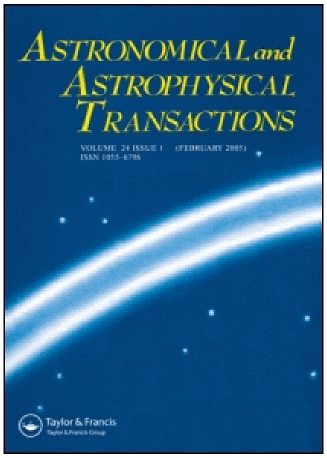

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Dynamic structure and rotation of Mercury

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Constant and periodic variations in the gravitational coefficients of the second harmonic of the gravitational potential of Mercury caused by the attraction of the Sun and its resonant rotation are studied on the basis of a planar model of translatory–rotary motion of Mercury as an elastic body on an elliptical orbit. The main variations in the coefficients C_{20}, C_{22} and S_{22} with periods that are a multiple of the orbit period have been obtained and evaluated for Mercury. Dynamic effects in the resonant rotation of Mercury considered as unchangeable non-spherical body (or as body with liquid core) are studied on the basis of two simple models: firstly, the plane motion on the unperturbed elliptical orbit; secondly, the rotation of Mercury on the precessing elliptical orbit. A few sets of possible values of Mercury's gravitational field parameters C_{20} and C_{22} are used in the paper for evaluations of the unperturbed, forced and resonant effects in Mercury's rotation (the Cassini positional parameters, amplitudes of periodic librations in longitude, and periods of resonant librations in the vicinity of Cassini–Colombo motion). The results are compared with similar characteristics of the resonant rotational motion of the Moon, which have been obtained in parallel on the basis of the same models of motion.

Keywords: Mercury's rotation; Mercury's gravitational field; Moon's rotation; Cassini–Colombo motion; Resonant librations

1. Introduction

For successful realization of the modern planned Mercury missions (Messenger and BepiColombo) a more in-depth understanding of the inner structure, rotational and inner dynamics and energization of Mercury is needed. Here the BepiColombo project has the leading role [1, 2]. New data are expected from this mission. More exact data about the gravitational field, figure and physical fields can be obtained from this mission and can make new dynamic studies possible. These will result in a new epoch of studying Mercury.

The main feature of Mercury's dynamics is the resonant character of its translational–rotational motion. Because we expect high-accuracy measurements in the framework of the space missions Messenger and BepiColombo, new studies of the main resonant features on the basis of new, more real models of Mercury are very important [3–6]. In future we plan to make a systematic study and to develop understanding of these fundamental features in

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Mercury's dynamics, taking into account real elastic and inelastic properties of these celestial bodies and their two-layer structure. This paper is a first step in that direction.

Colombo [7] was the first to consider the generalization of the Cassini laws to other objects in the Solar System such as Mercury. This work was developed by other workers: by Goldreich and Peale [8], by Peale [9, 10], by Beletskij [11], by Ward [12], in some of Barkin's [13–17] papers and by Barkin and Vestnik [18]. The four possible coplanar configurations were enumerated and designated as the Cassini states by Peale [9]. The regular motions of Cassini were obtained by different methods for the rigid model of a planet (Mercury, Moon and others). Beletskij [11] has effectively used average methods of perturbation theory for analysis of the resonant rotational motion of planets and satellites. Barkin [13–17] and Barkin and Vestnik [18] have developed the Hamiltonian formalism for models of translatory–rotary motion of a rigid body and in particular the applied Poincaré theory of periodic and conditionally periodic solutions for study of the resonant motions of Mercury, Moon and Venus. The evolution of rotation of celestial bodies to Cassini's positions was studied, taking into account the tidal deformations and friction of satellite [10–12] and others. The described studies need a modern development in accordance with present requirements produced to the realization of Mercury missions [1–3] and also ground radio-location studies of Mercury [19].

The studies of variations in Mercury's rotation in the vicinity of the above-mentioned resonant motion have been started by analytical and numerical methods for different models of Mercury (a rigid model, a deformable model, a two-layer model and a model with a liquid core). The first results of numerical modelling of the rotation of a rigid non-spherical Mercury model were obtained by Peale's group [3, 19] and Rambaux and Bois [20]. The influence of the liquid core on Mercury's resonant rotation was studied in a planar model [3] and in the general case of spatial motion [4–6]. The first studies of relative oscillations of Mercury's core and mantle were reported in [3, 21–23] on the basis of the methods and approaches developed for the Earth's (Moon's) rotation theory [4, 24]. These detailed studies let us establish a more precise relationships between the dynamic and inner structures of Mercury. A wide programme of studies of Mercury in connection with future space missions to this planet has been suggested in the report of the Moscow 40th Vernadskii–Brown Microsymposium on Planetology in Moscow [25] and in a report at the COSPAR Assembly in Paris [19]. The important results of the ground radio-location observations of Mercury were presented in the latter report. It was shown that the amplitude of the forced librations in longitude of Mercury can have a sufficiently large value of about 1. We have used these data for new evaluations of the coefficients of the second harmonic of Mercury's gravitational potential and to study the stationary resonant rotation of Mercury, its stability, resonant librations and others dynamic features.

In section 2 we study the possible variations in the gravitational field of Mercury due to its tidal solar deformations in conditions of orbital–rotational resonance. In the framework of the planar model of Mercury's motion we have shown that variations in the coefficients J_2 and C_{22} are periodic (with a period of orbital motion of 87.97 days) and are characterized by the considerable amplitude of the order of 5×10^{-8}, which is a few orders of magnitude larger than the corresponding tidal variations in the geopotential coefficients [26]. The tidal deformations of Mercury under resonant translatory–rotary motion give constant contributions to the coefficients J_2 and C_{22}. The ratio of these contributions was evaluated as $\langle \delta J_2 \rangle / \langle \delta C_{22} \rangle = 7.85$ which is sufficiently close to the ratio of coefficients, J_2/C_{22}, of 8 in the paper by Peale *et al.* [3]. One of the first attempts to evaluate this ratio was by Barkin [3] on the basis of a model problem about periodic translatory–rotary motion of the rigid satellite in a central gravitational field ($J_2/C_{22} = 35.3$).

The librations in longitude of Mercury have been studied on the basis of a planar model for the motion of a non-spherical celestial body with a liquid core in a central gravitational field. For the models of Mercury in this paper, the amplitudes of periodic librations were evaluated.

Owing to the influence of solar gravitational moments the angular velocity of Mercury varied by 0.03%. The phenomenon of non-perturbation of the rotation of Mercury in the vicinity of the pericentre of the orbit has been established. In a period of 15 days the angular velocity of Mercury has an almost permanent value.

The period of resonant librations in longitude of Mercury has been evaluated for a rigid model of the planet and for a two-layer model of the planet (with a rigid mantle and liquid core). The corresponding values for the periods are 16.0 years and 11.3 years. So, owing to the influence of the liquid core, the period of librations are reduced by 29.3%. The amplitude and phase of these librations can be determined only from observations. We assume that resonant librations are not damped and are perturbed by the mechanism of shell dynamics [21, 22]. This means that the resonant librations in longitude of Mercury can be determined from modern observations (including ground radio observations) in a few years.

The main regular features in Mercury's rotation were established and studied on the basis of its rigid model on the assumption that Mercury moves on the evaluated elliptical orbit. The plane of the orbit precesses with a permanent angular velocity, forming a constant angle with the main unmovable plane. The line of apses of the orbit also rotates with a permanent angular velocity. The theoretical value of inclination of the angular momentum of Mercury relative to the normal to the orbit plane was determined as $\rho_0 = 1'607$. This value is very close to another evaluation of this parameter $\rho_0 = 1'6$ [19]. The first evaluations of the inclination ρ_0 obtained earlier are $\rho_0 = 1'24$ and $\rho_0 = 1'67$ [27, 28]. The main regular features in the translatory–rotary motion of Mercury have been interpreted as the Cassini–Colombo laws [7, 11]. In these papers the detailed formulations of these laws and regular features were given. The definite stationary solution of the equations of rotational motion described in Andoyer variables was compared with Mercury's resonant motion. In this paper we are restricted by a consideration of Mercury as a non-spherical rigid-body model. The first steps to explain the resonant rotation of the Moon and Mercury as celestial bodies with a liquid core were made in previous papers by the present authors [4–6].

