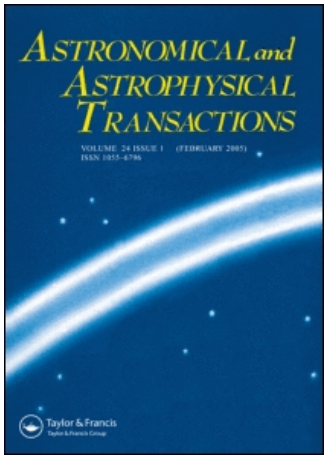

Astronomical & Astrophysical Transactions
The Journal of the Eurasian Astronomical Society
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713453505

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Online Publication Date: 01 October 2004
To cite this Article: Barkin, Yu. V. (2004) 'Comparative rotational dynamics of the Moon, Mercury and Titan', Astronomical & Astrophysical Transactions, 23:5, 481 - 492
To link to this article: DOI: 10.1080/10556790412331319659
URL: http://dx.doi.org/10.1080/10556790412331319659

Astronomical and Astrophysical Transactions
2004, Vol. 23, No. 5, pp. 481–492

COMPARATIVE ROTATIONAL DYNAMICS OF THE MOON, MERCURY AND TITAN

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(Received 4 October 2004)

Constant and periodic variations in the gravitational coefficients of second harmonics of gravitational potential of the Moon, Mercury and Titan caused by the attraction of their external bodies are described analytically. On the base of a plane model of translatory–rotary motion of the above-mentioned elastic bodies on elliptical orbits the main variations in the coefficients C_{20}, C_{22} and S_{22} with periods that are a multiple of the orbit period have been obtained and evaluated. Dynamic effects in resonant rotation of the Moon, Mercury and Titan considered as an unchangeable non-spherical bodies are studied on the basis of two simple models: firstly, the plane motion on the unperturbed elliptic orbit; secondly, the rotation of Titan on the precessing circular orbit. Peale's hydrostatic values of Titan's gravitational field parameters C_{20} and C_{22} (Peale, 1973) and two sets of their model values are used in paper to evaluate the unperturbed, forced and resonant effects in the rotation of Titan. For Mercury we also use an average equilibrium model of elastic body rotating in resonance 3:2 with an orbital motion. Another model of Mercury is based on the modern radio-location data (Peale *et al.*, 2004). For the Moon, well-known dynamical characteristics are used. Obtained Cassini's positional parameters, amplitudes of periodic librations in longitude, periods of resonant librations and other characteristics of studied celestial bodies are given in mutual comparison.

KEYWORDS: Titan's rotation, Titan's gravitational field, Moon's rotation, Mercury's rotation, Cassini's motion, resonant librations

1 GRAVITATIONAL PARAMETERS AND THEIR VARIATIONS

1.1 The Moon, Titan and Mercury characteristics

The translatory–rotary motion of Titan in the gravitational field of Saturn is resonant (synchronous) and in a definite sense is similar to the motion of the Moon. In this paper we shall consider some of the possible particularities in rotation of Titan using the analogy and constructive studies of the rotations of the Moon and Mercury given in the papers by Colombo (1966), Goldreich and Peale (1966), Beletskij (1972), Peale (1973, 1979) and Ward (1975) and in some papers by the present author (Barkin, 1976, 1978, 1979a, b, c, 1983). Although we have almost no data about the dynamic structure of Titan, for the first studies of its rotation we shall consider Titan as an ellipsoidal unchangeable body with some model values of the main parameters of its gravitational field J_2 and C_{22} (Table 1). For the non-dimensional moment of inertia we shall use the value $C/mR^2 = 0.35$. It is worth remarking

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ISSN 1055-6796 print; ISSN 1476-3540 online
DOI: 10.1080/10556790412331319659

Table 1. Model values of main coefficients of gravitational potentials.

Bodies, models	$J_2 \times 10^4$	$C_{22} \times 10^4$	J_2/C_{22}
Titan (I)	0.382	0.114	3.35
Titan (II)	0.330	0.120	3.34
Titan (III)	1.649	0.493	3.34
Mercury (I)	0.600	0.100	6.00
Mercury (II)	1.015	0.129	7.85
The Moon	2.027	0.223	9.09

that the physical properties of the Moon and Titan are somewhat different (*e.g.* the mean densities of these bodies are 3.34 and 1.37 g cm^{-3} respectively).

