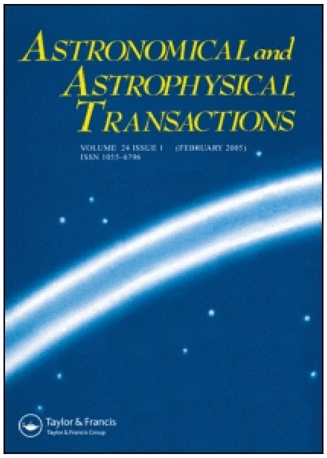

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Terrestrial tidal variations in the selenopotential coefficients

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The variations in the coefficients of the harmonics of second degree of the selenopotential caused by the terrestrial tides have been studied. In the paper we use analytical expressions for the tidal variations in the Stokes coefficients obtained for a model of the elastic celestial body with a concentric distribution of mass using the fundamental elastic parameter k_2 of the Moon. Taking into account the resonant properties of the Moon's motion, the variations in the selenopotential coefficients are presented in the form of Fourier series in the arguments of the theory of lunar orbital motion: l_M, l_S, F and D. The variations in the polar moment of inertia of the Moon due to the terrestrial tides lead to marked variations in the Moon's axial rotation, which also have been determined and tabulated. From the results obtained, it follows that the tide periodic variations in the gravitational coefficients of the Moon are an order larger than the corresponding tide variations in the geopotential coefficients.

Keywords: Celestial mechanics; Moon; Analytical methods

1. Tidal variations in the selenopotential coefficients

1.1 *Treatment of the problem*

The aim of this paper is to obtain and investigate analytical expressions of the temporal variations in the main selenopotential coefficients caused by the tidal terrestrial deformations in the form of a Fourier series in the angular variables of the libration theory of the Moon. For the construction of these formulae, we use an accurate model for the lunar orbital motion in accordance with the Cassini laws. Finally, we have tabulated the main terms of the periodic variations in the selenopotential coefficients. These variations have been obtained from the classical Takeuchi solution of the problem of elasticity for satellite deformations due to the gravitational action of a central planet [1, 2]. In the undeformed state of the satellite, the elastic parameters of the Moon are characterized by a concentric distribution of mass, and the variation in the density is linear with respect to the radial displacement and the volume divergence. The lunar core is small and we have not taken it into account. Variations in the coefficients of the

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harmonics of second degree of the selenopotential caused by the Earth have been determined following the work of Ferrándiz and Getino [2]. The effects of the elasticity in the Moon's deformations have been described with the help of the Love number $k_2 = 0.025$ [3, 4].

1.2 *Variations in the coefficients of the harmonic of second degree of the selenopotential*

We shall not present here the procedure of construction of the analytical formulae for the variation in the coefficients of the harmonics of second degree of the geopotential, which have been described in details by Ferrándiz and Getino [2]. These expressions applied to the Moon's tidal deformation are given by

$$\begin{aligned}
\delta J_2 &= -\frac{6D_\mathrm{t}}{mR^2}\left(\frac{a}{r}\right)^3 P_2(\sin\delta),\\
\delta C_{22} &= \frac{D_\mathrm{t}}{2mR^2}\left(\frac{a}{r}\right)^3 P_2^2(\sin\delta)\cos(2\alpha),\\
\delta S_{22} &= \frac{D_\mathrm{t}}{2mR^2}\left(\frac{a}{r}\right)^3 P_2^2(\sin\delta)\sin(2\alpha),\\
\delta C_{21} &= \frac{2D_\mathrm{t}}{mR^2}\left(\frac{a}{r}\right)^3 P_2^1(\sin\delta)\cos\alpha,\\
\delta S_{21} &= \frac{2D_\mathrm{t}}{mR^2}\left(\frac{a}{r}\right)^3 P_2^1(\sin\delta)\sin\alpha,
\end{aligned} \tag{1}$$

where δ and α are the latitude and longitude respectively of the centre $\mathrm{C_E}$ of mass of the Earth in the selenocentric reference system $\mathrm{C_M}\xi\eta\zeta$ with the origin at the Moon's centre $\mathrm{C_M}$ of mass and with the axes directed along the principal axes of inertia (considering an undeformed state of the Moon), r is the distance between the centres of mass of the Earth and the Moon, and a is the unperturbed value of the major semiaxis of the lunar orbit. The signs of the tesseral coefficients of the harmonics of second and third degree are opposite to the conventional usage by Getino and Ferrándiz [1].

In equation (1), the Legendre polynomials are determined by the following equations:

$$\begin{aligned}
P_2(\sin\delta) = \frac{3}{2}\sin^2\delta - \frac{1}{2},\quad P_2^2(\sin\delta) = 3(1-\sin^2\delta),\\
P_2^1(\sin\delta) = 3\sin\delta + (1-\sin^2\delta)^{1/2}.
\end{aligned} \tag{2}$$

Moreover, m and R are the mass and the mean radius, respectively of the Moon. The elastic parameter D_t is determined from the calculation of the following integral over the unperturbed radial distribution of densities [1]:

$$D_\mathrm{t} = \frac{\mathcal{G}M}{a^3}\frac{2\pi}{15}I, \tag{3}$$

$$I = \int_r \left(2\rho_0 r^4[5F_2(r) + r^2G_2(r)] - \frac{\mathrm{d}\rho_0}{\mathrm{d}r}r^5[2F_2(r) + r^2G_2(r)]\right)\mathrm{d}r, \tag{4}$$

where $\mathcal{G}$ is the gravitational constant, M and a are the mass of the perturbing body and the major semiaxis of its orbit respectively, $F_2(r)$ and $G_2(r)$ are the classical Takeuchi functions which are determined numerically as the solution of a known system of ordinary differential equations [5] and ρ_0 is the density of the concentric mass distribution. The parameter D_t can be obtained from different models of density distribution of the Moon [6]. The Earth's elastic parameter D_t has been calculated for a few models of the Earth: model 2 of Takeuchi and model 1066A [1].

The computation of D_t has been carried out in this paper through some relations between the rotational and tidal deformations of the Moon and considering a concrete value of the Love number k_2, to avoid the computation of the integral (4), which depends on the internal structure of the Moon. The Love number of the Moon is known with high accuracy from different dynamic studies of the Moon libration and its laser location [4]. The evaluation of the parameter D_t of the Moon is given below.

1.3 *Rotational variations in the selenopotential coefficients*

Let p, q and r be the projections of the angular velocity ω of the Moon on its principal axes (in absence of deformations) $C_M\xi\eta\zeta$. Owing to the non-inertial effects produced by rotation, the Moon is exposed to additional rotational deformations. Using the analytical expression of the vector of displacement for these deformations, the variations in the geopotential coefficients are determined in a similar manner to tidal variations. So, the final expressions for the variations in the moments of inertia of the Moon are expressed by the following formulae [1, 7]:

$$\begin{aligned}
\delta A_c &= -\frac{D_r}{\omega^2}(-\omega^2 + 3p^2),\\
\delta B_c &= -\frac{D_r}{\omega^2}(-\omega^2 + 3q^2),\\
\delta C_c &= -\frac{D_r}{\omega^2}(-\omega^2 + 3r^2),\\
\delta F_c &= 3\frac{D_r}{\omega^2}pq,\\
\delta E_c &= 3\frac{D_r}{\omega^2}pr,\\
\delta D_c &= 3\frac{D_r}{\omega^2}qr,
\end{aligned} \tag{5}$$

where ω is the angular velocity of the Moon; the elastic parameter D_r is determined from formulae similar to equations (3) and (4):

$$D_r = -\frac{4\pi}{15}\frac{\omega^2}{3}I, \tag{6}$$

where the integral I is determined by equation (4). In the general case, D_r is a function of time owing to the temporal dependence on the modulus of the angular velocity ω. In this paper, we consider a simple model of the Moon's rotation in accordance with the Cassini laws and with a constant angular velocity.

In accordance with the work of Getino and Ferrándiz [5], $D_r < 0$ is an elastic parameter having the dimensions of the moment of inertia and characterizing the satellite deformation due to its rotation. The increase in the polar moment of inertia follows from equation (5), from the assumption that $p = 0$, $q = 0$ and $r = \omega$, and with the help of the classical relations [8]

$$D_r = \delta A_r = \delta B_r = -\frac{\delta C_r}{2}, \quad D_r = -\frac{1}{9}k_2\frac{R^5\omega^2}{\mathcal{G}}, \tag{7}$$

where R is the equatorial radius of the Moon, ω is the mean angular velocity of the Moon rotation, k_2 is the Love number and $\mathcal{G}$ is the gravitational constant. The evaluation of the increment $\delta C_c/mR^2$ is given in table 1.

Table 1. Parameters and dynamic characteristics of the Moon.