For comparison with all the resonant effects and phenomena of Mercury rotation discussed we analyse in parallel also similar effects of the Moon's rotational motion.

2. Gravitational parameters and their variations

2.1 *Models of Mercury, parameters of gravitational potential and inner structure*

The translatory–rotary motion of Mercury has an exotic resonant character of type 3:2. The first constructive studies of Mercury's resonant rotation were reported in [7–18, 28]. Using numerical methods, Rambaux and Bois [20] have studied some aspects of the planar and spherical motion of Mercury considering this planet as a rigid non-spherical body. In particular, they evaluated the inclination of angular momentum and its long-period variations, two to three periods of the resonant librations of Mercury [20]. The more real and complicated Mercury models have also been studied in the last few years. Different aspects of the dynamic influence of the liquid core of Mercury on its rotation have been studied by Peale *et al.* [3], Barkin and Ferrandiz [4–6], Ferrandiz and Barkin [23] and others. The rotational dynamics of Mercury as a system of two non-spherical rigid shells (core–mantle system) interacting with each other owing to a thin elastic layer and perturbed by external celestial bodies were studied by Peale *et al.* [3], Barkin [21, 22] and Ferrandiz and Barkin [23]. In our paper the different models of Mercury are considered with the purpose of carrying out dynamic studies of Mercury's resonant rotation and tidal variations in its dynamic structure.

Table 1. Evaluations of the main coefficients of Mercury's gravitational potential.

Reference	Method	$J_2 \times 10^6$	$C_{22} \times 10^6$
Esposito *et al.* [29]	Mariner 10 observations	80 ± 60	–
Barkin [13]	Theory	80	3.3
Barkin [18]	Theoretical values (model I)	77.34	9.86
Anderson *et al.* [30]	Mariner 10 observations (model II)	60 ± 20	10 ± 5
Margot *et al.* [19]	Theoretical values, radio-location of Mercury (model III)	101.6	12.94

For the first studies of Mercury's rotation in this paper we shall consider this planet as an ellipsoidal unchangeable body with some model values of the main parameters of its gravitational field J_2 and C_{22} and others dynamic characteristics (tables 1 and 2). As a first approximation, Mercury can be considered as a non-spherical rigid body with small dynamic oblatenesses. In table 1 are given known evaluations of the two main coefficients of Mercury's gravitational field, J_2 and C_{22}, obtained on the basis of observations of the Mariner 10 mission [29, 30] and by some theoretical studies [13, 18]. Planned missions to Mercury (BepiColombo and Messenger) promise to obtain new and accurate data about the dynamics and structure of this planet [1, 2, 25].

Evaluations of the dynamic parameters of Mercury give $C/mR^2 = 0.35$ and $C_m/C = 0.5 \pm 0.07$ [3, 31]. Here C and C_m are the moments of inertia of full Mercury and of its core, m is the mass of Mercury and R is the mean radius of Mercury. For the dimensionless moment of inertia we shall use $C/mR^2 = 0.35$.

Model I is based on the predicted value of oblateness $(C - B)/B$ [18] and on the resonant value of the ratio J_2/C_{22} for the deformed elastic Mercury in the Sun's gravitational field (section 1.1). Model II is based on the data from Mariner 10 observations. It is characterized by a significant error in the value of the coefficient C_{22} (about 50%). The modern model III is based on the evaluation of the amplitude of forced libration in longitude of Mercury obtained by Peale's group [19] in a study of Mercury's rotation with the help of high-accuracy radio-location observations (Earth-based measurements) of this planet. For all models of Mercury we use the same value of the dimensionless moment of inertia, $I = 0.34$, and Peale's [31] evaluation of the ratio of the moment of inertia of Mercury to the moment of inertia of its mantle, $C/C_m = 2$.

2.2 *Elastic models of Mercury and its satellites*

In a definite approximation, Mercury can be considered and modelled as an elastic body. Because of the gravitational attraction of the Sun the elastic Mercury moving in a elliptic orbit is deformed. In the considered case of resonant motion this leads to definite contributions

Table 2. Main dynamic characteristics of the models of Mercury.

Parameter	Model I	Model II	Model III
$(C - A)/B$	2.855×10^{-4}	2.353×10^{-4}	3.748×10^{-4}
$(C - B)/B$	1.695×10^{-4}	1.177×10^{-4}	2.226×10^{-4}
$(B - A)/B$	1.160×10^{-4}	1.177×10^{-4}	1.523×10^{-4}
δ	1.684	2.000	1.684
C_{22}	9.857×10^{-6}	10.0×10^{-6}	12.942×10^{-6}
J_2	77.344×10^{-6}	60.0×10^{-6}	101.553×10^{-6}
J_2/C_{22}	7.8469	6.00	7.8469

to the J_2 and C_{22} coefficients. These variations can be obtained on the basis of the well-known classical Takeuchi solution of the problem of elasticity for satellite deformations due to gravitational action of the central planet [26, 32]. In the undeformed state of a satellite a concentric mass distribution is assumed. In the case of resonant translatory–rotary motion of a satellite the constant components of variations in the coefficients J_2 and C_{22} can be described by the following formulae:

$$\langle J_2\rangle = -3\frac{D_\mathrm{r}}{mR^2}\left(1+\frac{6}{N^2}\frac{m^*}{m^*+m}X_0^{-3.0}(e)\right), \quad \langle C_{22}\rangle = -\frac{9}{N^2}\frac{D_\mathrm{r}}{mR^2}\frac{m^*}{m^*+m}X_N^{-3.2}(e). \quad (1)$$

For Mercury, $N = 3$ and $3T_\mathrm{rot} = 2T_\mathrm{orb}$. Taking into account that the ratio of the mass of Mercury to the mass of the Sun, m/m^*, is small, from equations (1) we find that

$$\langle J_2\rangle = -\frac{D_\mathrm{r}}{mR^2}[3+2X_0^{-3.0}(e)], \quad \langle C_{22}\rangle = -\frac{D_\mathrm{r}}{mR^2}X_3^{-3.2}(e),$$
$$\left(\frac{\langle J_2\rangle}{\langle C_{22}\rangle}\right)_\mathrm{Mercury} = \frac{3+2X_0^{-3.0}(e)}{X_3^{-3.2}(e)}. \quad (2)$$

For the considered bodies the commensurability of the unperturbed angular velocity $\omega = \Omega$ and the mean motion n has occurred: $3\Omega = 2n$ for Mercury and $\Omega = n$ for the Moon. In equations (1) and (2), m is the mass of Mercury (or the Moon), R is the mean radius of Mercury (or the Moon) and m^* is the mass of the central body (the Sun for Mercury, and the Earth for the Moon). $X_0^{-3.0}(e)$ and $X_N^{-3.2}(e)$ are the Hanzen coefficients depending only on the eccentricity of orbit e [33]. $D_\mathrm{r} < 0$ is an elastic parameter having the dimensions of moment of inertia and characterizing Mercury's (the Moon's) deformation due to its rotation. $-2D_\mathrm{r} = \Delta C$. Here ΔC is an increment in the polar moment of inertia of the satellite due to its rotational deformation. This parameter is calculated for the concrete model of the density distribution of a satellite by well-known formulae [32]. Also this coefficient can be evaluated on the basis of [34]

$$\frac{D_\mathrm{r}^{(0)}}{mR^2} = -k_2\frac{R^3}{9fm}\Omega^2 = -\frac{1}{9}k_2\frac{\Omega^2}{N_0^2} = -\frac{1}{9}k_2\frac{T_{N_0}^2}{T_\mathrm{rot}^2}. \quad (3)$$

In equation (3), k_2 is the Love number characterizing the elastic properties and structure of the satellite. $N_0 = (fm)^{1/2}/R^{3/2}$ is a fundamental frequency and $T_{N_0} = 2\pi/N_0$ the corresponding period of a celestial body. T_{N_0} is equal to the period of orbital motion of some fictive satellite on a circular orbit with a 'surface radius' R. $T_\mathrm{rot} = 2\pi/\Omega$ is an unperturbed (resonant) period of planet rotation. Let us also introduce the orbital period of the satellite, $T_\mathrm{orb} = 2\pi/n$, where $n = [f(m^*+m)]^{1/2}/a^{*3/2}$ is the mean motion.