Peale's evaluations (model I) were obtained for hydrostatic equilibrium of Titan on a circular orbit. Another two sets of the parameters J_2 and C_{22} (Titan models II and III) were adopted on the basis of an analogy with the values of the parameter J_2/C_{22} for a group of large satellites (Table 2) and for a hypothetical value of the parameter J_2. A Mercury model I has been constructed on the base of Mariner-10 observational data (Anderson *et al.*, 1987; Peale *et al.*, 2003). A model (III) is based on the modern radio-location observation data on the Mercury libration (Peale *et al.*, 2004). Data about this model have been presented on the last COSPAR Assembly in Paris. The known values of gravitational coefficients J_2 and C_{22} of the Moon are used in the paper. For evaluations of elastic effects in variations of gravitational coefficients, we use the following values of Love number: $k_2 = 0.025$ for the Moon (Konopliv *et al.*, 1998), potential value $k_2 = 0.37$ for Mercury (Dehant *et al.*, 2004) and hypothetical value $k_2 = 0.5$ for Titan.

In Table 2 are given the theoretical values of the gravitational coefficients J_2 and C_{22} for the natural satellites of the Solar System obtained in two ways: firstly, on the assumption of hydrostatic equilibrium motion of the satellites (Peale, 1973) and, secondly, for their ellipsoidal homogeneous models (indicated as (ell)). The latter values were calculated with the formulae

$$C_{20} = \frac{a^2 + b^2 - 2c^2}{2(a^2 + b^2)} I, \quad C_{22} = \frac{a^2 - b^2}{20 R_{\mathrm{m}}^2}, \quad I = \frac{a^2 + b^2}{5 R_{\mathrm{m}}^2},$$
$$R_{\mathrm{m}} = (abc)^{1/3}, \quad \delta = \frac{4C_{22}}{2C_{22} - C_{20}}. \tag{1}$$

Here $a > b > c$ are the semiaxes of the approximate ellipsoid, R_{m} is the mean radius, I is the non-dimensional moment of inertia of satellite and δ is a characteristic of the three-axis nature of the satellite. Ratios of the gravitational coefficients are also presented in Table 2. The data in this table illustrate the good agreement between the coefficients calculated for two different models of satellites (with a few exceptions). Here we have used known values of the semiaxes $a > b > c$ of satellite ellipsoidal models (http://pds.jpl.nasa.gov/planets).

1.2 Elastic models of satellites

As a definite approximation the satellites of planet can be modelled as elastic bodies. Because of the gravitational attraction of the parent planet the elastic satellite moving in an elliptical orbit is deformed. In the considered case of resonant motion this leads to definite contributions to J_2 and C_{22}. They are obtained on the basis of the well-known classical Takeuchi solution for the problem of elasticity (Ferrandiz and Getino, 1993) for a satellite with a concentric mass distribution (in an undeformed state) and are described by the following formulae:

$$J_2 = -3\frac{D_{\mathrm{r}}}{mR^2}\left(1 + \frac{6}{N^2}\frac{M}{M+m}X_0^{-3.0}(e)\right), \quad C_{22} = -\frac{9}{N^2}\frac{D_{\mathrm{r}}}{mR^2}\frac{M}{M+m}X_N^{-3.2}(e). \tag{2}$$

Table 2. Coefficients of the second harmonic of the gravitational potential of the hydrostatic and ellipsoidal models of satellites.

