$\mu = \frac{4\pi^2 a_i^3}{T_i^2},$$

and it is clearer that the universal gravitational constant

$$f = \frac{\mu}{M+m} \tag{25}$$

takes different values for varions planets, regardless of how small their masses are. In order to understand such an algebraic conclusion, let us present a simple mathematical example. In the linear equation $y = kx$ in the plane $(x, y)$, the coefficient of direction $k$ is constant, but it can take all numerical values from the interval $(-\infty, +\infty)$. Also, the elastic forces for a small deformation, measurable with different balances, including the torsion balance, are the linear functions $F = -cx$, where $c$ is a constant restitution coefficient but still has different numerical values for different bodies $c_i \neq c_j$. It is clear that, in the form

$$F = f\frac{Mm}{\rho^2}, \tag{26}$$

the value $f$ is a constant, but it is not certain whether it has the same numerical value for all bodies in a cosmos, *e.g.* in the Sun's planetary system. In order to explain this even better, let us start from Newton's theorems.

## 3. Force of attraction of two bodies

In the previous section, the main analysis referred to whether $f$ in equation (2) or (7) is the same known number, for example, or whether it is a function, determined by equation (23). In order to avoid this problem of a unique gravitational constant $f$ let us avoid equation (7) as a previous condition in the motion of two bodies and let us find $f$ based on Newton's theorems (5) and (6) and Kepler's laws (17) and (18). With regard to any pole, we can write

$$\boldsymbol{r}_2 - \boldsymbol{r}_1 = \boldsymbol{\rho} \longrightarrow \ddot{\boldsymbol{r}}_2 - \ddot{\boldsymbol{r}}_1 = \ddot{\boldsymbol{\rho}}. \tag{27}$$

By substitution of $\ddot{\boldsymbol{r}}_1$ and $\ddot{\boldsymbol{r}}_2$ from equations (5) and (6) into equation (27), the form

$$\boldsymbol{F}_1 = -\boldsymbol{F}_2 = \frac{Mm}{M+m}\ddot{\boldsymbol{\rho}} \tag{28}$$

is obtained or

$$\mathcal{M}\ddot{\boldsymbol{\rho}} = \mathcal{M}\left(\frac{\mathrm{D}\dot{\rho}}{\mathrm{d}t}\boldsymbol{g}_\rho + \frac{\mathrm{D}\dot{\theta}}{\mathrm{d}t}\boldsymbol{g}_\theta\right) = F_\rho \boldsymbol{g}_\rho + F_\theta \boldsymbol{g}_\theta, \tag{29}$$

where $\mathcal{M} = Mm/(M+m)$ is the reduced mass. By substituting $\ddot{\rho}$ and $\dot{\theta}$ from equation (19) into equation (29), the formula

$$F = \frac{4\pi^2 a^3}{(M+m)T^2}\frac{Mm}{\rho^2} = f\frac{Mm}{\rho^2} \tag{30}$$

is obtained.

By comparing equation (30) with the equation (2) or (3), it is clearly confirmed that the coefficient $f$ has the form (23).

## 4. Conclusion

On the basis of the above, it is concluded that 'the generalized Kepler's third law' (4) is neither Kepler's, nor a law, as it is not based on physical measures, and it is not formulated on the complete mathematical analysis of equation (20).

Finally, the aim [18] of this paper is not to change a law of nature here but to revert to the original propositions of Kepler. Isaac Newton wrote (p. 504 of [5]): 'Rule IV. In experimental philosophy, we are to look upon a proposition collected by general induction form phenomena as accurately or very nearly true, not with standing any contrary hypotheses that may be imagined, till such time as other phenomena occur, by which they may either be made more accurate, or liable to exceptions.'

## References

[1] S.I. Seleshnikov, *Astronomiya i Kosmonautika* (Naukova Dumka, Kiev, 1967), p. 330.
[2] G.N. Duboshin, *Nebesnaya Mehanika-Osnovnye Zadachi i Metody* (Fizmatgiz, Moscow, 1963), p. 526.
[3] *Van Nostrand's Scientific Encylopedia*, 2nd edition (Van Nostrand, New York, 1938, 1947), p. 824.
[4] A. Sommerfeld, Mechanics, (Academic Press, New York, 1952) (translated from German by M. O. Stern; Von Arnold Sommerfeld, Mechanik (Becher-Erler, Laipzig, 1943) p. XII + 275).
[5] I. Newton, *Matematicheskie Nachala Natural'noj Filosofii*, (Nauka, Moscow, 1989), p. 689 (translated by Krylov A.N. from I. Newton, *Philosophiæ Naturalis Principia Mathematica*, 3rd edition (London, 1726).
[6] H. Goldstein, *Classical Mechanics*, (Addison-Wesley, Cambridge, Massachusetts, 1951), p. 399.
[7] V.I. Arnol'd, *Matematicheskie metody klassicheskoj mechaniki*, 3rd edition, pererabotannoe and dopolnennoe (Nauka, Moscow, 1989), p. 472.
[8] E. Danilović and M.O. Raspopović, *Fizika za I razred gimnazije* (Naučna knjiga, Beograd, 2004), p. 192.
[9] T. Andjelić and T.P. Andjelich, *Uvod u Astrodinamiku* (Mathematical Institute, Beograd, 1983), p. 158.
[10] P. Appel', *Teoreticheskaya Mechanika*, Vol. (Fizmatgiz, Moscow, 1960) (translated by I.G. Malkin from P. Appell, *Treté de Mecanique Rationnelle*, 5th edition (Gauthier-Wilars, Paris, 1909), p. 515).
[11] P. Appell, *Treté de Mécanique Rationnell*, Vol. 1, 3rd edition (Gauthier- Villars, Paris, 1909).
[12] A.G. Chertov, *Edinicy Fizicheskih Velichin* (Vysshaya Shkola, Moscow, 1977), p. 287.
[13] *Enciklopedija Leksikografskog Instituta FNRJ* (Kniga 3, Zagreb), p. 240.
[14] *Astronomicheskij Ezhegodnik na 1999* (Astronomical Yearbook for 1999) (Pulkovskaya Observatory, St Petersburg, 1999), p. 650.
[15] B. Shevarlić and Z. Brčić, *Opšta Astronomja* (Naučna kniga, Beograd, 1981), p. 635.
[16] N.N. Buhgol'c, *Osnovnoj Kurs Teoreticheskoj Mechaniki*, Part I, 6th edition pererabotannoe and dopolnennoe by S.M. Targom (Nauka, Moscow, 1965), p. 388.
[17] N.A. Kil'chevski, *Kurs Teoreticheskoj Mechaniki*, Vol. I (Nauka, Moscow, 1997), p. 415.
[18] V.A. Vujičić, Prikladnaya Mechanika, **40**(3) 136–144 (2004) (English translation) Int. Appl. Mech. **40**(3) 351–359 (2004).