Tidal deformations are characterized by the parameter $D_\mathrm{t}^{(0)}$, which is similar to the elastic parameter $D_\mathrm{r}^{(0)}$ in equation (3). A simple relation between these two characteristics exists:

$$\frac{D_\mathrm{t}^{(0)}}{D_\mathrm{r}^{(0)}} = -\frac{3}{2}\frac{m^*}{m^*+m}\frac{T_\mathrm{rot}^2}{T_\mathrm{orb}^2}. \quad (4)$$

For Mercury's motion we obviously have

$$3T_\mathrm{rot} = 2T_\mathrm{orb}, \quad D_\mathrm{t}^{(0)} = -\frac{2}{3}\frac{m^*}{m^*+m}D_\mathrm{r}^{(0)}. \quad (5)$$

The ratio of the mass of Mercury to the mass of the Sun, m/m^*, is small and we believe that $3D_\mathrm{t}^{(0)} = -2D_\mathrm{r}^{(0)}$. The ratios $\langle\delta J_2\rangle/\langle\delta C_{22}\rangle$ for models I–III of Mercury and for the Moon are presented in table 2.

2.3 *Tidal variations in Mercury's gravitational field*

Above we have discussed the constant components of the gravitational parameters J_2 and C_{22} caused by tidal and rotational deformations of satellites. Here we also present the general formulae for periodic tidal variations in the coefficients of the second harmonic of Mercury's gravitational potential ($N = 3$):

$$
\begin{aligned}
\delta J_2 &= 3\frac{D_\mathrm{t}}{mR^2}\sum_{\sigma=1}^{\infty} X_\sigma^{-3.0}(e)\cos(\sigma M),\\
\delta C_{22} &= \frac{3D_\mathrm{t}}{2mR^2}\sum_{\sigma=1}^{\infty}[X_\sigma^{-3.2}(e)\cos(\sigma M-2g)+X_{-\sigma}^{-3.2}(e)\cos(\sigma M+2g)],\\
\delta S_{22} &= \frac{3D_\mathrm{t}}{2mR^2}\sum_{\sigma=1}^{\infty}[X_\sigma^{-3.2}(e)\sin(\sigma M-2g)+X_{-\sigma}^{-3.2}(e)\sin(\sigma M+2g)].
\end{aligned}
\tag{6}
$$

By the summation in δC_{22} in equation (6) the resonant term is omitted.

The trigonometric series in equations (6) are given in multiples of the mean orbital anomaly $M = nt + M_0$ ($M_0 = M(0)$ is an initial value). Here $X_\sigma^{-3.0}$ and $X_\sigma^{-3.2}$ are known eccentricity functions which have been presented in polynomial form by Jarnagin [33]. $k_2 = 0.37$ is the theoretical value of the Love number for Mercury [25]. Taking into account equations (3)–(5) and the simple resonant relation $3M = 2g$ and using the values of necessary parameters presented in table 3 on the basis of equations (6) we obtain the following expressions for tidal variations in the gravitational coefficients C_{20}, C_{22} and S_{22} of Mercury:

$$
\begin{aligned}
(\delta J_2)_{\mathrm{periodic}} &= 10^{-8}[2.6991\cos M+0.8203\cos(2M)+0.2450\cos(3M)\\
&\quad +0.0723\cos(4M)+0.0211\cos(5M)+0.0061\cos(6M)\\
&\quad +0.0018\cos(7M)+0.0005\cos(8M)+0.0002\cos(9M)],\\
(\delta C_{22})_{\mathrm{periodic}} &= 10^{-8}[5.0907\cos M+0.1489\cos(2M)+0.2220\cos(3M)\\
&\quad +0.0816\cos(4M)+0.0283\cos(5M)],\\
(\delta S_{22})_{\mathrm{periodic}} &= 10^{-8}[-2.3734\ \sin M+1.0012\sin(2M)+0.2220\sin(3M)\\
&\quad +0.0816\sin(4M)+0.0283\sin(5M)].
\end{aligned}
\tag{7}
$$

The results obtained describe marked changes in the dynamic structure of Mercury due to the gravitational influence of the Sun. The curve of variations in the coefficients C_{22} and S_{22} with the orbital period of Mercury (87.97 days) on the ($X = (\delta C_{22})_{\mathrm{periodic}}$, $Y = (\delta S_{22})_{\mathrm{periodic}}$) plane has a 'whale' form (figure 1).

The ratio $\langle\delta J_2\rangle/\langle\delta C_{22}\rangle = 7.85$ for Mercury is sufficiently close to the similar relation in [3]: $J_2/C_{22} = 8$. In table 3 for comparison we present also the corresponding parameters for the Moon. The variations $\langle\delta J_2\rangle$ and $\langle\delta C_{22}\rangle$ were obtained here for a Love number $k_2 = 0.37$ for Mercury and for the known value $k_2 = 0.025$ for the Moon [35]. The values of geopotential coefficients $(J_2)_{\mathrm{hydro}}$ and $(C_{22})_{\mathrm{hydro}}$ for hydrostatic equilibrium of the considered bodies were obtained with the value $k_2 = 1$. From our results it follows that the tide periodic variations in the gravitational coefficients of Mercury and the Moon is a few orders of magnitude larger than the corresponding tide variations in Earth's geopotential coefficients [26].

Table 3. Parameters and dynamic characteristics of Mercury and the Moon.

Parameter	Mercury	Moon
k_2	0.37	0.025
fm (km^3/s^2)	22,031.97	4902.801
R (km)	2439.7	1737.5
$m^*/(m^* + m)$	1	0.987849
T_{rot} (days)	58.6456	27.2122
T_{orb} (days)	87.9684	27.2122
$T_{\mathrm{rot}}/T_{\mathrm{orb}}$	2/3	1/1
N_0^2 (s^{-2})	1.517204×10^{-6}	0.934695×10^{-6}
N_0 (s^{-1})	1.231749×10^{-3}	0.966796×10^{-3}
$T_{N_0} = 2\pi/N_0$ (days)	0.0590397	0.0752196
D_{r}/mR^2	$-k_2 0.1126 \times 10^{-6}$	$-k_2 0.8490 \times 10^{-6}$
D_{r}/mR^2	-0.4167×10^{-7}	-0.2122×10^{-7}
D_{t}/mR^2	0.2778×10^{-7}	0.3184×10^{-7}
$D_{\mathrm{t}}/D_{\mathrm{r}}$	$-2/3$	$-(3/2)m^*/(m^* + m)$
$D_{\mathrm{t}}/(D_{\mathrm{t}})_{\mathrm{Earth}}$	10.95	12.55
$\Delta C/mR^2$	0.8333×10^{-7}	0.4245×10^{-7}
$X_0^{-3.0}(e)$	1.066953	1.004622
$X_N^{-3.2}(e)$	0.654261	0.992335
$\langle \delta J_2 \rangle$	2.139×10^{-7}	1.585×10^{-7}
$\langle \delta C_{22} \rangle$	0.2726×10^{-7}	0.4681×10^{-7}
$\langle \delta J_2 \rangle/\langle \delta C_{22} \rangle$	7.8469	3.3849
$J_2/(J_2)_{\mathrm{hydro}}$	103.8	31.98
$(J_2)_{\mathrm{hydro}}$	0.5781×10^{-6}	6.3383×10^{-6}