MP	*Satellite*	J_2	C_{22}	J_2/C_{22}
Ju	Ganymede	2.009×10^{-4}	0.603×10^{-4}	3.33
Sa	Titan	0.382×10^{-4}	0.114×10^{-4}	3.35
Ju	Callisto	0.539×10^{-4}	0.161×10^{-4}	3.35
Ju	Io	2.086×10^{-3}	0.626×10^{-3}	3.33
Ju	Io (ell)	1.986×10^{-3}	0.622×10^{-3}	3.19
Ea	Moon	2.011×10^{-4}	0.201×10^{-4}	9.99
Ju	Europa	0.532×10^{-3}	0.159×10^{-2}	3.34
Ne	Triton	0.641×10^{-3}	0.193×10^{-3}	3.33
Ur	Titania	1.796×10^{-4}	0.538×10^{-4}	3.34
Sa	Rhea	0.672×10^{-3}	0.201×10^{-3}	3.34
Ur	Oberon	0.749×10^{-4}	0.224×10^{-4}	3.34
Sa	Iapetus	1.650×10^{-6}	0.494×10^{-6}	3.33
Ur	Umbriel	0.791×10^{-3}	0.238×10^{-3}	3.32
Ur	Ariel	2.132×10^{-3}	0.640×10^{-3}	3.33
Ur	Ariel (ell)	1.247×10^{-3}	0.553×10^{-3}	2.26
Sa	Diona	1.904×10^{-3}	0.571×10^{-3}	3.33
Sa	Tethys	4.830×10^{-3}	1.444×10^{-3}	3.35
Sa	Tethys (ell)	4.597×10^{-3}	1.402×10^{-3}	3.28
Sa	Enceladus	0.847×10^{-2}	0.254×10^{-2}	3.34
Sa	Enceladus (ell)	1.247×10^{-2}	0.361×10^{-2}	3.45
Ur	Miranda	6.825×10^{-3}	2.048×10^{-3}	3.33
Ur	Miranda (ell)	7.496×10^{-3}	2.646×10^{-3}	2.83
Ne	Proteus (ell)	2.289×10^{-2}	4.881×10^{-3}	4.69
Sa	Mimas	2.093×10^{-2}	0.628×10^{-2}	3.33
Sa	Mimas (ell)	2.251×10^{-2}	6.574×10^{-3}	3.42
Sa	Hyperion	2.286×10^{-4}	0.684×10^{-4}	3.34
Sa	Hyperion (ell)	1.335×10^{-1}	3.196×10^{-2}	4.18
Sa	Phoebe (ell)	2.714×10^{-2}	4.657×10^{-3}	5.83
Sa	Janus	3.640×10^{-2}	1.085×10^{-2}	3.36
Sa	Janus (ell)	8.249×10^{-2}	1.971×10^{-3}	41.9
Ju	Amalthea	2.601×10^{-2}	0.780×10^{-2}	3.34
Ju	Amalthea (ell)	1.816×10^{-1}	7.962×10^{-2}	2.28
Sa	Epimetheus (ell)	4.959×10^{-2}	2.468×10^{-2}	2.01
Sa	Prometheus (ell)	1.551×10^{-1}	0.593×10^{-2}	26.2
Sa	Pandora (ell)	1.705×10^{-1}	3.058×10^{-2}	5.58
Sa	Atlas (ell)	1.028×10^{-1}	0.873×10^{-2}	11.8
Ma	Phobos	6.860×10^{-2}	1.995×10^{-2}	3.44
Ma	Phobos (ell)	0.990×10^{-1}	2.663×10^{-2}	3.72
Sa	Telesto (ell)	2.146×10^{-1}	2.740×10^{-2}	7.83
Ju	Adrastea (ell)	1.382×10^{-1}	3.382×10^{-2}	4.09
Sa	Calypso (ell)	2.111×10^{-1}	8.214×10^{-2}	2.57
Ma	Deimos	6.160×10^{-2}	1.785×10^{-2}	3.45
Ma	Deimos (ell)	8.290×10^{-2}	2.399×10^{-2}	3.46

The case $N = 2$ corresponds to synchronous motion of the satellite and $N = 3$ to the case of Mercury's resonant motion. For the considered bodies the condition of commensurability for the angular rotation velocity ω and the mean orbital motion n is $N\omega = 2n$. In equation (2), m and R are the mass and mean radius respectively of the satellite and M is the mass of the central planet. $X_0^{-3.0}(e)$ and $X_N^{-3.2}(e)$ are the Hanzen coefficients depending only on the eccentricity e of the satellite orbit. D_r is an elastic parameter having the dimensions of moment of inertia and characterizing the satellite deformation due to its rotation. This parameter is calculated for a concrete model of density distribution of satellite by the formulae given by Ferrandiz and Getino (1993). This coefficient can also be evaluated from the formula

$$D_\mathrm{r} = -\frac{1}{9} k_2 \frac{m+M}{m} \left(\frac{\omega}{n}\right)^2 \left(\frac{R}{a}\right)^3 m R^2 \quad \text{or} \quad D_\mathrm{r} = -\frac{1}{9G} k_2 \frac{m}{M} \omega^2 R^5, \tag{3}$$

which follows from the expression for the elastic parameter characterizing tidal deformation D_t of an elastic satellite (Moritz and Muller, 1987) and its relation to the parameter D_r:

$$D_t = \frac{1}{6} k_2 m R^2 \frac{M}{m} \left(\frac{R}{a}\right)^3, \quad D_t = -\frac{3}{2}\left(\frac{n}{\omega}\right)^2 \frac{M}{m+M} D_r. \tag{4}$$