Parameter	Value
k_2	0.025
fm (kg · m^3/s^2)	4902.801
R (km)	1737.5
$M/(M+m)$	0.987 849
T_{rot} (days)	27.2122
T_{orb} (days)	27.2122
$T_{N_0} = 2\pi/N_0$ (days)	0.075 219 6
D_{r}/mR^2	$-k_2\, 0.8490 \times 10^{-6}$
D_{r}/mR^2	-0.2122×10^{-7}
D_{t}/mR^2	0.3184×10^{-7}
$D_{\mathrm{t}}/D_{\mathrm{r}}$	$-3/2\, M/(M+m)$
$\delta C_{\mathrm{r}}/mR^2$	0.4245×10^{-7}

The temporal variations in the components of the tensor of inertia caused by the rotational deformation of the Moon can be computed from equations (5).

1.4 *Elastic parameters of the Moon k_2, D_{r} and D_{t}*

From equations (3), (4) and (6) we obtain the relation

$$D_{\mathrm{t}} = -\frac{3}{2}\left(\frac{n}{\omega}\right)^2 \frac{M}{m+M} D_{\mathrm{r}}, \tag{8}$$

where $n^2 = \mathcal{G}(M+m)/a^3$, n being the unperturbed mean orbital motion of the Moon. Taking into account equation (7), we obtain

$$D_{\mathrm{t}} = \frac{1}{6} k_2 \left(\frac{n}{\omega}\right)^2 \frac{M}{m+M} \frac{R^5\omega^2}{\mathcal{G}}. \tag{9}$$

Using equations (8) and (9) we can express two of the parameters k_2, D_{r} and D_{t} as functions of the third. In particular, the elastic parameter D_{r} is expressed through the elastic characteristic k_2 by the formulae

$$D_r = -\frac{1}{9} k_2 (m+M) \left(\frac{\omega}{n}\right)^2 \frac{R^5}{a^3} R^2. \tag{10}$$

In the case of the Moon's synchronous motion, we can put $n = \omega$ and equations (8) and (10) become

$$D_{\mathrm{r}} = -\frac{1}{9} k_2 (m+M) \frac{R^5}{a^3}, \quad D_{\mathrm{t}} = \frac{1}{6} k_2 M \frac{R^5}{a^3}. \tag{11}$$

The numerical values of the parameters (11) and some other characteristics of the Moon and its orbital motion are presented in table 1. In order to compute the amplitudes of the variations in the selenopotential coefficients (1), we first determine the models of the orbital and rotational motion of the Moon.

With the help of equation (11), the variation in the selenopotential coefficients can be arranged as follows:

$$
\begin{aligned}
\delta J_2 &= -k_2 \frac{M}{m}\left(\frac{R}{a}\right)^3 \left(\frac{a}{r}\right)^3 P_2(\sin\delta),\\
\delta C_{22} &= \frac{1}{12} k_2 \frac{M}{m}\left(\frac{R}{a}\right)^3 \left(\frac{a}{r}\right)^3 P_2^2(\sin\delta)\cos(2\alpha),\\
\delta S_{22} &= \frac{1}{12} k_2 \frac{M}{m}\left(\frac{R}{a}\right)^3 \left(\frac{a}{r}\right)^3 P_2^2(\sin\delta)\sin(2\alpha),\\
\delta C_{21} &= \frac{1}{3} k_2 \frac{M}{m}\left(\frac{R}{a}\right)^3 \left(\frac{a}{r}\right)^3 P_2^1(\sin\delta)\cos\alpha,\\
\delta S_{21} &= \frac{1}{3} k_2 \frac{M}{m}\left(\frac{R}{a}\right)^3 \left(\frac{a}{r}\right)^3 P_2^1(\sin\delta)\sin\alpha.
\end{aligned}
\tag{12}
$$

1.5 *Fourier series for the variation in the selenopotential coefficients*

The variation in the selenopotential coefficients are expressed by equation (1) through the spherical functions of the coordinates (r, δ and α) of the perturbing body [5, 9]. Now we present these variations as periodic functions of time by means of the developments of the spherical functions.

1.5.1 The Moon's rotation. For the description of the orientation of the Moon, we have adopted Andoyer variables referred to the selenocentric ecliptic of date reference system $C_M xyz$ with axis $C_M x$ directed to the mean equinox of date. We denote the Andoyer variables of the Moon by L, G, H, l, g and h. G is the modulus of the angular momentum vector $\boldsymbol{G}$ of the absolute rotation of the Moon about its centre of mass. Let θ be the angle between the angular momentum and the polar axis $C_M\zeta$, and ρ be the angle between the axis $C_M z$ and $\boldsymbol{G}$. Then, $L = G\cos\theta$ is the projection of $\boldsymbol{G}$ on the polar axis of inertia $C_M\zeta$ of the Moon, and $H = G\cos\rho$ is the projection of the same vector on the $C_M z$ axis (orthogonal to the ecliptic of date).

The Euler angles g, θ and l determine the orientation of the principal axes $C_M\xi\eta\zeta$ of inertia of the Moon in the intermediate reference system $C_M G_1 G_2 G_3$ connected with the angular momentum vector $\boldsymbol{G}$ (the axis $C_M G_3$ is directed along the vector $\boldsymbol{G}$, and the axis $C_M G_1$ along the node line of the intermediate plane $C_M G_1 G_2$ with respect to the coordinate plane $C_M xy$). ρ and h are the inclination and longitude, respectively of the ascending node of the intermediate plane $C_M G_1 G_2$ with respect to the reference plane $C_M xy$.

In this paper, we consider only the main particularities of the resonant libration of the Moon in accordance with the Cassini laws. The corresponding values of the Andoyer variables in this case are

$$
S = l + g = F, \quad h = \pi, \quad \theta_0 = 0, \quad \rho_0 = \rho_0(\beta, \gamma, A_\nu^{(j)}) = 1^\circ.5450, \quad \Omega = -\pi. \tag{13}
$$

1.5.2 Spherical functions of the Moon. The spherical harmonics of second degree which appear in equation (12) are given by a Fourier series in the five arguments of the theory of the lunar orbital motion and in the angular Andoyer variables l, g and h [1, 9]. Taking into account the smallness of the angle between the angular momentum $\boldsymbol{G}$ of the Moon and its polar axis of inertia *i.e.* $\theta \approx 0$, we obtain the following expressions for the spherical functions:

$$\begin{aligned}
\left(\frac{a}{r}\right)^3 P_2(\sin\delta) &= 3\sum_{\nu} B_{\nu}\cos\Theta_{\nu},\\
\left(\frac{a}{r}\right)^3 P_2^1(\sin\delta)\cos\alpha &= 3\sum_{\tau}\sum_{\nu} C_{\nu}(\tau)\sin(S-\tau\Theta_{\nu}),\\
\left(\frac{a}{r}\right)^3 P_2^1(\sin\delta)\sin\alpha &= 3\sum_{\tau}\sum_{\nu} C_{\nu}(\tau)\cos(S-\tau\Theta_{\nu}),\\
\left(\frac{a}{r}\right)^3 P_2^2(\sin\delta)\cos(2\alpha) &= -3\sum_{\tau}\sum_{\nu} D_{\nu}(\tau)\cos(2S-\tau\Theta_{\nu}),\\
\left(\frac{a}{r}\right)^3 P_2^2(\sin\delta)\sin(2\alpha) &= 3\sum_{\tau}\sum_{\nu} D_{\nu}(\tau)\sin(2S-\tau\Theta_{\nu}),
\end{aligned}\tag{14}$$

or

$$\begin{aligned}
\left(\frac{a}{r}\right)^3 P_2(\sin\delta) &\approx 3\sum_{\nu} B_{\nu}\cos\Theta_{\nu},\\
\left(\frac{a}{r}\right)^3 P_2^1(\sin\delta)\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix} &\approx 3\sum_{\tau}\sum_{\nu} C_{\nu}(\tau)\begin{bmatrix}\sin\\ \cos\end{bmatrix}(S-\tau\Theta_{\nu}),\\
\left(\frac{a}{r}\right)^3 P_2^2(\sin\delta)\begin{bmatrix}\cos(2\alpha)\\ \sin(2\alpha)\end{bmatrix} &\approx 3\sum_{\tau}\sum_{\nu} D_{\nu}(\tau)\begin{bmatrix}-\cos\\ \sin\end{bmatrix}(2S-\tau\Theta_{\nu}),
\end{aligned}\tag{15}$$

where

$$\begin{aligned}
B_{\nu} &= -\frac{1}{6}(3\cos^2\rho-1)A_{\nu}^{(0)} - \frac{1}{2}\sin(2\rho)\,A_{\nu}^{(1)} - \frac{1}{4}\sin^2\rho\,A_{\nu}^{(2)},\\
C_{\nu}(\tau) &= -\frac{1}{4}\sin(2\rho)\,A_{\nu}^{(0)} + \frac{1}{2}(1+\tau\cos\rho)(-1+2\tau\cos\rho)A_{\nu}^{(1)}\\
&\quad + \frac{\tau}{4}\sin\rho\,(1+\tau\cos\rho)A_{\nu}^{(2)},\\
D_{\nu}(\tau) &= -\frac{1}{2}\sin^2\rho\,A_{\nu}^{(0)} + \tau\sin\rho\,(1+\tau\cos\rho)A_{\nu}^{(1)} - \frac{1}{4}(1+\tau\cos\rho)^2 A_{\nu}^{(2)}.
\end{aligned}\tag{16}$$

These inclination functions (16) were introduced by Kinoshita [9] in his analytical theory of the rotation of the Earth.