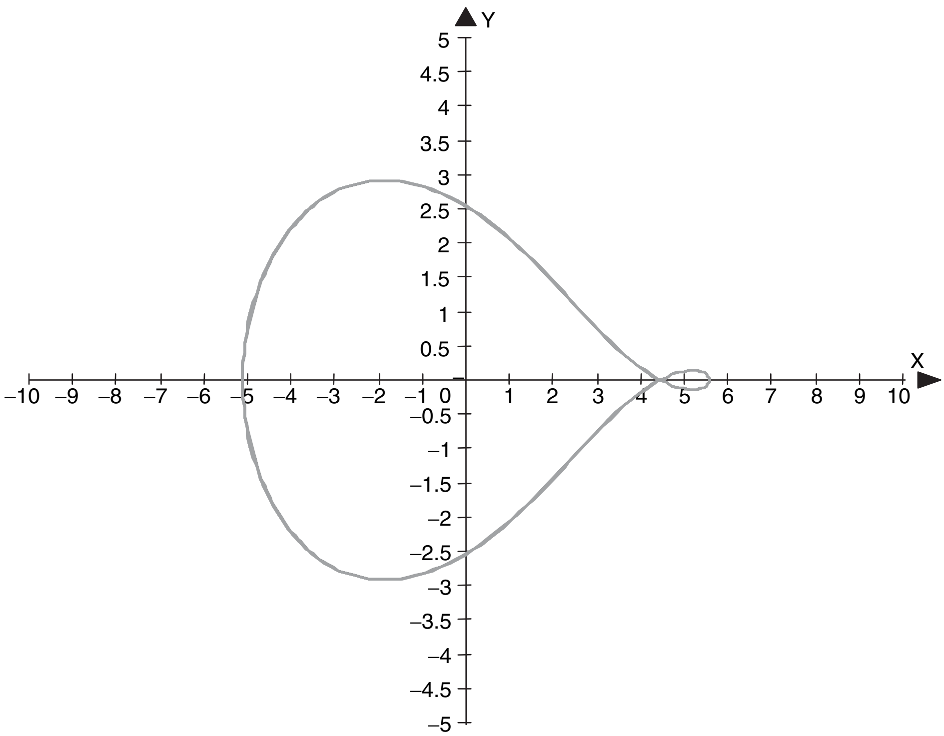

Figure 1. The 'whale' form of the parametric curve of $(\delta C_{22})_{\mathrm{periodic}}(M)$ against $(\delta S_{22})_{\mathrm{periodic}}(M)$. 1 unit $= 10^{-8}$.

3. Rotation

3.1 *Planar librations of Mercury with a liquid core and variations in its angular velocity*

3.1.1 Reference systems and variables. Let us study the librations of Mercury in the framework of the simple model of planar librations of a non-spherical rigid body with a liquid core on an unperturbed elliptical orbit. Let Mercury move on a Keplerian elliptical orbit in the gravitational field of the Sun. O*XYZ* is the Cartesian reference system with the origin at the centre of the Sun. The centre of mass of Mercury moves in the fixed plane O*XY* of the elliptical orbit and the axis OX is directed towards the pericentre of the orbit. C*XYZ* is a similar reference system with the origin at the centre of the mass of Mercury and with axes that are parallel to corresponding axes of the system O*XYZ*. We model here Mercury as a two-layer planet consisting of a rigid non-spherical mantle and a liquid core. We assume that the liquid is homogeneous and ideal and occupies an ellipsoidal central cavity with semiaxes $a > b > c$. The axes of the Cartesian reference system Cxyz with the origin at the centre of mass of the planet are directed along the corresponding axes of the cavity and coincide with the principal axes of inertia of the mantle (and full Mercury). The axis Cz and shorter semiaxis c of the cavity correspond to the maximal moment of inertia of the liquid core C_c and is orthogonal to the orbit plane. We shall assume that the liquid in the cavity executes a simple Poincaré motion. So, Mercury rotates about the axis Cz (CZ). The angle g is measured between the axes CX and Cx. The angular velocity of Mercury's rotation about the principal axis Cz is $r = r_m = \dot{g}$. The simple motion of liquid is modelled as axial rigid rotation with respect to the Poincaré reference system C$x_c y_c z_c$ with the origin at the centre of mass of Mercury. The axis Cz_c coincides with the polar axes CX and Cx. The orientation of the cavity with respect to the C$x_c y_c z_c$ reference system is determined by an angle g_c between the axis Cx_c and a cavity axis Cx. The corresponding angular velocity of cavity rotation here will be equal to $r_c = \dot{g}_c$. The main dynamic characteristics of the liquid core and the full planet, C and C_c, are the principal polar moments of inertia of Mercury and its liquid core, m_c is the mass of the liquid core:

$$C_c = \frac{1}{5} m_c (a^2 + b^2), \quad D_c = \frac{2}{5} m_c ab.$$

Here D_c is a characteristic of the core similar to the product of inertia of some fictive body. If the cavity is axysimmetric, therefore $a = b$ and $C_c = D_c$.

3.1.2 Force function of the problem. This function in the considered problem is identified with the second harmonic of the gravitational potential of Mercury and the Sun. In accordance with [18], for this function we have the following trigonometric development:

$$U_2 = n^2 \frac{m^*}{m^* + m} \frac{C}{I} \sum_{\sigma=0}^{\infty} \left\{ \frac{1}{2} J_2 X_{\sigma}^{-3.0} \cos(\sigma M) + 3C_{22} [X_{\sigma}^{-3.2} \cos(\sigma M - 2g) + X_{-\sigma}^{-3.2} \cos(\sigma M + 2g)] \right\}. \tag{8}$$

We shall consider the gravitational coefficients J_2 and C_{22} of Mercury as small parameters (table 2). The force function (8) is a periodic function of time with the period $T_{\mathrm{orb}} = 2\pi/n$ (table 4).

3.1.3 Canonical equations of the planar librations of Mercury on an elliptical orbit. The kinetic energy of the body with a liquid core with variables g, g_c, r and r_c is given by the expression [24]

$$2T = Cr_m^2 + C_c r_c^2 - 2D_c r_m r_c. \tag{9}$$

Let us assume that the body moves under the action of potential forces with a force function

$$U = U(g, t). \tag{10}$$

The canonical momentums conjugated to introduced generalized coordinates (to Euler angles) are defined by

$$G = \frac{\partial T}{\partial \dot{g}_s} = \nu_s, \tag{11}$$

where

$$G = Cr_m - D_c r_c \tag{12}$$

and

$$G_c = C_c r_c - D_c r_m \tag{13}$$

are the projections of the vector of the full angular momentum of the body with liquid core (equation (9)) (with respect to its centre of mass) and of the angular momentum vector of the liquid core (equation (10)) (with respect to the centre of mass of the liquid core) on the axes of the Cartesian reference system $Cxyz$ and $C_c x_c y_c z_c$ respectively. In the canonical variables (11)–(13), the equations of motion of the Poincaré problem have the following canonical form:

$$\frac{dg_s}{dt} = \frac{\partial K}{\partial G_s}, \quad \frac{dG_s}{dt} = -\frac{\partial K}{\partial g_s}. \tag{14}$$

On the basis of equations (5)–(10) for the Hamiltonian K we obtain the following expression:

$$K = \frac{1}{2}\left(G^2 \frac{C_c}{\Delta} + G_c^2 \frac{C}{\Delta}\right) + GG_c \frac{D_c}{\Delta} - U(g, t), \tag{15}$$

where the force function is defined by the trigonometric series (8), and $\Delta = C_c C - D_c^2$. The right-hand sides of equations (8), (14) and (15) are periodic functions of time. This means that the Poincaré theory of periodic solutions can be applied to these equations.