For Titan models I and II and for Mercury model (II), the static ratios J_2/C_{22} are obtained by neglecting the non-equilibrium components of gravitational coefficients. The obtained values of ratio J_2/C_{22} for the Moon and Titan (3.39 and 3.34) are close to similar ratios J_2/C_{22} of the many synchronous satellites of the Solar System (Table 2). They are in good agreement with the values of these ratios obtained for Peale's equilibrium models of synchronous satellites and, for some, their ellipsoidal homogeneous models (Table 2). Formally all ellipsoidal models of satellites, presented in the list in Table 2, can be separated into three groups:

1. Io (3.19), Tephys (3.28), Mimas (3.42), Enceladus (3.45), Deimos (3.46), Phobos (3.72);
2. Epimetheus (2.01), Arial (2.26), Amalthea (2.28), Callipso (2.57), Miranda (2.83);
3. Adrastea (4.09), Hyperion (4.18), Proteus (4.69), Pandora (5.58), Pheba (5.83).

Taking into account the uncertainness in sizes of ellipsoidal models we can assert that all values of J_2/C_{22} for equilibrium and for ellipsoidal models of mentioned satellites are coordinated. Only Telesto (7.83), Atlas (11.8), Prometheus (26.2), Janus (41.9) are characterized by large values of parameter J_2/C_{22}.

1.3 Tidal variations of gravitational fields of the Moon, Mercury and Titan

Above we have discussed the constant components of the gravitational parameters J_2 and C_{22}. Here we also present general formulae for variations in the coefficients of the second harmonic of the gravitational potential of synchronous satellites ($N = 2$) and Mercury ($N = 3$):

$$\begin{gathered} \delta J_2 = \sum_{\sigma=1}^{\infty} J_2^{(\sigma)} \cos(\sigma M), \quad \delta C_{22} = \sum_{\sigma=1}^{\infty} C_{22}^{(\sigma)} \cos(\sigma M), \quad \delta S_{22} = \sum_{\sigma=1}^{\infty} S_{22}^{(\sigma)} \sin(\sigma M), \\ J_2^{(\sigma)} = K_2 X_\sigma^{3.0}(e), \quad C_{22}^{(\sigma)} = S_{22}^{(\sigma)} = \frac{1}{2} K_2 \left[X_{\sigma+N}^{3.2}(e) + X_{N-\sigma}^{3.2}(e)\right]. \end{gathered} \tag{5}$$

Here $X_\sigma^{-3.0}$ and $X_\sigma^{-3.2}$ are known eccentricity functions. The parameter K_2 depends on the elastic properties of satellite and is expressed through the Love number k_2 by the formula

$$K_2 = 3k_2 \frac{m}{M} \left(\frac{R}{a}\right)^3, \tag{6}$$

where a is a major semiaxis of the elliptical orbit of the satellite and R is the mean radius of the satellite.

Trigonometric series in (5) are situated by multiples of mean orbital anomaly $M = nt + M_0$ ($M_0 = M(0)$ is an initial value). Using the values of necessary parameters of the considered celestial bodies presented in Table 3 on the base of formulae (5) and (6), we obtain the following

Table 3. Parameters for models of the Moon, Mercury and Titan.

	Value for the following		
Parameter	*Titan*	*Mercury*	*Moon*
N	2	3	2
e	0.028 87	0.205 614	0.054 9
$X_0^{-3.2} = (1-e^2)^{-3/2}$	1.001 252	1.066 941	1.004 538
$X_N^{-3.2}$	0.997 917	0.654 214	0.992 472
$M/(m+M)$	0.999 763	1	0.987 850
m/M	0.000 237	1	0.012 3
J_2/C_{22}	3.34	7.85	3.39

expressions for tidal variations of the gravitational coefficients C_{20}, C_{22}, S_{22} of potential of considered celestial bodies:

Titan:

$$\begin{aligned}(\delta J_2)_{periodic} &= 10^{-7}[4.2869\cos M + 0.1856\cos(2M) + 0.0079\cos(3M)\\ &\quad + 0.0003\cos(4M)],\\ (\delta C_{22})_{periodic} &= 10^{-7}[4.2738\cos M + 0.3497\cos(2M) + 0.0209\cos(3M)\\ &\quad + 0.0011\cos(4M)],\\ (\delta S_{22})_{periodic} &= 10^{-7}[5.7013\sin M + 0.3497\sin(2M) + 0.0209\sin(3M)\\ &\quad + 0.0011\sin(4M)].\end{aligned} \tag{7^{T}}$$