In the case of the Moon's motion, $\rho = 1^{\circ}.5450$ [10] is the mean angle of inclination of the angular momentum $\boldsymbol{G}$ of the Moon with respect to the ecliptic of the epoch, and $S = l + g = F$ is the rotation angle of the Moon. $A_{\nu}^{(0)}$, $A_{\nu}^{(1)}$ and $A_{\nu}^{(2)}$ are the coefficients of the standard intermediate trigonometric developments of the spherical functions [9]

$$\begin{aligned}
\frac{1}{2}\left(\frac{a}{r}\right)^3(1-3\sin^2\beta) &= \sum_{\nu} A_{\nu}^{(0)}\cos\Theta_{\nu},\\
\left(\frac{a}{r}\right)^3 \sin\beta\cos\beta\sin\lambda_{\mathrm{m}} &= \sum_{\nu} A_{\nu}^{(1)}\cos\Theta_{\nu},\\
\left(\frac{a}{r}\right)^3 \cos^2\beta\cos(2\lambda_{\mathrm{m}}) &= \sum_{\nu} A_{\nu}^{(2)}\cos\Theta_{\nu}.
\end{aligned}\tag{17}$$

Table 2. Numerical values of the coefficients $A^{(0)}_{\nu,0}$.

Ω	l_M	l_S	F	D	Period	$A^{(0)}_{\nu,0}(\times 10^{-7})$
0	0	0	0	0	∞	−2 493 184.1204
0	1	0	0	0	27.555	−407 882.6985
0	1	0	0	−2	−31.812	−78 005.5495
0	0	0	0	2	14.765	−67 668.0154
0	2	0	0	0	13.777	−33 430.6143
0	0	0	2	0	13.606	−29 930.3150
0	1	0	0	2	9.614	−10 725.4322
0	1	0	2	0	9.108	−5 738.8253
0	0	1	0	−2	−15.387	−4 454.7752
0	2	0	0	−2	205.892	−3 617.2702
0	1	1	0	−2	−34.847	−3 334.4046
0	3	0	0	0	9.185	−2 685.5491
0	1	−1	0	0	29.803	−2 505.9637
0	1	0	−2	0	−26.878	2 384.4187
0	0	0	0	1	29.531	2 155.9159
0	1	1	0	0	25.622	2 090.2524
0	1	0	0	−4	−10.085	−2 080.1502
0	0	0	2	−2	173.310	1 971.5022
0	2	0	0	2	7.127	−1 280.7415
0	2	0	0	−4	−15.906	−1 269.7707

The coefficients $A^{(0)}_{\nu}$, $A^{(1)}_{\nu}$ and $A^{(2)}_{\nu}$ [9] are linear functions of time owing to the secular effects of the planetary motion:

$$A^{(\sigma)}_{\nu} = A^{(\sigma)}_{\nu,0} + A^{(\sigma)}_{\nu,1} t. \tag{18}$$

New computations of $A^{(\sigma)}_{\nu,0}$ and $A^{(\sigma)}_{\nu,1}$ have been performed, and they are presented in tables 2–7.

For the computation of the developments (17) we have taken into account the relation between the latitude and longitude of the Earth and Moon, $\beta = \beta_E$ and $\lambda_m = \lambda_{mE}$ in the present selenocentric ecliptic reference system $C_M xyz$ (with the origin at the Moon's centre C_M of mass). Moreover, the latitude δ_M and longitude α_M of the Moon in the geocentric

Table 3. Numerical values of the coefficients $A^{(0)}_{\nu,1}$.

Ω	l_M	l_S	F	D	Period	$A^{(0)}_{\nu,1}(\times 10^{-7})$
0	0	1	0	−2	−15.387	11.1849
0	1	1	0	−2	−34.847	8.3754
0	1	−1	0	0	29.803	6.3128
0	1	1	0	0	25.622	−5.2657
0	0	1	0	0	365.260	−1.9700
0	1	−1	0	2	9.874	1.9598
0	0	1	0	2	14.192	−1.8781
0	1	−1	0	−2	−29.263	−1.2841
0	0	2	0	−2	−16.064	1.0497
0	2	−1	0	0	14.317	1.0397
0	2	1	0	0	13.276	−0.8549
0	0	1	0	1	27.322	0.8211
0	1	2	0	−2	−38.522	0.5691
0	1	1	0	2	9.367	−0.4314
0	1	1	0	−4	−10.371	0.3612
0	0	0	0	2	14.765	−0.3025
0	2	1	0	−2	131.671	0.3010
0	2	−1	0	2	7.269	0.2535
0	0	1	0	−4	−7.535	0.2370
0	1	−2	0	−2	−27.093	−0.2316

Table 4. Numerical values of the coefficients $A^{(1)}_{\boldsymbol{\nu},0}$.

Ω	l_M	l_S	F	D	Period	$A^{(1)}_{\boldsymbol{\nu},0}(\times 10^{-7})$
1	0	0	0	0	−6798.384	1 352 490.7592
1	0	0	2	0	13.633	−1 337 751.1126
1	1	0	2	0	9.121	−256 073.9173
1	1	0	0	0	27.667	111 888.1472
1	−1	0	0	0	−27.443	110 564.5973
1	−1	0	2	2	9.543	−48 636.3908
1	0	0	2	2	7.088	−40 898.4227
1	0	0	2	−2	177.844	37 407.0473
1	−1	0	2	0	26.985	37 110.3000
1	2	0	2	0	6.852	−33 891.2400
1	−1	0	0	2	31.961	23 410.5344
1	1	0	0	−2	−31.664	21 153.0584
1	0	0	0	2	14.797	−20 176.1271
1	0	0	0	−2	−14.733	18 115.5431
1	1	0	2	−2	23.858	11 847.2828
1	1	0	2	2	5.638	−9 880.9445
1	−2	0	0	0	−13.749	9 055.3535
1	2	0	0	0	13.805	7 440.7290
1	1	0	0	2	9.627	−4 589.1155
1	3	0	2	0	5.488	−3 814.2433

reference system $C_E xyz$ (with origin at the Earth's center C_E of mass) are given by the following equalities: $\delta_E = -\delta_M$; $\alpha_E = \alpha_M + \pi$.

In equations (14)–(17), Θ_ν is a linear combination of the arguments of the theory of the lunar orbital motion with the integer coefficients

$$\Theta_{\boldsymbol{\nu}} = \nu_1 l_M + \nu_2 l_S + \nu_3 F + \nu_4 D + \nu_5 \Omega, \tag{19}$$

where l_M is the mean anomaly of the Moon, l_S is the mean anomaly of the Sun, $F = L_M - \Omega$, $D = L_M - L_S$, Ω is the mean longitude of the ascending node of the lunar orbit, L_M is the

Table 5. Numerical values of the coefficients $A^{(1)}_{\boldsymbol{\nu},1}$.

Ω	l_M	l_S	F	D	Period	$A^{(1)}_{\boldsymbol{\nu},1}(\times 10^{-7})$
1	0	1	2	0	13.143	−7.3227
1	0	−1	2	2	7.229	6.7924
1	0	−1	2	0	14.162	5.5961
1	−1	−1	2	2	9.799	5.4799
1	1	−1	2	0	9.354	5.1389
1	0	1	0	0	385.998	5.1298
1	1	1	2	0	8.898	−4.6104
1	0	1	2	−2	119.607	−3.8762
1	0	−1	0	0	−346.636	−3.7949
1	0	1	0	−2	−15.353	−2.9468
1	−1	−1	0	2	35.026	−2.6298
1	1	1	0	−2	−34.669	−2.2009
1	1	−1	0	0	29.934	−2.1884
1	0	−1	2	−2	346.604	1.8650
1	1	−1	2	2	5.726	1.8459
1	1	1	0	0	25.719	1.7281
1	0	1	0	2	14.221	−1.6632
1	1	1	2	−2	22.395	−1.3366
1	−1	1	0	0	−29.673	−1.3331
1	0	1	2	2	6.953	−1.3192

Table 6. Numerical values of the coefficients $A^{(2)}_{\boldsymbol{\nu},0}$.