Equations for variables G and g are separated from the canonical system (14) and (15) and integrated independently:

$$\frac{dG}{dt} = \frac{\partial U}{\partial g}, \quad \frac{dg}{dt} = \frac{1}{\Delta}(C_c G + D_c G_c). \tag{16}$$

The variables G_c and g_c are determined from the first integral and a simple quadrature:

$$G_c = \text{constant}, \quad g_c = \frac{1}{\Delta}\int [CG_c + D_c G]dt + \text{constant}. \tag{17}$$

3.1.4 Perturbations of the first order. If the small parameters of the problem are zero, then $U = 0$, and from equations (14) and (15) we obtain the generating periodic solution

$$G = C\Omega, \quad 2\Omega = 3n, \quad g = nt + g_0, \quad G_c = -D_c\Omega, \quad g_0 = M_0. \tag{18}$$

Here Ω is the resonant value of the angular velocity.

Periodic perturbations of the first order are determined by the simple quadratures

$$\delta G = \int \frac{\partial U}{\partial g}\,\mathrm{d}t + \text{constant}, \qquad \delta g = \frac{C_\mathrm{c}}{\Delta}\int\left(\int \frac{\partial U}{\partial g}\,\mathrm{d}t\right)\mathrm{d}t + \text{constant}, \tag{19}$$

where

$$\frac{C_\mathrm{c}}{\Delta} = \frac{1}{C - C_\mathrm{c}(D_\mathrm{c}^2/C_\mathrm{c}^2)} \approx \frac{1}{C - C_\mathrm{c}} = \frac{1}{C_\mathrm{m}}$$

and C_m is the polar moment of inertia of the mantle of Mercury. The integrals in equation (19) are calculated for the generating values of variables in equation (18). For perturbations of the first order of the angular velocity of Mercury and for its rotation period we have the simple relations

$$\delta\omega = \frac{\delta G}{C}, \qquad \delta T = -\frac{\delta\omega}{\Omega}T_\mathrm{rot}.$$

Calculating the integrals we obtain the following formulae for the perturbations (19):

$$\begin{aligned}
\delta g &= -6C_{22}\frac{C}{I[C - C_\mathrm{c}(D_\mathrm{c}^2/C_\mathrm{c}^2)]}\sum_{\substack{\sigma=1\\ \sigma\neq 3}}^{\infty}\left(\frac{X_\sigma^{-3.2}}{(\sigma-N)^2}\sin[(\sigma-N)M]\right.\\
&\quad\left. - \frac{X_{-\sigma}^{-3.2}}{(\sigma+N)^2}\sin[(\sigma+N)M]\right),\\
\delta\omega &= -6C_{22}n\frac{1}{I}\frac{C}{[C - C_\mathrm{c}(D_\mathrm{c}^2/C_\mathrm{c}^2)]}\sum_{\substack{\sigma=1\\ \sigma\neq 3}}^{\infty}\left(\frac{X_\sigma^{-3.2}}{\sigma-N}\cos[(\sigma-N)M]\right.\\
&\quad\left. - \frac{X_{-\sigma}^{-3.2}}{\sigma+N}\cos[(\sigma+N)M]\right),
\end{aligned} \tag{20}$$

or

$$\delta g = \sum_{\sigma=1}^{\infty} g_\sigma \sin(\sigma M), \qquad \frac{\delta\omega}{n} = \sum_{\sigma=1}^{\infty}\omega_\sigma\cos(\sigma M),$$

$$g_\sigma = -6C_{22}\frac{C}{I[C - C_\mathrm{c}(D_\mathrm{c}^2/C_\mathrm{c}^2)]}(X_{\sigma+N}^{-3.2} - X_{N-\sigma}^{-3.2})\frac{1}{\sigma^2}, \qquad \omega_\sigma = \sigma g_\sigma.$$

In the considered Mercury case the first terms of perturbations δg in equation (20) can be presented in the following, more detailed form:

$$\begin{aligned}
\delta g = \frac{C_{22}}{I}\frac{C}{C - C_\mathrm{c}(D_\mathrm{c}^2/C_\mathrm{c}^2)}\Bigg[&6\left(1 - 11e^2 + \frac{959}{48}e^4 - \frac{3641}{288}e^6 + \frac{11359}{2880}e^8\right)\sin M\\
&- \frac{3}{4}e\left(1 + \frac{421}{12}e^2 - \frac{32{,}515}{384}e^4 + \frac{2{,}186{,}863}{32{,}256}e^6 - \frac{428{,}399{,}713}{15{,}482{,}880}e^8\right)\sin(2M)\\
&- \frac{1}{24}e^4\left(533 - \frac{13{,}827}{10}e^2 + \frac{728{,}889}{560}e^4\right)\sin(3M)\\
&+ \frac{1}{128}e^3\left(1 - \frac{57{,}073}{20}e^2 + \frac{7{,}678{,}157}{960}e^4 - \frac{298{,}080{,}597}{34{,}560}e^6\right)\sin(4M) + \cdots\Bigg].
\end{aligned}$$

If $C = C_\mathrm{m}$, perturbations in the rotation of the resonant satellite (equation (20)) are obtained as a particular case from a more general treatment of the problem about a planar translatory–rotary motion of the rigid satellite considered in [13, 15].

Equation (20) was obtained for the orbital–rotational resonances $Nn = 2\omega$ (the case $N = 2$ corresponds to synchronous motions of satellites and the case $3n = 2\omega$ to Mercury's motion). For a construction of perturbations the Poincaré theory of periodic solutions has been used in [13, 17, 18]. It was shown that the coefficients of perturbations proportional to C_{22}/I and the ratios of any from amplitudes in perturbations δg (and $\delta\omega$) for the concrete celestial body depend only on the eccentricity of the orbit. For example

$$g_{1/\sigma} = \frac{g_1}{g_\sigma} = \frac{X_{N+1}^{-3.2} - X_{N-1}^{-3.2}}{X_{\sigma+N}^{-3.2} - X_{N-\sigma}^{-3.2}}\sigma^2, \quad \omega_{1/\sigma} = \frac{\omega_1}{\omega_\sigma} = \frac{X_{N+1}^{-3.2} - X_{N-1}^{-3.2}}{X_{\sigma+N}^{-3.2} - X_{N-\sigma}^{-3.2}}\sigma. \tag{21}$$

So the ratios of the amplitudes of the main perturbations in the librations in longitude of Mercury are $A_{1/2} = -9.4850$, $A_{1/3} = 96.248$ and $A_{1/4} = -477.76$ (see table 4 later).

For parameters of model I for Mercury (tables 2 and 3) on the basis of equation (20) we obtain the following explicit expressions for the periodic perturbations of the angular rotation g, the angular velocity ω and the instant period of the rotation of Mercury:

$$\begin{aligned}
\delta g &= 53''670 \sin M - 5''658 \sin(2M) - 0''558 \sin(3M) \\
&\quad - 0''107 \sin(4M) - 0''013 \sin(5M), \\
\frac{\delta\omega}{\omega} &= [1.7347 \cos M - 0.3658 \cos(2M) - 0.0541 \cos(3M) \\
&\quad - 0.0139 \cos(4M) - 0.0022 \cos(5M) \times 10^{-4}, \\
\delta T &= -T_{\mathrm{rot}}[1.7347 \cos M - 0.3658 \cos(2M) - 0.0541 \cos(3M) \\
&\quad - 0.0139 \cos(4M) - 0.0022 \cos(5M)] \times 10^{-4}.
\end{aligned} \tag{22}$$

Here $T_{\mathrm{rot}} = 58.64562241$ days is the exact resonant value of the rotational period. The instant period of the rotation of Mercury is the following periodic function of time:

$$\begin{aligned}
T = 58.6456[1 &- 0.0007347 \cos M + 0.0000366 \cos(2M) \\
&+ 0.0000054 \cos(3M) + 0.0000014 \cos(4M)].
\end{aligned} \tag{23}$$

The curve of the time dependence of the period (23) is presented in figure 2. Two features of Mercury's rotation are shown here.