Obtained results describe remarkable changes in the dynamical structure of Titan due to the gravitational influence of Saturn. This conclusion is in agreement with the prediction of high endogenous activity of Titan (Barkin, 2002). Main variations of the coefficients C_{22} and S_{22} with period of orbital motion of Titan (15.544 days, (Sinclair, 1977)) can be interpreted on the plane $((\delta C_{22})_{periodic}, (\delta S_{22})_{periodic})$ by ellipse with eccentricity $e_t = 0.6619$.

For comparison, here we present similar expressions of periodic variations of gravitational coefficients of second harmonic: C_{20}, C_{22}, S_{22} for the Moon. Base parameters for calculations are presented in Table 3.

The Moon:

$$\begin{aligned}(\delta J_2)_{periodic} &= [0.7964\cos M + 0.0661\cos(2M) + 0.0054\cos(3M)\\ &\quad + 0.0004\cos(4M)]\cdot 10^{-8},\\ (\delta C_{22})_{periodic} &= [0.7875\cos M + 0.1237\cos(2M) + 0.0142\cos(3M)\\ &\quad + 0.0015\cos(4M)]\cdot 10^{-8},\\ (\delta S_{22})_{periodic} &= [1.0520\sin M + 0.1237\sin(2M) + 0.0142\sin(3M)\\ &\quad + 0.0015\sin(4M)]\cdot 10^{-8}.\end{aligned} \tag{7^{M}}$$

The ellipse of variations of the coefficients C_{22} and S_{22} with anomalistic period of the Moon (27.212 days) on the plane $((\delta C_{22})_{periodic}, (\delta S_{22})_{periodic})$ is characterized by eccentricity $e_t = 0.6630$.

Also, remarkable variations in the gravitational field of Mercury are described by the following formulae:

$$
\begin{aligned}
(\delta J_2)_{periodic} &= 10^{-8}[2.6991\cos M + 0.8203\cos(2M) + 0.2450\cos(3M) \\
&\quad + 0.0723\cos(4M) + 0.0211\cos(5M) + 0.0061\cos(6M) \\
&\quad + 0.0018\cos(7M) + 0.0005\cos(8M) + 0.0002\cos(9M)], \qquad (7)^{\mathrm{Me}} \\
(\delta C_{22})_{periodic} &= 10^{-8}[5.0907\cos M + 0.1489\cos(2M) \\
&\quad + 0.2220\cos(3M) + 0.0816\cos(4M) + 0.0283\cos(5M)], \\
(\delta S_{22})_{periodic} &= 10^{-8}[-2.3734\cdot\sin M + 1.0012\sin(2M) \\
&\quad + 0.2220\sin(3M) + 0.0816\sin(4M) + 0.0283\sin(5M)].
\end{aligned}
$$

Formulae $(7)^{\mathrm{T}}$, $(7)^{\mathrm{M}}$ and $(7)^{\mathrm{Me}}$ describe the periodic variations of the main coefficients of gravitational potentials of Titan, the Moon and Mercury with corresponding orbital periods.

2 ROTATION

2.1 Forced librations of the Moon, Mercury and Titan

Let us study the librations of Titan (and its native bodies, namely the Moon and Mercury) using the simple model of plane librations of a non-spherical rigid body on an unperturbed elliptical orbit. Perturbations in the rotation of the resonant satellite are obtained as a particular case from a more general treatment of the problem about plane translatory–rotary motion of a rigid satellite (Barkin, 1976, 1979a, b, c). So perturbations with the angle rotation g and angular velocity ω are determined by the following formulae:

$$
\begin{aligned}
&\delta g = \sum_{\sigma=1}^{\infty} g_\sigma \ \sin(\sigma M), \quad \frac{\delta\omega}{n} = \sum_{\sigma=1}^{\infty} \omega_\sigma \ \cos(\sigma M), \\
&g_\sigma = -6\frac{C_{22}}{I}\left(X_{\sigma+N}^{-3.2} - X_{N-\sigma}^{-3.2}\right)\frac{1}{\sigma^2}, \quad \omega_\sigma = \sigma g_\sigma.
\end{aligned} \tag{8}
$$