Ω	l_M	l_S	F	D	Period	$A^{(2)}_{\boldsymbol{\nu},0}(\times 10^{-7})$
2	0	0	2	0	13.661	29 779 864.7328
2	1	0	2	0	9.133	5 701 663.2047
2	−1	0	2	2	9.557	1 082 953.6375
2	0	0	2	2	7.096	910 505.1331
2	−1	0	2	0	27.093	−841 838.9109
2	2	0	2	0	6.859	754 480.7914
2	1	0	2	2	5.643	219 988.9098
2	1	0	2	−2	23.942	−219 620.3081
2	0	0	0	0	−3399.192	121 869.6333
2	0	1	2	0	13.168	−102 418.8593
2	0	−1	2	0	14.192	90 148.3040
2	2	0	2	−2	12.811	−89 471.6492
2	3	0	2	0	5.492	84 883.3776
2	0	−1	2	2	7.236	61 429.5056
2	1	−1	2	0	9.367	53 206.6172
2	−1	−1	2	2	9.814	49 978.1440
2	1	1	2	0	8.910	−47 970.6980
2	−1	0	2		45.802	42 507.3616
2	2	0	2	2	4.684	36 415.2679
2	0	0	2	1	9.340	−30 754.6027

mean longitude of the Moon, L_S is the mean longitude of the Sun and $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ is a vector of integer indexes ($\nu_5 \geqslant 0$).

The numerical values of the coefficients $A^{(j)}_{\boldsymbol{\nu}}$ have been obtained in several papers [5, 9, 11] devoted to the Earth's rotation theory (tables 2–7). The construction of developments (14)–(19) has been carried out under the assumption that the angle θ between the angular momentum $\boldsymbol{G}$ of the Moon and its polar axis $C\zeta$ is small (and therefore $\sin\theta \approx 0$ and $\cos\theta \approx 1$). In equations (16), ρ is the angle between ecliptic plane and the intermediate plane, which is orthogonal to the vector $\boldsymbol{G}$ of the Moon ($\rho = 1^\circ.5450$).

Table 7. Numerical values of the coefficients $A^{(2)}_{\boldsymbol{\nu},1}$.

Ω	l_M	l_S	F	D	Period	$A^{(2)}_{\boldsymbol{\nu},1}(\times 10^{-7})$
2	0	1	2	0	13.168	257.7192
2	0	−1	2	0	14.192	−226.7827
2	0	−1	2	2	7.236	−154.2896
2	1	−1	2	0	9.367	−133.9696
2	−1	−1	2	2	9.814	−125.6000
2	1	1	2	0	8.910	120.8013
2	1	−1	2	2	5.731	−41.8401
2	0	1	2	2	6.961	32.2851
2	2	−1	2	0	6.991	−29.3533
2	−1	1	2	2	9.313	26.5882
2	1	1	2	−2	22.469	26.4398
2	2	1	2	0	6.733	25.6584
2	0	−2	2	2	7.383	−14.7594
2	0	1	2	1	9.107	−11.6934
2	1	1	2	2	5.557	10.4847
2	2	1	2	−2	12.377	9.0944
2	−1	−2	2	2	10.085	−8.9997
2	−1	−1	2	4	5.895	−8.9954
2	−1	1	2	0	25.222	8.5247
2	2	−1	2	2	4.744	−7.5575

Then, from equations (12) and (13), we obtain

$$\begin{aligned}
\delta J_2 &= -18\frac{D_{\mathrm{t}}}{mR^2}\sum_{\boldsymbol{\nu}} B_{\boldsymbol{\nu}}\cos\Theta_{\boldsymbol{\nu}},\\
\delta C_{22} &= -\frac{3}{2}\frac{D_{\mathrm{t}}}{mR^2}\sum_{\tau}\sum_{\boldsymbol{\nu}} D_{\boldsymbol{\nu}}(\tau)\cos(2S-\tau\Theta_{\boldsymbol{\nu}}),\\
\delta S_{22} &= \frac{3}{2}\frac{D_{\mathrm{t}}}{mR^2}\sum_{\tau}\sum_{\boldsymbol{\nu}} D_{\boldsymbol{\nu}}(\tau)\sin(2S-\tau\Theta_{\boldsymbol{\nu}}),\\
\delta C_{21} &= 6\frac{D_{\mathrm{t}}}{mR^2}\sum_{\tau}\sum_{\boldsymbol{\nu}} C_{\boldsymbol{\nu}}(\tau)\sin(S-\tau\Theta_{\boldsymbol{\nu}}),\\
\delta S_{21} &= 6\frac{D_{\mathrm{t}}}{mR^2}\sum_{\tau}\sum_{\boldsymbol{\nu}} C_{\boldsymbol{\nu}}(\tau)\cos(S-\tau\Theta_{\boldsymbol{\nu}})
\end{aligned}\tag{20}$$

can be reduced to the following form:

$$\begin{aligned}
\delta J_2 &= J_2^{(0,0,0,0)} + \sum_{|\boldsymbol{\nu}|\geqslant 1} J_2^{(\boldsymbol{\nu})}\cos\theta_{\boldsymbol{\nu}} + \dot{J}_2 t + t\sum_{|\boldsymbol{\nu}|\geqslant 1}\dot{J}_2^{(\boldsymbol{\nu})}\cos\theta_{\boldsymbol{\nu}},\\
\delta C_{22} &= C_{22}^{(0,0,0,0)} + \sum_{|\boldsymbol{\nu}|\geqslant 1} C_{22}^{(\boldsymbol{\nu})}\cos\theta_{\boldsymbol{\nu}} + \dot{C}_{22} t + t\sum_{|\boldsymbol{\nu}|\geqslant 1}\dot{C}_{22}^{(\boldsymbol{\nu})}\cos\theta_{\boldsymbol{\nu}},\\
\delta S_{22} &= S_{22}^{(0,0,0,0)} + \sum_{|\boldsymbol{\nu}|\geqslant 1} S_{22}^{(\boldsymbol{\nu})}\sin\theta_{\boldsymbol{\nu}} + \dot{S}_{22} t + t\sum_{|\boldsymbol{\nu}|\geqslant 1}\dot{S}_{22}^{(\boldsymbol{\nu})}\sin\theta_{\boldsymbol{\nu}},\\
\delta C_{21} &= C_{21}^{(0,0,0,0)} + \sum_{|\boldsymbol{\nu}|\geqslant 1} C_{21}^{(\boldsymbol{\nu})}\sin\theta_{\boldsymbol{\nu}} + \dot{C}_{21} t + t\sum_{|\boldsymbol{\nu}|\geqslant 1}\dot{C}_{21}^{(\boldsymbol{\nu})}\sin\theta_{\boldsymbol{\nu}},\\
\delta S_{21} &= S_{21}^{(0,0,0,0)} + \sum_{|\boldsymbol{\nu}|\geqslant 1} S_{21}^{(\boldsymbol{\nu})}\cos\theta_{\boldsymbol{\nu}} + \dot{S}_{21} t + t\sum_{|\boldsymbol{\nu}|\geqslant 1}\dot{S}_{21}^{(\boldsymbol{\nu})}\cos\theta_{\boldsymbol{\nu}},
\end{aligned}\tag{21}$$

where $\theta_{\boldsymbol{\nu}} = \nu_1 l_{\mathrm{M}} + \nu_2 l_{\mathrm{S}} + \nu_3 F + \nu_4 D$, $\nu_1 \neq \pm 1$.

In equations (21), $J_2^{(0,0,0,0)}$, $C_{22}^{(0,0,0,0)}$, $S_{22}^{(0,0,0,0)}$, $C_{21}^{(0,0,0,0)}$ and $S_{21}^{(0,0,0,0)}$ are the constant components of the variations in the selenopotential coefficients; $J_2^{(\boldsymbol{\nu})}$, $C_{22}^{(\boldsymbol{\nu})}$, $S_{22}^{(\boldsymbol{\nu})}$, $C_{21}^{(\boldsymbol{\nu})}$ and $S_{21}^{(\boldsymbol{\nu})}$ are the constant coefficients of the periodic variations in the corresponding selenopotential coefficients. The secular terms in equations (21) are characterized by the constant velocities $\dot{J}_2$, $\dot{C}_{22}$, $\dot{S}_{22}$, $\dot{C}_{21}$ and $\dot{S}_{21}$ of change and can be obtained from the following analytical formulae:

$$\begin{aligned}
\dot{J}_2 &= -18\frac{D_{\mathrm{t}}}{mR^2}\sum_{\nu_5}\dot{B}_{0,0,0,0,\nu_5}\cos(\nu_5\pi),\\
\dot{C}_{22} &= -\frac{3}{2}\frac{D_{\mathrm{t}}}{mR^2}\sum_{\nu_5}\sum_{\tau\nu_3=2}\dot{D}_{0,0,\nu_3,0,\nu_5}(\tau)\cos[2S-\tau(\nu_3 F-\nu_5\pi)],\\
\dot{S}_{22} &= \frac{3}{2}\frac{D_{\mathrm{t}}}{mR^2}\sum_{\nu_5}\sum_{\tau\nu_3=2}\dot{D}_{0,0,\nu_3,0,\nu_5}(\tau)\sin[2S-\tau(\nu_3 F-\nu_5\pi)],\\
\dot{C}_{21} &= 6\frac{D_{\mathrm{t}}}{mR^2}\sum_{\nu_5}\sum_{\tau\nu_3=2}\dot{C}_{0,0,\nu_3,0,\nu_5}(\tau)\sin[S-\tau(\nu_3 F-\nu_5\pi)],\\
\dot{S}_{21} &= 6\frac{D_{\mathrm{t}}}{mR^2}\sum_{\nu_5}\sum_{\tau\nu_3=2}\dot{C}_{0,0,\nu_3,0,\nu_5}(\tau)\cos[S-\tau(\nu_3 F-\nu_5\pi)],
\end{aligned}\tag{22}$$

where

$$
\begin{aligned}
\dot{B}_{\nu} &= -\frac{1}{6}(3\cos^2\rho - 1)\, A^{(0)}_{\nu,1} - \frac{1}{2}\sin(2\rho) A^{(1)}_{\nu,1} - \frac{1}{4}\sin^2\rho\, A^{(2)}_{\nu,1},\\
\dot{C}_{\nu}(\tau) &= -\frac{1}{4}\sin(2\rho) A^{(0)}_{\nu,1} + \frac{1}{2}(1+\tau\cos\rho)(-1+2\tau\cos\rho) A^{(1)}_{\nu,1}\\
&\quad + \frac{\tau}{4}\sin\rho(1+\tau\cos\rho)\, A^{(2)}_{\nu,1},\\
\dot{D}_{\nu}(\tau) &= -\frac{1}{2}\sin^2\rho\, A^{(0)}_{\nu,1} + \tau\sin\rho(1+\tau\cos\rho) A^{(1)}_{\nu,1} - \frac{1}{4}(1+\tau\cos\rho)^2 A^{(2)}_{\nu,1}.
\end{aligned}
\tag{23}
$$

Finally, $\dot{J}_2^{(\nu)}$, $\dot{C}_{22}^{(\nu)}$, $\dot{S}_{22}^{(\nu)}$, $\dot{C}_{21}^{(\nu)}$ and $\dot{S}_{21}^{(\nu)}$ in equation (21) are the constant coefficients of the mixed terms in the variations in the selenopotential coefficients.

From the analysis of the numerical values of coefficients $A^{(0)}_{\nu,1}$, $A^{(1)}_{\nu,1}$ and $A^{(2)}_{\nu,1}$ in tables 2–7, it follows that the secular components of all the selenopotential coefficients of the second-degree harmonics are equal to zero.

In tables 8–17, the values of the main coefficients $J_2^{(\nu)}$, $C_{22}^{(\nu)}$ and $C_{21}^{(\nu)}$ (1 unit $= 10^{-8}$) are listed as well as their respective arguments θ_ν and their corresponding periods $T_\nu = 2\pi/\dot{\theta}_\nu$

Table 8. Tidal periodic variations in the zonal coefficients J_2 and angular velocity of the Moon's rotation.

N	l_M	l_S	F	D	Period	$J_2^{(\nu)}$ ($\times10^{-8}$)	ω ($\times10^{-8}$)
1	0	0	0	0	0.000	9.4460	−16.0646
2	1	0	0	0	27.555	1.5453	−2.6281
3	1	0	0	−2	31.812	0.2954	−0.5024
4	0	0	0	2	14.765	0.2584	−0.4395
5	0	0	2	0	13.606	0.1933	−0.3287
6	2	0	0	0	13.777	0.1267	−0.2155
7	1	0	0	2	9.614	0.0410	−0.0697
8	1	0	2	0	9.108	0.0370	−0.0629
9	0	1	0	−2	15.387	0.0170	−0.0289
10	2	0	0	−2	205.892	0.0137	−0.0233
11	1	1	0	−2	34.847	0.0126	−0.0214
12	1	0	−2	0	26.878	−0.0113	0.0192
13	3	0	0	0	9.185	0.0102	−0.0173
14	1	−1	0	0	29.803	0.0095	−0.0162
15	0	0	2	−2	173.310	−0.0094	0.0160
16	0	0	0	1	29.531	−0.0082	0.0139
17	1	0	0	−4	10.085	0.0080	−0.0136
18	1	1	0	0	25.622	−0.0079	0.0134
19	1	0	−2	−2	9.530	0.0069	−0.0117
20	0	0	2	2	7.081	0.0059	−0.0100
21	2	0	0	2	7.127	0.0049	−0.0083
22	2	0	2	0	6.846	0.0049	−0.0083
23	2	0	0	−4	15.906	0.0048	−0.0082
24	0	0	0	4	7.383	0.0040	−0.0068
25	1	−1	0	2	9.874	0.0030	−0.0051
26	0	1	0	0	365.260	−0.0030	0.0051
27	0	1	0	2	14.192	−0.0029	0.0049
28	1	0	2	−2	23.775	−0.0026	0.0044
29	1	−1	0	−2	29.263	−0.0019	0.0032
30	2	−1	0	0	14.317	0.0016	−0.0027
31	1	0	2	2	5.633	0.0014	−0.0024
32	1	0	0	1	14.254	−0.0014	0.0024
33	2	1	0	0	13.276	−0.0013	0.0022
34	0	1	0	1	27.322	0.0012	−0.0020
35	1	0	−2	2	32.764	−0.0012	0.0020

Table 9. Periodic variations in C_{21}.

N	l_M	l_S	F	D	Period	$C_{21}^{(\nu)}$ $(\times 10^{-8})$
1	0	0	1	0	27.212	−2.2209
2	1	0	1	0	13.691	−0.3028
3	1	0	−1	0	2190.350	0.0613
4	1	0	−1	−2	14.666	0.0577
5	0	0	1	2	9.572	−0.0488
6	2	0	1	0	9.146	−0.0355
7	0	0	1	−2	32.281	0.0329
8	1	0	1	2	7.104	−0.0106
9	1	0	1	−2	188.201	−0.0095
10	2	0	−1	0	27.906	0.0052
11	1	0	−1	2	14.866	−0.0044
12	0	0	3	0	9.071	−0.0037
13	0	1	−1	−2	9.829	0.0032
14	3	0	1	0	6.867	−0.0032
15	0	1	1	0	25.325	0.0027
16	2	0	1	−2	24.036	0.0020
17	1	1	−1	−2	15.280	0.0018
18	1	−1	1	0	14.224	−0.0018
19	1	1	1	0	13.197	0.0016
20	1	0	−1	−4	7.358	0.0016
21	2	0	1	2	5.648	−0.0014
22	0	0	1	1	14.162	0.0011
23	2	0	−1	−4	10.038	0.0010
24	1	1	2	1	6.838	−0.0010
25	0	1	1	−2	35.410	0.0010
26	0	1	−1	0	29.403	0.0009

Table 10. Periodic variations in S_{21}.

N	l_M	l_S	F	D	Period	$S_{21}^{(\nu)}$ $(\times 10^{-8})$
1	1	0	−1	0	2190.350	−0.1227
2	1	0	1	0	13.691	0.1204
3	1	0	1	−2	188.201	−0.0283
4	1	0	−1	−2	14.666	0.0227
5	2	0	1	0	9.146	0.0205
6	0	0	1	2	9.572	0.0188
7	0	0	1	−2	32.281	−0.0141
8	0	0	1	0	27.212	−0.0105
9	2	0	−1	0	27.906	−0.0076
10	1	0	1	2	7.104	0.0058
11	2	0	1	−2	24.036	−0.0042
12	0	0	3	0	9.071	−0.0037
13	0	1	−1	0	29.403	0.0035
14	3	0	1	0	6.867	0.0032
15	0	1	1	0	25.325	−0.0027
16	1	−1	1	0	14.224	0.0018
17	1	1	−1	−2	15.280	0.0018
18	1	1	1	0	13.197	−0.0016
19	1	0	−1	−4	7.358	0.0016
20	2	0	1	2	5.648	0.0014
21	0	1	−1	−2	9.829	0.0012
22	0	0	1	1	14.162	−0.0011
23	0	1	1	−2	35.410	−0.0010
24	1	1	2	1	6.838	−0.0010
25	2	0	−1	−4	10.038	0.0010

Table 11. Periodic variations in C_{22}.