(i) Because of eccentricity variations in the gravitational moment of the Sun the angular velocity of Mercury and the corresponding instant period of the rotation of Mercury are varied with an amplitude of about 0.0341%. So the period of rotation has a minimal value of 58.638 days at the moment of crossing the pericentre of the orbit and a maximal value of 56.658 days at the moment of crossing the apocentre of the orbit.
(ii) In the time intervals of 7.4 days before passing the pericentre of the orbit and 7.4 days after it, the angular velocity of Mercury with a high accuracy maintains a constant pericentre value. This interval of time of 14.8 days can be called 'the period of Mercury's non-perturbation'.

In reality the ratio of the change in the angular velocity in the afore-mentioned period to the change in the angular velocity in the half-orbital period (at the pericentre and apocentre) is about 0.869%.

In this paper we shall not discuss the question about the constant components of perturbations of the first order the nature of which is connected to the third and higher harmonics of the force function of the problem and to the tidal dissipation of the elastic energy of Mercury.

Table 4. Amplitudes of the planar librations of Mercury and the Moon.

Parameter, amplitude	Mercury (model I)	Mercury (model II)	Mercury (model III)	Moon
C_{22}	9.8566×10^{-6}	10.0×10^{-6}	12.94×10^{-6}	22.3×10^{-6}
I	0.34	0.34	0.34	0.392
e	0.205614	0.205614	0.205614	0.055
$C/C_{\rm n}$	2	2	2	1.00060
g_1	40"8813	41"4760	53"6700	−15"2070
g_2	−4"3112	−4"3739	−5"6599	−0"4471
g_3	−0"4249	−0"4311	−0"5579	−0"0228
g_4	−0"0861	−0"0874	−0"1131	−0"0013
g_5	−0"0191	−0"0194	−0"0250	−0"0001
ω_1	1.3213×10^{-4}	1.3405×10^{-4}	1.7347×10^{-4}	-0.7373×10^{-4}
ω_2	-0.2787×10^{-4}	-0.2827×10^{-4}	-0.3659×10^{-4}	-0.0434×10^{-4}
ω_3	-0.0412×10^{-4}	-0.0418×10^{-4}	-0.0541×10^{-4}	-0.0033×10^{-4}
ω_4	-0.0111×10^{-4}	-0.0113×10^{-4}	-0.0146×10^{-4}	-0.0003×10^{-4}
ω_5	-0.0031×10^{-4}	-0.0031×10^{-4}	-0.0041×10^{-4}	-0.00002×10^{-4}

Evaluations of the amplitudes of the librations and variations in angular velocity for all three considered models of Mercury are given in table 4. For a comparison of discussed perturbed rotational effects in this table we also present similar characteristics of the librations in longitude of the Moon. Mercury and the Moon present two different types of celestial body: a body with a very large liquid core and a body with a small liquid core. The forced librations of Mercury concern the case of a strong influence of the liquid core and the librations of the

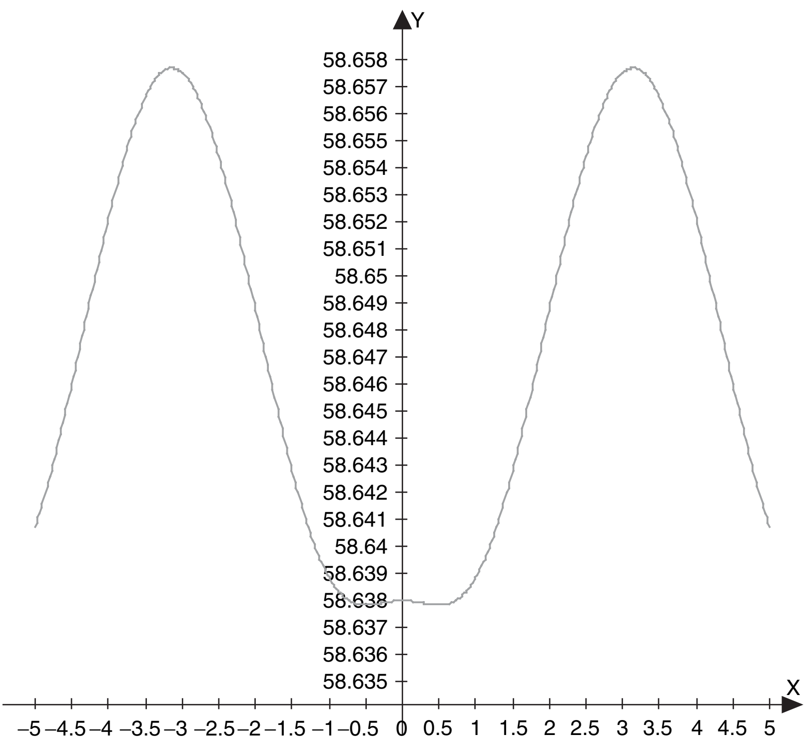

Figure 2. Seasonal variation in Mercury's period: $Y = T$ (in days); $X = M$ (in radians).

Moon concern the case of a weak influence of the liquid core. This means that the amplitudes of the Moon's librations from table 4 are close to similar characteristics of the corresponding rigid model of the Moon.

Firstly the evaluations of the amplitudes of the librations in longitude of Mercury have been made in [18] on the basis of the model problem about resonant translatory–rotary motion of the rigid body (Mercury) in the gravitational field of the central body (the Sun). The hypothetical relation between the coefficients C_{20} and C_{22} has been obtained from the Poincaré condition of the existence of periodic solution of the problem [13] and, as the basic value of the coefficient, $C_{20} = 80 \times 10^{-6}$ has been used [29]. So the amplitudes of librations of rigid Mercury have been evaluated as $g_1 = -10''.098$ and $g_2 = 2''.414$. The corresponding amplitudes of variations in Mercury's angular velocity are $\omega_1/\omega = -0.489 \times 10^{-4}$ and $\omega_2/\omega = 0.119 \times 10^{-4}$. The librations in longitude of Mercury (table 4) are characterized by the following values of the ratios of amplitudes (equation (10)): $g_{1/2} = -9.4850$, $g_{1/3} = 96.248$, $g_{1/4} = -477.76$, $\omega_{1/2} = -4.7425$, $\omega_{1/3} = 32.083$ and $\omega_{1/4} = -119.44$. In [18], $g_{1/2} = -8.37$ and $\omega_{1/2} = -4.07$ have been obtained.

For comparison of the dynamic effects in the last column of table 4 we present similar characteristics of the planar resonant librations of the Moon. The librations in longitude of the Moon (table 4) are characterized by the following values of the ratios of amplitudes (equation (10)): $g_{1/2} = 34.258$, $g_{1/3} = 677.8$, $\omega_{1/2} = 17.12$ and $\omega_{1/3} = 223.4$. For the Moon's librations the ratios $g_{1/2} = 34.0$, $g_{1/3} = 674$, $\omega_{1/2} = 17.0$ and $\omega_{1/3} = 218$ have been evaluated earlier in [15, 17].

In accordance with the Poincaré theory of periodic solutions we can confirm that periodic perturbations of the first order also contain constant components. The dependence of the above-mentioned corrections are determined from values of the coefficients of third and higher harmonics of gravitational potential of corresponding satellite.

3.2 *Resonant librations in longitude*

The period of resonant librations in longitude of Mercury with a liquid core in the framework of the considered planar problem about the rotational motion of a satellite on an elliptical orbit is defined by the formula

$$T_{\mathrm{res}}^{(\mathrm{liq})} = \frac{T_{\mathrm{orb}}}{2[m^*/(m+m^*)]^{1/2}[3C_{22}(C/IC_{\mathrm{m}})X_N^{-3.2}(e)]^{1/2}}. \tag{24}$$

From equation (24) the expression for the period $T_{\mathrm{res}}^{(\mathrm{rid})}$ of librations of the rigid model of Mercury is obtained from $C = C_{\mathrm{m}}$ [17]. Here the Hanzen coefficient $X_N^{-3.2}(e)$ is a known function of orbital eccentricity e.