Equations ((8)) were obtained for orbital–rotational resonances: $Nn = 2\omega$ (the case $N = 2$ corresponds to synchronous motions and the case $3n = 2\omega$ to the motion of Mercury). For construction of perturbations, the Poincare theory of periodic solutions has been used in papers by the present author (Barkin, 1976, 1979a, b, c). So the coefficients of perturbations depend on the eccentricity of the orbit and proportional to C_{22}/I. From equations ((8)) it follows that the ratios of any amplitudes in the perturbations δg (and $\delta\omega$) for a concrete satellite depend only on the eccentricity of orbit. For example

$$
A_{1/\sigma} = \frac{X_{N+1}^{-3.2} - X_{N-1}^{-3.2}}{X_{\sigma+N}^{-3.2} - X_{N-\sigma}^{-3.2}}\,\sigma^2. \tag{9}
$$

Some of the values of the ratios (9) are presented later in Table 4 for perturbations in the librations of the Moon, Mercury and Titan.

Table 4. Amplitudes (and their ratios) of the plane librations.

Parameters, amplitudes	*Value for the following*		
	Mercury (II)	*Moon*	*Titan (III)*
C_{22}	12.94×10^{-6}	22.3×10^{-6}	49.6×10^{-6}
I	0.34	0.392	0.35
e	0.205614	0.055	0.02887
$A_{1/2}$	−9.4850	34.258	65.218
$A_{1/3}$	96.248	677.80	2457.4
$A_{1/4}$	−477.76	11494	79788
g_1	$-53''6700/2$	$-15''1979$	$-20''1130$
g_2	$5''6599/2$	$-0''4469$	$-0''3084$
g_3	$-0''5579/2$	$-0''0228$	$-0''0082$
g_4	$-0''1131/2$	$-0''0013$	$-0''0003$
ω_1	$1.7347 \times 10^{-4}/2$	-0.7368×10^{-4}	-0.9751×10^{-4}
ω_2	$-0.3659 \times 10^{-4}/2$	-0.0433×10^{-4}	-0.0299×10^{-4}
ω_3	$-0.0541 \times 10^{-4}/2$	-0.0033×10^{-4}	-0.0012×10^{-4}
ω_4	$-0.0146 \times 10^{-4}/2$	-0.0003×10^{-4}	-0.0001×10^{-4}

In the case of Mercury the first terms of perturbations (8) can be presented in the following more detailed form:

$$\begin{aligned}\delta g = \frac{C_{22}}{I}\Bigg[&6\left(1 - 11e^2 + \frac{959}{48}e^4 - \frac{3641}{288}e^6 + \frac{11\,359}{2880}e^8\right)\sin\ M \\ &- \frac{3}{4}e\left(1 + \frac{421}{12}e^2 - \frac{32\,515}{384}e^4 + \frac{2\,186\,863}{32\,256}e^6 - \frac{428\,399\,713}{15\,482\,880}e^8\right)\sin(2M) \\ &- \frac{1}{24}e^4\left(533 - \frac{13\,827}{10}e^2 + \frac{728\,889}{560}e^4\right)\sin(3M) \\ &+ \frac{1}{128}e^3\left(1 - \frac{57\,073}{20}e^2 + \frac{7\,678\,157}{960}e^4 - \frac{298\,080\,597}{34\,560}e^6\right)\sin(4M) + \cdots\Bigg] \qquad (10)\end{aligned}$$

The perturbations in rotational motion described above have been obtained for corresponding rigid non-spherical models of Mercury (II), the Moon and Titan (III). But the formulae easily lets us take into account the influence on librations of liquid core. In this case, amplitudes of perturbations (8) will be increased proportionately to the ratio of polar moments of inertia of the full body and its mantle C/C_m. So the expected value of this ratio for Mercury is $C/C_m = 2$ (Peale *et al.*, 2002). It means that, due to the existence of liquid core amplitudes of Mercury, the librations presented in Table 4 are doubled.

2.2 The Cassini motions of the Moon, Titan and Mercury

Let us consider now the space rotational motion of the satellite in a central gravitational field assuming that the satellite moves in an elliptical orbit with precessing line of nodes and with a moving pericentre of orbit (Colombo, 1966; Goldreich and Peale, 1966; Beletskij, 1972; Peale, 1973; Ward, 1975; Barkin, 1978, 1979a, 1983; Peale, 1979). We shall describe in Andoyer variables the rotational motion of a satellite under gravitational moments of the central body and shall use the equations of motion in the Hamiltonian form. As it has been shown in the paper by the present author (Barkin, 1978) in the particular case of a precessing circular orbit the right-hand sides of equations will be periodic functions of time and the Poincare theory of periodic solutions can be applied to study the resonant synchronous rotation of a satellite.