N	l_M	l_S	F	D	Period	$C_{22}^{(\nu)}$ ($\times 10^{-8}$)
1	0	0	0	0	∞	4.7278
2	1	0	0	0	27.555	0.7715
3	0	0	0	2	14.765	0.1486
4	1	0	0	−2	31.812	0.1371
5	2	0	0	0	13.777	0.1198
6	1	0	0	2	9.614	0.0349
7	0	0	2	0	13.606	0.0319
8	2	0	0	−2	205.892	−0.0169
9	3	0	0	0	9.185	0.0135
10	0	1	0	−2	15.387	0.0098
11	1	−1	0	0	29.803	0.0084
12	1	1	0	0	25.622	−0.0076
13	1	0	0	−4	10.085	0.0067
14	1	1	0	−2	34.847	0.0062
15	2	0	0	2	7.127	0.0058
16	0	0	0	1	29.531	−0.0049
17	2	0	0	−4	15.906	0.0044
18	0	0	0	4	7.383	0.0036
19	3	0	0	−2	24.302	−0.0029
20	1	−1	0	2	9.874	0.0026
21	1	0	2	0	9.108	0.0026
22	0	0	2	−2	173.310	−0.0021
23	0	1	0	2	14.192	−0.0020
24	0	1	0	0	365.260	−0.0020
25	2	−1	0	0	14.317	0.0019
26	1	−1	0	−2	29.263	−0.0017
27	2	1	0	0	13.276	−0.0016
28	4	0	0	0	6.889	0.0014
29	1	0	0	1	14.254	−0.0013
30	2	2	4	2	3.419	0.0010
31	1	0	0	4	5.823	0.0010

(in days). Graphs of the functions of the temporal behaviour of the variations δJ_2, δC_{22} and δC_{21} for the period 2005–2010 are presented in figures 1–3 (time is represented in Julian days).

A preliminary study of the variations of the selenopotential coefficients was carried out earlier considering a plane resonant motion of the Moon on the unperturbed Keplerian orbit [12]. The results obtained here are in good agreement with the above mentioned paper (table 18).

2. Tidal variations in the angular velocity of the Moon

For the tidal deformations of the Moon, the classical relation $\delta A + \delta B + \delta C = 0$ between the variations in the axial moment of inertia is satisfied. The tidal mass transformations change the polar moment of inertia of the Moon. Considering the rotation of the Moon as an isolated body with a changeable structure, we obtain the following relation between the variations in the polar moment of inertia and the angular velocity of the Moon from the conservation of the angular momentum [1]:

$$\frac{\delta\omega}{\omega_0} = -\frac{\delta C}{C} = -\frac{2}{3I}\delta J_2, \tag{24}$$

Table 12. Periodic variations in S_{22}.

N	l_M	l_S	F	D	Period	$S_{22}^{(\nu)}$ ($\times 10^{-8}$)
1	0	0	0	0	∞	−4.7278
2	1	0	0	0	27.555	1.0389
3	1	0	0	−2	31.812	−0.2067
4	0	0	0	2	14.765	0.1404
5	2	0	0	0	13.777	0.1198
6	1	0	0	2	9.614	0.0349
7	0	0	2	0	13.606	−0.0319
8	0	1	0	0	365.260	−0.0306
9	3	0	0	0	9.185	0.0135
10	2	0	0	−2	205.892	−0.0115
11	0	1	0	−2	15.387	−0.0098
12	1	1	0	−2	34.847	−0.0096
13	1	−1	0	0	29.803	0.0084
14	1	1	0	0	25.622	−0.0076
15	1	0	0	−4	10.085	−0.0067
16	1	0	−2	0	26.878	0.0066
17	2	0	0	2	7.127	0.0058
18	0	0	0	1	29.531	−0.0049
19	2	0	0	−4	15.906	−0.0044
20	0	0	0	4	7.383	0.0036
21	3	0	0	−2	24.302	−0.0029
22	1	−1	0	2	9.874	0.0026
23	1	0	2	0	9.108	−0.0026
24	0	0	2	−2	173.310	−0.0021
25	0	1	0	2	14.192	−0.0020
26	2	−1	0	0	14.317	0.0019
27	1	−1	0	−2	29.263	0.0017
28	2	1	0	0	13.276	−0.0016
29	4	0	0	0	6.889	0.0014
30	1	0	0	1	14.254	−0.0013
31	2	2	4	2	3.419	−0.0010
32	1	0	0	4	5.823	0.0010

or

$$\frac{\delta\omega}{\omega_0} = -\frac{2}{3I}\sum_{\boldsymbol{\nu}} J_2^{(\boldsymbol{\nu})}\cos\theta_{\boldsymbol{\nu}},$$
$$= \sum_{\boldsymbol{\nu}} \omega^{(\boldsymbol{\nu})}\cos\theta_{\boldsymbol{\nu}},$$

where

$$\omega^{(\boldsymbol{\nu})} = -\frac{2}{3I} J_2^{(\boldsymbol{\nu})}. \tag{25}$$

In table 8, we show the variations in the angular velocity. The value of the dimensionless moment of inertia is $I = C/mR^2 = 0.392$. In figure 4, we give the graph of the temporal change in the Moon's angular velocity for the year 2005.

3. Conclusions

(1) Although the lunar Love number k_2 has a small value in comparason with the Earth's value, the tidal periodic variations in the selenopotential coefficients are an order larger than the corresponding variations in the geopotential coefficients [2].

Table 13. Tidal mixed periodic variations in the zonal coefficients J_2 and the angular velocity of the Moon's rotation.

N	l_M	l_S	F	D	Period	$\dot{J}_2^{(\nu)}$ ($\times 10^{-8}$)	ω ($\times 10^{-12}$)
1	0	1	0	−2	15.387	−0.4260	0.7245
2	1	1	0	−2	34.847	−0.3172	0.5395
3	1	−1	0	0	29.803	−0.2391	0.4066
4	1	1	0	0	25.622	0.1995	−0.3393
5	1	−1	0	2	9.874	−0.0747	0.1270
6	0	1	0	0	365.260	0.0745	−0.1267
7	0	1	0	2	14.192	0.0723	−0.1230
8	1	−1	0	−2	29.263	0.0486	−0.0827
9	0	2	0	−2	16.064	−0.0399	0.0679
10	2	−1	0	0	14.317	−0.0394	0.0670
11	2	1	0	0	13.276	0.0324	−0.0551
12	0	1	0	1	27.322	−0.0311	0.0529
13	1	2	0	−2	38.522	−0.0216	0.0367
14	1	1	0	2	9.367	0.0166	−0.0282
15	1	1	0	−4	10.371	−0.0138	0.0235
16	0	0	0	2	14.765	0.0116	−0.0197
17	2	1	0	−2	131.671	−0.0114	0.0194
18	0	1	2	−2	117.539	0.0104	−0.0177
19	0	1	−2	−2	7.221	−0.0097	0.0165
20	2	−1	0	2	7.269	−0.0097	0.0165
21	0	1	0	−4	7.535	−0.0091	0.0155
22	1	−2	0	−2	27.093	0.0088	−0.0150
23	1	−2	0	0	32.451	−0.0079	0.0134
24	1	1	−2	−2	9.785	−0.0077	0.0131
25	1	−2	0	2	10.148	−0.0075	0.0128
26	0	1	2	0	13.117	0.0070	−0.0119
27	1	−1	2	0	9.341	−0.0068	0.0116
28	2	1	0	−4	16.630	−0.0061	0.0104
29	1	1	2	0	8.887	0.0060	−0.0102
30	1	0	0	−2	31.812	0.0053	−0.0090
31	1	1	0	1	13.719	−0.0052	0.0088
32	0	1	−2	0	14.133	−0.0048	0.0082
33	3	−1	0	0	9.422	−0.0047	0.0080
34	0	1	−2	2	329.791	−0.0044	0.0075
35	1	2	0	0	23.942	0.0043	−0.0073
36	3	1	0	0	8.960	0.0039	−0.0066
37	0	2	0	0	182.630	0.0031	−0.0053
38	1	1	2	−2	22.322	0.0028	−0.0048
39	1	0	0	2	9.614	0.0027	−0.0046
40	1	−1	2	2	5.722	−0.0026	0.0044
41	2	1	0	2	6.991	0.0025	−0.0043
42	0	3	0	−2	16.803	−0.0024	0.0041
43	1	−1	0	−4	9.814	0.0022	−0.0037
44	1	−1	0	4	5.917	−0.0020	0.0034
45	1	2	0	−4	10.674	−0.0018	0.0031
46	0	1	2	2	6.946	0.0018	−0.0031
47	2	−1	2	0	6.976	−0.0017	0.0029
48	0	2	0	2	13.661	0.0017	−0.0029
49	1	−1	−2	2	35.992	0.0016	−0.0027
50	1	−1	0	−1	3232.862	−0.0016	0.0027
51	0	1	0	4	7.236	0.0016	−0.0027
52	2	−2	0	0	14.901	−0.0016	0.0027
53	2	1	2	0	6.720	0.0014	−0.0024
54	1	−1	−2	−2	9.288	0.0014	−0.0024
55	0	2	0	−4	7.694	−0.0014	0.0024
56	1	−1	−2	0	25.036	0.0013	−0.0022
57	1	3	0	−2	43.064	−0.0010	0.0017
58	1	1	−2	0	29.013	−0.0010	0.0017
59	1	0	0	0	27.555	0.0010	−0.0017

Table 14. Mixed variations in C_{21}.