In table 5 are given the values of both periods $T_{\mathrm{res}}^{(\mathrm{rid})}$ and $T_{\mathrm{res}}^{(\mathrm{liq})}$ for the three models of Mercury (tables 1 and 2) and for the Moon. We have employed models for Mercury with a liquid core using $C = 2C_{\mathrm{m}}$ to carry out calculations and for the Moon using $C = 1.00060C_{\mathrm{m}}$ to carry out calculations [4–6]. Here we adopt a dimensionless moment of inertia $I = 0.34$ for Mercury and $I = 0.392$ for the Moon. The values of resonant periods for the Mercury models and for the Moon are given in table 5. In the Mercury case the existence of the liquid core leads to a strong reduction in the resonant period and in the Moon case to a weak reduction.

So the differences between the resonant periods of librations for a rigid body and for a body with a liquid core consist are about 29.3% for Mercury and 0.03% for the Moon. The earlier evaluations of the resonant period of the rotation of the rigid Mercury model were 19 and 21 years [36].

Table 5. Resonant periods for the Mercury models and for the Moon.

Parameter	Mercury (model I)	Mercury (model II)	Mercury (model III)	Moon
C_{22}	9.857×10^{-6}	10.0×10^{-6}	12.942×10^{-6}	22.3×10^{-6}
$T_{\text{res}}^{(\text{liq})}$ (years)	11.288	11.207	9.851	2.891
$T_{\text{res}}^{(\text{rid})}$ (years)	15.964	15.849	13.932	2.892
$T_{\text{rid}}^{(\text{res})} - T_{\text{res}}^{(\text{rid})}$ (years)	4.676	4.642	4.081	0.000857
$T_{\text{res}}^{(\text{rid})}$ (%)	29.289	29.289	29.292	0.0296

3.3 *The Cassini–Colombo motion of Mercury*

Let us consider now the space resonant rotational motion of a rigid celestial body (satellite) in a central gravitational field assuming that it moves on an elliptical orbit with a precessing line of nodes and with a moving pericentre of the orbit. This model problem was first formulated by Colombo [7] and then has been intensively studied by Goldreich and Peale [8], Peale [10], Beletskij [11], Ward [12] and Barkin [13, 15, 17].

Let O*XYZ* be the main reference system with the origin at the Sun's centre. The coordinate plane OXY is a unmoveable Laplacian plane. Let Oxyz be the orbital reference system connected to the line of nodes of the orbit (axis Ox) on the plane OXY (ecliptic plane). Oxy is an orbit plane and the axis Oz is orthogonal to it. Let the inclination of orbit plane i be a constant which precesses with the constant angular velocity n_Ω with respect to normal to the main coordinate plane OXY. The major semiaxis a and eccentricity e of the orbit in the considered model are constant. Let also O$\xi\eta\zeta$ be the axes of Mercury directed along its principal axes of inertia. The axial moments of inertia of Mercury, A, B and C, correspond to the axes Cξ, Cη and Cζ $(B > A > C)$.

We shall describe the rotational motion of Mercury under the action of the gravitational moments of the Sun in the Andoyer variables

$$l, g, h, \theta, \rho, G, \tag{25}$$

referred to the moving orbital plane [11, 14]. Here l, g and θ are the standard Euler angles determining the orientation of the body axes O$\xi\eta\zeta$ with respect to the intermediate reference system connected to the angular momentum $\boldsymbol{G}$ of the satellite. $G = |\boldsymbol{G}|$ is the modulus of the angular momentum vector. ρ is the angle between the plane Q orthogonal to the vector $\boldsymbol{G}$ and the orbit plane. h is the longitude of the ascending node of the plane Q calculated along Mercury's orbit from the ascending node of this plane on the ecliptic plane.

In [14, 17] the canonical forms of the equations of rotational motion of a rigid celestial body have been used. As has been shown in the particular case of a precessing circular orbit of a satellite (with a constant velocity of precession and with a constant inclination i to the principal plane) the right-hand sides of the equations will be periodic functions of time, and the Poincaré theory of periodic solutions can be applied to the study of resonant synchronous rotation of a satellite. The existence of a definite class of periodic solutions of the problem has been proved. The corresponding generating periodic solutions are obtained directly from the analytical conditions of existence of periodic solutions and have allowed us to explain the main regular features of the satellite motion which can be called the generalized Cassini laws [14, 15].

The above-mentioned method is not applicable directly to the study of the resonant rotation of a celestial body in an elliptical orbit. In this case the equations of motion described in the Andoyer variables are not periodic on the time and the Poincaré theory is not applicable directly. An effective approach can be realized for the study of the resonant rotation of Mercury in an elliptical precessing orbit on the basis of the average methods of Beletskij [11] style.

The existence of a definite class of stationary solutions of the average equations of the problem described in the Andoyer variables has been proved. These solutions correspond to a definite class of periodic motions of Mercury with respect to the precessing orbital reference system Oxyz. One of these solutions is determined by the formulae

$$l_0 = 0, \quad g_0 = 0, \ \frac{\pi}{2}, \ \ h_0 = 0, \pi, \ \ \theta_0 = \frac{\pi}{2}, \ \ \rho_0 = \rho_0(\chi, \delta), \tag{26}$$

and by the condition of commensurability of the unperturbed angular velocity Ω of Mercury and its mean orbital motion:

$$3n = 2(\Omega - n_\omega), \tag{27}$$

where $\Omega = G_0/B$ and G_0 are unperturbed values of the angular velocity of Mercury and its angular momentum, n is the mean motion of Mercury and n_ω is an angular velocity of the motion of the line of apses.

Solutions (26) and (27) describe the following regular features in Mercury's motion.

(i) With respect to the orbital reference system connected to the line of nodes of the orbit, Mercury rotates about its polar axis of inertia corresponding to the maximal principal moment of inertia B with permanent angular velocity $\Omega - n_\omega$ equal to 3/2 of its mean orbital motion.

(ii) The largest axis of Mercury's ellipsoid of inertia corresponding to the minimal moment of inertia C is directed along the heliocentric radius vector of the Sun at the moment of passages of the pericentre of the orbit. The middle axis of Mercury's ellipsoid of inertia corresponding to the middle moment of inertia A is directed along the heliocentric radius vector of the Sun at the moment of passages of the apocentre of the orbit. Thus the orientations of the above-mentioned axes' (at the apocentre and the pericentre of the orbit) in the orbital are changed in the opposite way in two consecutive orbital cycles.

(iii) The ascending nodes of the orbit plane and intermediate plane orthogonal to the angular momentum vector of Mercury on the ecliptic plane coincide. This intermediate plane coincides with the equatorial plane of the ellipsoid of inertia of the satellite. The angles between the orbital plane, intermediate plane and main (Laplacian) plane are constant.

(iv) The angular velocity and angular moment vectors of Mercury coincide and form a constant angle $\rho_0 = \rho_0(i, n/n_\Omega, e)$ with the normal to the orbit plane. These vectors and the normal to the orbit plane and to the Laplacian plane are situated in one plane orthogonal to the orbit plane.

(v) The angle between the angular momentum vector of Mercury and the normal to its orbit plane $\rho_0 = \rho_0(i, n/n_\Omega, e)$ is constant and determined from the equation [11, 17]

$$\cos i + \varepsilon_1 \frac{\cos \rho_0}{\sin \rho_0} \sin i + \frac{2n}{INn_\Omega}[-\cos \rho_0 C_{20} X_0^{-3.0} + (1 + \cos \rho_0) C_{22} X_3^{-3.2} \varepsilon_2] = 0, \tag{28}$$

where $\varepsilon_1 = \cosh 0 = \pm 1$, $\varepsilon_2 = \cos[2(g_0 + h_0 - \omega_0)] = \pm 1$. ω_0 is the unperturbed value of longitude of the pericentre of the Mercury orbit. This equation is equivalent to the Peale equation [20, 37].