The existence of a definite class of periodic solutions of the problem has been proved. The corresponding generating periodic solutions obtained directly from analytical conditions of the existence of periodic solutions explain the main regularities of the satellite motions which can be called the generalized Cassini laws. The generating values of angular Andoyer variables

$$l_0 = 0, \quad g_0 = \frac{\pi}{2}, \quad h_0 = \pi, \quad \theta_0 = \frac{\pi}{2}, \quad \rho_0 = \rho_0(\chi, \delta) \tag{11}$$

and the condition of commensurability for the unperturbed angular velocity ω of the satellite and its mean orbital motion given by

$$Nn = 2\omega \tag{12}$$

describe the following regularities in a synchronous motion of satellites ($N = 2$).

(i) The satellite rotates with a permanent angular velocity equal to its mean orbital motion about its polar axis of inertia corresponding to the maximal principal moment B of inertia.

(ii) The largest axis of the satellite ellipsoid of inertia corresponding to minimal moment c of inertia is directed to the centre of planet at the moment of crossing of the nodes of the orbit.

(iii) The ascending node of the orbit plane on the ecliptic coincides with the descending node of the intermediate plane orthogonal to the angular momentum vector of the satellite. This intermediate plane coincides with the equatorial plane of the ellipsoid of inertia of the satellite. The angles between the orbital plane, intermediate plane and ecliptic plane are constant.

(iv) The vectors of angular velocity and angular momentum coincide and form an angle $\rho_0 = \rho_0(\chi, \delta)$ with the normal to the plane of the satellite's orbit; then the satellite crosses the orbit's ascending node. The value of this angle is determined from the equation (Barkin, 1978, 1979a)

$$-\chi \sin(\rho_0 - i) + (4 - 3\delta)\sin(2\rho_0) - 2\delta \sin \rho_0 = 0 \tag{13}$$

and depends on the two following parameters of the problem:

$$\chi = \frac{16}{3}\frac{n_\Omega}{n}\frac{I}{C_{20} + 2C_{22}}, \quad \delta = \frac{4C_{22}}{C_{20} + 2C_{22}} \tag{14}$$

(χ is a measure of the mobility of the orbit plane and δ is a measure of the triaxial nature of the ellipsoid of inertia; i is a constant inclination of the orbit plane). Here n_Ω is the constant angular velocity of orbit precession. Equations (13) and (14) have been used to determine the angles of inclination of the rotation axes of the Moon (Barkin, 1978) and Titan. The necessary parameters of the motion and calculated values of inclinations are presented in Table 4. For Titan the inclinations ρ_0 have been determined for three models of non-sphericity of this satellite.

In the vicinity of the discovered generating Cassini motions of the Moon and Titan the perturbed periodic motions of these resonant satellites can be constructed.

Another approach was realized for the study of Mercury's resonant motion on an elliptical precessing orbit based on the application of average methods in the style of Beletskij (1973). This approach of course can be applied for analysis of synchronous rotations. Equations of motion are also described in Andoyer variables but in this model they are not periodic with respect to the time, and the Poincare theory is not applicable directly. However, specially

averaged equations determine some class of stationary motions which can be illustrated as a generalization of the Cassini laws. In this case the main condition for a stationary state is the Beletskij equation for inclination ρ_0 which can be written in the following form:

$$\cos i + \varepsilon_1 \frac{\cos \rho_0}{\sin \rho_0} \sin i + \frac{2n}{INn_\Omega} \left[-\cos \rho_0 \, C_{20} X_0^{-3.0} + (1 + \cos \rho_0) \, C_{22} X_3^{-3.2} \varepsilon_2 \right] = 0, \quad (15)$$

where $\varepsilon_1 = \cosh_0 = \pm 1$, $\varepsilon_2 = \cos[2(g_0 + h_0 - \omega_0)] = \pm 1$. This equation is equivalent to the equation derived by Peale (1979) (see also Rambaux and Bois (2004)).