N	l_M	l_S	F	D	Period	$\dot{C}_2^{(v)}$ $(\times 10^{-12})$
1	0	1	−1	−2	9.829	−0.0780
2	1	1	−1	−2	15.280	−0.0617
3	1	−1	1	0	14.224	0.0543
4	1	1	1	0	13.197	−0.0479
5	0	1	1	0	25.325	−0.0451
6	0	1	1	−2	35.410	−0.0292
7	0	1	−1	0	29.403	−0.0220
8	0	1	−1	2	29.660	−0.0219
9	1	−1	1	2	7.245	0.0191
10	0	1	1	2	9.327	−0.0145
11	2	−1	1	0	9.381	0.0121
12	1	−1	−1	0	313.055	−0.0118
13	1	−1	−1	−2	14.100	0.0110
14	1	1	−1	0	438.360	0.0104
15	2	1	1	0	8.923	−0.0103
16	1	1	1	−2	124.205	0.0081
17	0	2	−1	−2	10.101	−0.0073
18	0	1	1	1	13.633	0.0059
19	1	−1	−1	2	15.496	0.0046
20	1	2	−1	−2	15.947	−0.0043
21	1	1	−1	2	14.284	−0.0037
22	1	1	1	2	6.968	−0.0037
23	1	1	−1	−4	7.509	−0.0036
24	2	1	1	−2	22.552	−0.0032
25	2	−1	1	2	5.737	0.0028
26	0	1	−1	−4	5.901	−0.0022
27	2	−1	−1	0	30.214	−0.0021
28	1	−1	1	−2	388.247	−0.0019
29	2	1	−1	−4	10.322	−0.0019
30	0	0	1	2	9.572	−0.0017
31	0	1	1	−4	10.420	−0.0017
32	0	1	−1	1	6792.345	−0.0017
33	1	−2	1	2	7.391	0.0016
34	0	2	1	−2	39.212	−0.0016
35	2	1	−1	0	25.925	0.0015
36	1	−2	1	0	14.801	0.0015
37	0	2	1	0	23.683	−0.0015
38	1	1	1	−4	16.757	−0.0015
39	3	−1	1	0	6.999	0.0015
40	1	2	1	−2	92.687	0.0013
41	3	1	1	0	6.740	−0.0013
42	1	−2	−1	−2	13.576	0.0012
43	1	1	1	1	9.121	0.0012
44	0	2	−1	0	31.977	−0.0011
45	1	1	2	1	6.838	0.0010
46	0	0	1	−2	32.281	0.0010
47	1	0	−1	−2	14.666	0.0010

(2) The main tidal variations can be clearly assigned and are given by

$$\delta J_2 = 1.5453 \times 10^{-8} \cos l_M,$$

$$\delta C_{22} = 0.7715 \times 10^{-8} \cos l_M,$$

$$\delta S_{22} = 1.0389 \times 10^{-8} \sin l_M,$$

$$\delta C_{21} = -2.2209 \times 10^{-8} \cos F,$$

$$\delta S_{21} = 0.1204 \times 10^{-8} \sin(l_M + F) - 0.1227 \times 10^{-8} \sin(l_M - F).$$

Table 15. Mixed variations in S_{21}.

N	l_M	l_S	F	D	Period	$\dot{S}_{21}^{(\nu)}$ ($\times 10^{-12}$)
1	0	1	1	0	25.325	0.0923
2	0	1	−1	0	29.403	−0.0882
3	0	1	−1	−2	9.829	−0.0348
4	1	−1	1	0	14.224	−0.0341
5	1	1	1	0	13.197	0.0315
6	1	1	1	−2	124.205	0.0297
7	1	1	−1	−2	15.280	−0.0295
8	0	1	1	−2	35.410	0.0196
9	1	−1	−1	0	313.055	0.0192
10	1	1	−1	0	438.360	−0.0142
11	1	−1	1	2	7.245	−0.0115
12	2	−1	1	0	9.381	−0.0083
13	0	1	1	2	9.327	0.0079
14	2	1	1	0	8.923	0.0073
15	1	−1	−1	−2	14.100	0.0068
16	1	−1	1	−2	388.247	−0.0047
17	1	1	1	2	6.968	0.0037
18	1	1	−1	−4	7.509	−0.0036
19	0	2	−1	−2	10.101	−0.0033
20	2	1	1	−2	22.552	0.0032
21	2	−1	1	2	5.737	−0.0028
22	0	1	1	1	13.633	−0.0027
23	0	1	−1	−4	5.901	−0.0022
24	1	2	−1	−2	15.947	−0.0021
25	2	−1	−1	0	30.214	0.0021
26	2	1	−1	−4	10.322	−0.0019
27	0	1	−1	2	29.660	−0.0017
28	0	1	1	−4	10.420	−0.0017
29	0	1	−1	1	6792.345	0.0017
30	0	0	1	2	9.572	0.0017
31	0	2	1	−2	39.212	0.0016
32	1	−2	1	2	7.391	−0.0016
33	1	1	1	−4	16.757	−0.0015
34	0	2	1	0	23.683	0.0015
35	1	−2	1	0	14.801	−0.0015
36	2	1	−1	0	25.925	−0.0015
37	3	−1	1	0	6.999	−0.0015
38	1	1	−1	2	14.284	0.0013
39	1	2	1	−2	92.687	0.0013
40	3	1	1	0	6.740	0.0013
41	1	−2	−1	−2	13.576	0.0012
42	1	1	1	1	9.121	−0.0012
43	0	2	−1	0	31.977	−0.0011
44	0	0	1	−2	32.281	−0.0010
45	1	1	2	1	6.838	0.0010
46	1	0	−1	−2	14.666	0.0010
47	1	−1	−1	2	15.496	0.0002

So, the largest variations in the selenopotential coefficients δJ_2, δC_{22} and δS_{22} occur with a period of 27.555 days. On the coordinate plane (δC_{22}, δS_{22}), these variations are interpreted as an ellipse with eccentricity 0.6697. The coefficient δC_{21} presents the greatest value of the amplitude, while the coefficient δS_{21} is the smallest and presents oscillations with periods of 13.691 and 2190.350 days (5.9969 years).

(3) Owing to the terrestrial tides, the period of the Moon rotation shows the variation (with anomalistic period)

$$\delta T = T 2.6281 \times 10^{-8} \cos l_M = 61.7872\,\text{ms}.$$

Table 16. Mixed variations in C_{22}.

N	l_{M}	l_{S}	F	D	Period	$\dot{C}_{22}^{(v)}$ ($\times 10^{-12}$)
1	0	1	0	−2	15.387	−0.2494
2	1	−1	0	0	29.803	−0.1993
3	1	1	0	0	25.622	0.1879
4	1	1	0	−2	34.847	−0.1574
5	1	−1	0	2	9.874	−0.0664
6	0	1	0	2	14.192	0.0526
7	0	1	0	0	365.260	0.0491
8	2	−1	0	0	14.317	−0.0466
9	2	1	0	0	13.276	0.0407
10	1	−1	0	−2	29.263	0.0401
11	0	2	0	−2	16.064	−0.0234
12	2	1	0	−2	131.671	0.0174
13	1	1	0	2	9.367	0.0166
14	0	1	0	1	27.322	−0.0161
15	1	1	0	−4	10.371	−0.0143
16	2	−1	0	2	7.269	−0.0120
17	1	2	0	−2	38.522	−0.0110
18	0	1	0	−4	7.535	−0.0090
19	0	0	0	2	14.765	0.0080
20	2	1	0	−4	16.630	−0.0075
21	3	−1	0	0	9.422	−0.0073
22	1	−2	0	0	32.451	−0.0073
23	1	−2	0	2	10.148	−0.0069
24	3	1	0	0	8.960	0.0063
25	1	−2	0	−2	27.093	0.0053
26	1	1	0	1	13.719	−0.0051
27	1	0	0	−2	31.812	0.0046
28	2	1	0	2	6.991	0.0035
29	1	2	0	0	23.942	0.0033
30	0	0	0	0	23.942	0.0031
31	1	0	0	0	27.555	0.0029
32	1	0	0	2	9.614	0.0028
33	1	−1	0	−4	9.814	0.0028
34	1	−1	0	4	5.917	−0.0026
35	3	1	0	−2	22.786	0.0025
36	1	2	0	−4	10.674	−0.0024
37	2	−2	0	0	14.901	−0.0019
38	0	1	−2	0	14.133	0.0018
39	0	1	0	4	7.236	0.0018
40	3	−1	0	2	5.752	−0.0017
41	0	2	0	−4	7.694	−0.0017
42	0	3	0	−2	16.803	−0.0015
43	0	1	2	0	13.117	−0.0015
44	0	2	0	2	13.661	0.0014
45	2	−1	0	−4	15.242	0.0014
46	2	−2	0	2	7.417	−0.0013
47	2	−1	0	−2	471.891	0.0013
48	1	−1	0	−1	3232.862	0.0012
49	3	−1	0	−2	26.034	0.0011
50	2	0	0	0	13.777	0.0011
51	0	1	2	−2	117.539	0.0011
52	2	2	0	−4	17.423	−0.0010
53	2	2	0	−2	96.782	0.0010
54	0	1	−2	2	329.791	−0.0010
55	2	2	4	2	3.419	−0.0010
56	0	1	0	−1	32.128	0.0010

Table 17. Mixed variations in S_{22}.