So the inclination ρ_0 can be evaluated on the basis of the known parameters of the mobility n/n_Ω of the orbit plane, the inclination of orbit plane and its eccentricity e and by the concrete generating values $g_0 - \omega_0$ of the angular Andoyer variables h_0. The precession of Mercury's orbit relative to the normal to the Laplacian plane has a regressive character $n_\Omega < 0$.

In the case of small values of inclination ρ_0 the approximate solution of equation (28) will be

$$\rho_0 = \frac{-\varepsilon_1 \sin i}{\cos i + (2n/INn_\Omega)[-C_{20}X_0^{-3.0}(e) + 2C_{22}X_3^{-3.2}(e)\varepsilon_2]}. \tag{29}$$

The period of orbital motion of Mercury is 87.969 days and the period of progressive precession of the line of node of the orbit plane on the Laplacian plane is 278,898 years [7, 10]. The inclination of the plane of Mercury's orbit with respect to the ecliptic plane (the Laplace plane) is about $i = 7.0028806$ and the eccentricity $e = 0.205614$. In equation (17) we have

$$\frac{n_\Omega}{n} = -0.863563 \times 10^{-6}, \quad X_0^{-3.0} = 1.06365, \quad X_3^{-3.2} = 0.6537974, \quad I = 0.34,$$

and, for values $\varepsilon_1 = \cosh_0 = 1$, $\varepsilon_2 = \cos 2(g_0 + h_0 - \omega_0) = 1$ and for accepted parameters of the gravitational field of Mercury model II (tables 1 and 2), we obtain $\rho_0 = 0°0268$. This means that the mean angle between the normal to the ecliptic plane and Mercury's rotation axis is $\rho + i = 7.0297$. The first evaluations of the inclination of Mercury's rotation axis made by Barkin in 1983–1985 gave $\rho_0 = 0°0217$ and $0°0315$ ($1'27$ and $1'67$ respectively). All the established characteristics of the Cassini–Colombo motion of Mercury (and the Moon for comparison) are summarized in table 5.

The generating solution (26)–(29) describes the periodic rotational motion of a satellite with respect to an orbital reference system Oxyz connected to the line of nodes of the orbit (axis Ox) on the main fixed plane Oxy (ecliptic plane for the Moon). Then axis Oz is orthogonal to the orbital plane.

3.4 *Resonant librations*

In accordance with the general properties of resonant rotational motion, Mercury executes some free oscillations in the neighbourhood of the above-described stationary periodic solutions of the problem (in the neighbourhood of the Cassini–Colombo motion). In this paragraph we evaluate the periods of these librations. It is worth remarking that the amplitudes and phases of these librations can be determined only on the basis of observational data. Only the above-mentioned characteristics of the Moon were determined from laser ranging [37].

Analytical expressions for the resonant periods of librations in the considered problem are obtained from general studies of the resonant translatory–rotary motions of rigid celestial bodies presented in Barkin's [17] dissertation (see also [16]):

$$T_l = \frac{T_0}{\kappa[(C_{20} - 2C_{22})(C_{20} + 2C_{22})]^{1/2}[\Lambda_1(\rho)\Lambda_2(\rho)]^{1/2}}, \quad \kappa = \frac{3}{4N}\left(1 + \frac{m}{m^*}\right),$$

$$\begin{aligned}\Lambda_{1,2}(\rho) = {} & 2(2 - 3\sin^2\rho)X_0^{-3.0} + \frac{2}{3}N^2\left(1 + \frac{m}{m^*}\right) \pm [\sin^2\rho\, X_N^{-3.0} \\ & + (X_N^{-3.2} + X_{-N}^{-3.2})(1 + \cos^2\rho) + 2\cos\rho(X_N^{-3.2} - X_{-N}^{-3.2})],\end{aligned} \tag{30}$$

$$T_g = \frac{T_0}{[3(C_{22}/I)\varepsilon_2(1 + \cos\rho)^2 X_N^{-3.2}]^{1/2}}, \tag{31}$$

$$T_h = \frac{T_0}{\{\varepsilon_1(n_\Omega/n)\sin i \sin\rho_0\,[\varepsilon_1(n_\Omega/n)\sin i\,\mathrm{cosec}^3\rho_0 + (1/I)(-C_{20}X_0^{-3.0} + C_{22}X_N^{-3.2}\varepsilon_2)]\}^{1/2}} \tag{32}$$

In (30) and (31), $\rho = \rho_0$ is the inclination of Mercury's axis; $X_N^{-3.0}$ and $X_N^{-3.2}$ are the Hanzen coefficients (functions of eccentricity of orbit); m and m^* are masses of Mercury

Table 6. Periods of resonant librations of Mercury and the Moon.

Parameter	Mercury (model II)	Moon
C_{20}	-60×10^{-6}	-202.7×10^{-6}
C_{22}	10×10^{-6}	22.3×10^{-6}
I	0.34	0.392
n_Ω/n	-0.86356×10^{-6}	-3983.2×10^{-6}
i	7.00288	5.1454
ρ	0.02677	6.7433
h_0	0	π
T_{orb} (days)	87.969	27.212
T_l (years)	1012.8 1066 (RamBo)	74.998
T_g (years)	16.087 15.847 (RamBo)	2.871
T_h (years)	18.657 (e) (RamBo)	20.113
T_{Eul} (years)	1447.6 964.88 (RamBo)	147.70

and the Sun. Equation (30) defines the period of perturbed pole motion, equation (31) the period of librations in longitude and equation (32) the period of librations of the angular momentum vector. It should be noted that the Euler periods in the case of small amplitudes of the pole motion of synchronous satellites ($N = 2$) are approximately twice the corresponding perturbed period (30) (table 6). For comparison in table 6 we present also similar characteristics of resonant librations of the Moon (calculated also from equations (30)–(32)). The periods of the Moon's resonant librations are in good agreement with their values obtained on the basis of laser ranging data, $T_g = 2.9$ years and $T_l = 75$ years [37], and with the periods from analytical theories of the Moon's rotation, $T_l = 75.23$, $T_g = 2.88$ and $T_h = 24.14$ [38], $T_l = 75.20$, $T_g = 2.88$ and $T_h = 24.68$ [17, 18], and $T_l = 75.205$, $T_g = 2.916$ and $T_h = 24.297$ [28]. The values of the periods of Mercury's resonant librations are in good agreement with the values obtained by numerical stimulations of the equations of translatory–rotary motion of Mercury [20] (they are labelled RamBo in table 6).

In this section we are restricted by consideration of only the rigid-body model for Mercury.

In a future paper we plan to study the influence of the liquid core on the resonant librations of the Mercury in the neighbourhood of the Cassini–Colombo motion using the general approach developed by Barkin and Ferrandiz [5, 6, 24].

4. Conclusions

In this paper, new results about the variations in the dynamic structure of Mercury and about its resonant rotation are reported. Model parameters of the second harmonic of the gravitational potential of Mercury are constructed on the basis of theoretical constructions and therefore on the basis of observational data. The obtained results describe some new aspects and features of Mercury's rotation in comparison with similar characteristics of the resonant rotation of the Moon. Tidal variations in the gravitational potential are sufficiently significant and can be studied in the framework of the planned missions to Mercury.

We assume also that the amplitudes and phases of resonant librations in longitude and in the space position of angular momentum can be observed in the close future from radio-location ground observations of Mercury and also as the result of space missions. The significant role of a liquid core in the expected forced resonant librations of Mercury are confirmed by our studies.

The obtained results present important interest for the effective realization of future missions to Mercury, the Moon and Titan [1, 2] and for study of the inner structure of these celestial bodies. The prediction of the high endogenous activity of Titan has been given earlier on the basis of the shell dynamics mechanism [21, 22]. For the considered model, Titan has the third highest value of endogenous energy (power) after the very active satellite Io and Europa. This means that sufficiently high tectonic activity of the Titan can be observed in reality. This conclusion is very important and some confirmation of it can be obtained from data from the Cassini–Huygens expedition to Saturn in 2005.

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