N	l_M	l_S	F	D	Period	$S_{22}^{(\nu)}$ ($\times 10^{-12}$)
1	0	1	0	0	365.260	0.7699
2	1	1	0	−2	34.847	0.2414
3	0	1	0	−2	15.387	0.2406
4	1	−1	0	0	29.803	−0.2263
5	1	1	0	0	25.622	0.1959
6	1	−1	0	2	9.874	−0.0664
7	0	1	0	2	14.192	0.0500
8	2	−1	0	0	14.317	−0.0466
9	1	−1	0	−2	29.263	−0.0443
10	2	1	0	0	13.276	0.0407
11	0	2	0	−2	16.064	0.0234
12	0	1	0	1	27.322	−0.0211
13	1	2	0	−2	38.522	0.0176
14	0	2	0	0	182.630	0.0169
15	1	1	0	2	9.367	0.0166
16	1	1	0	−4	10.371	0.0143
17	2	−1	0	2	7.269	−0.0120
18	2	1	0	−2	131.671	0.0114
19	0	1	0	−4	7.535	0.0090
20	0	0	0	2	14.765	0.0080
21	2	1	0	−4	16.630	0.0075
22	3	−1	0	0	9.422	−0.0073
23	1	−2	0	0	32.451	−0.0073
24	1	−2	0	2	10.148	−0.0069
25	3	1	0	0	8.960	0.0063
26	1	−2	0	−2	27.093	−0.0053
27	1	1	0	1	13.719	−0.0051
28	1	0	0	−2	31.812	−0.0046
29	2	−1	0	−2	471.891	0.0043
30	2	1	0	2	6.991	0.0035
31	1	2	0	0	23.942	0.0033
32	0	0	0	0	23.942	−0.0031
33	0	1	2	−2	117.539	0.0031
34	1	0	0	0	27.555	0.0029
35	1	0	0	2	9.614	0.0028
36	1	−1	0	−4	9.814	−0.0028
37	1	−1	0	4	5.917	−0.0026
38	3	1	0	−2	22.786	0.0025
39	1	2	0	−4	10.674	0.0024
40	2	−2	0	0	14.901	−0.0019
41	0	1	−2	0	14.133	0.0018
42	0	1	0	4	7.236	0.0018
43	0	2	0	−4	7.694	0.0017
44	3	−1	0	2	5.752	−0.0017
45	0	1	2	0	13.117	0.0015
46	0	3	0	−2	16.803	0.0015
47	0	2	0	2	13.661	0.0014
48	2	−1	0	−4	15.242	−0.0014
49	2	−2	0	2	7.417	−0.0013
50	1	−1	0	−1	3232.862	0.0012
51	3	−1	0	−2	26.034	0.0011
52	2	0	0	0	13.777	0.0011
53	2	2	0	−4	17.423	0.0010
54	0	1	0	−1	32.128	−0.0010
55	2	2	4	2	3.419	0.0010
56	0	1	−2	2	329.791	0.0010
57	2	2	0	−2	96.782	0.0010

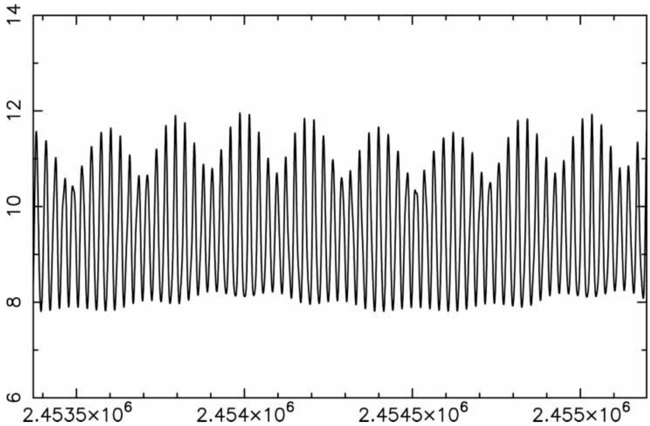

Figure 1. Tidal variations δJ_2 in the period 2005–2010.

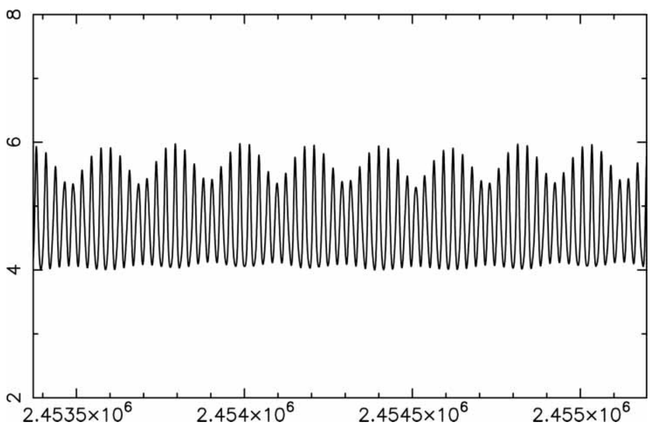

Figure 2. Tidal variations δC_{22} in the period 2005–2010.

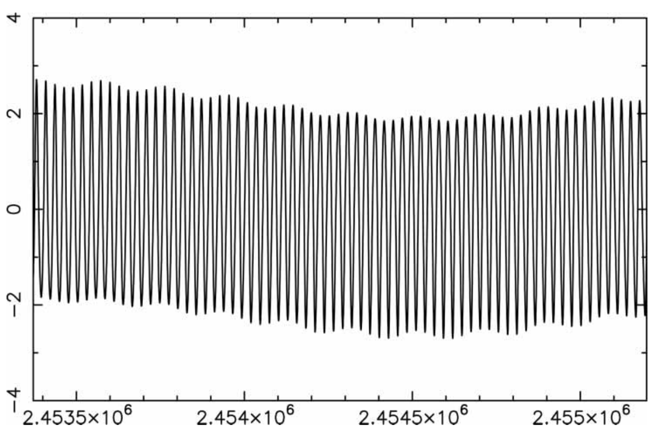

Figure 3. Tidal variations δC_{21} in the period 2005–2010.

Table 18. Comparison of the amplitudes of the variations in the selenopotential coefficients for the planar elliptical orbit and the perturbed orbit.

	Amplitude	
Coefficient	Planar orbit	Real orbit
$J_2^{(0,0,0,0)}$	9.7410	9.3764
$J_2^{(1,0,0,0)}$	1.5453	1.5338
$J_2^{(2,0,0,0)}$	0.1267	0.1258
$J_2^{(3,0,0,0)}$	0.0102	0.0102
$C_{22}^{(0,0,0,0)}$	4.6810	4.7278
$C_{22}^{(1,0,0,0)}$	0.7875	0.7715
$C_{22}^{(2,0,0,0)} = S_{22}^{(2,0,0,0)}$	0.1237	0.1198
$C_{22}^{(3,0,0,0)} = S_{22}^{(3,0,0,0)}$	0.0142	0.0135
$S_{22}^{(1,0,0,0)}$	1.0520	1.0389

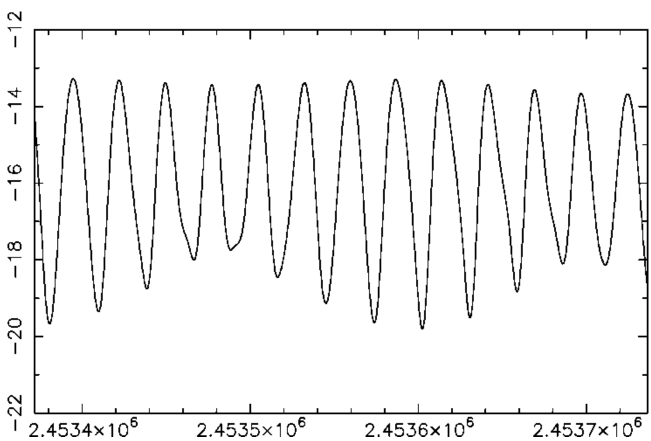

Figure 4. Tidal variations in the Moon's angular velocity in 2005.

The analysis of the geometry and kinematics of the temporal behaviour of the principal axes of inertia of the Moon due to tidal variations in its tensor of inertia has attracted great interest. A marked change in the orientations of these axes can be expected [13].

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