In the case of small values of the inclination ρ_0, equation (15) is easily solved:

$$\rho_0 = \frac{-\varepsilon_1 \sin i}{\cos i + (2n/IN\omega) \left[-C_{20} X_0^{-3.0}(e) + 2C_{22} X_3^{-3.2}(e) \varepsilon_2 \right]}. \quad (16)$$

The period of orbital motion of Titan is 15.945 days and the period of regressive precession of the line of nodes of the orbit plane on the Laplacian plane is 720.865 years (Peale, 1973). The inclination of the plane of Titan's orbit with respect to the equator plane of Saturn is about $i = 0.308$ and the eccentricity $e = 0.028\,87$. In equation (16) we have

$$I = 0.35, \ \frac{n_\Omega}{n} = -0.605\,59 \times 10^{-4}, \ X_0^{-3.0} = (1 - e_0^2)^{-3/2} = 1.001\,252,$$
$$X_2^{-3.2} = 1 - \frac{5}{2} e_0^2 + \frac{13}{16} e_0^4 = 0.997\,917,$$

and for values $\varepsilon_1 = \cosh_0 = -1$, $\varepsilon_2 = \cos[2(g_0 + h_0 - \omega_0)] = 1$ and for accepted parameters of the models of gravitational field (Table 2) we obtain $\rho = 5.553$, $\rho = 4.043$ and $\rho = 0.076$ for models I, II and III respectively from Table 1. This means that the mean angles between the normal to the equator plane of Saturn and the axis of rotation of Titan can be about $\rho - i = 5.245$, $\rho - i = 3.735$ and $\rho - i = -0.232$ respectively.

Interpretations of the Cassini motions of the Moon and Titan are identical and are different only in the concrete values of the parameters. The Cassini–Colombo motion of Mercury is different from those mentioned above.

The period of orbital motion of Mercury is 87.969 days and the period of progressive precession of the line of nodes of the orbit plane on the Laplacian plane is 278 898 years (Colombo, 1966; Peale, 1973). The inclination of the plane of Mercury's orbit with respect to the ecliptic plane (the Laplace plane) is about $i = 7.002\,880\,6$ and eccentricity $e = 0.205\,614$. In equation (16) we have

$$\frac{n_\Omega}{n} = 0.863\,563 \times 10^{-6}, \quad X_0^{-3.0} = 1.063\,65, \quad X_3^{-3.2} = 0.653\,797\,4, \quad I = 0.34,$$

and for values $\varepsilon_1 = \cosh_0 = 1$, $\varepsilon_2 = \cos[2(g_0 + h_0 - \omega_0)] = 1$ and for accepted parameters of the models of gravitational field (Table 2) we obtain $\rho_0 = 0.000\,467\,3$. This means that the mean angle between the normal to the ecliptic plane and the rotation axis of Mercury is $\rho + i = 7.003\,347\,9$. All established characteristics of the Cassini motions of the Moon, Titan and Mercury are summarized in Table 5.

2.3 Resonant librations

In accordance with the general properties of the resonant motion the synchronous satellites execute some free oscillations in the neighbourhood of above-described stationary periodic

Table 5. Periods of resonant librations of the Moon, Titan and Mercury.

	Value for the following			
Parameters	*Moon*	*Mercury (I)*	*Titan (I)*	*Titan (II)*
$n_1 : n$	1:1	3:2	1:1	1:1
e	0.054 9	0.205 61	0.028 87	0.028 87
$X_0^{-3.0}$	1.004 55	1.063 65	1.001 25	1.001 25
$X_N^{-3.2}$	0.992 45	0.653 80	0.997 92	0.997 92
$X_N^{-3.0}$	0.006 78	0.029 36	0.001 88	
C_{20}	-202.7×10^{-6}	-60×10^{-6}	-38.2×10^{-6}	-38.2×10^{-6}
C_{22}	22.3×10^{-6}	10×10^{-6}	11.4×10^{-6}	12.0×10^{-6}
δ	−0.564 11	1.000	−2.961 04	−3.380 28
I	0.392	0.34	0.35	0.35
n_Ω/n	-3983.2×10^{-6}	$0.863\,56 \times 10^{-6}$	-60.559×10^{-6}	-60.559×10^{-6}
χ	52.64	−0.040 30	7.340 49	7.960 81
i	5.6953	7.0029	0.308	0.308
ρ	6.1342	0.000 467	5.553	4.043
T_{orb} (days)	27.212	87.969	15.946	15.946
T_{l} (years)	74.713	1012.8 (e)	251.92	258.51
	74.998 (e)	1066 (RamBo)	251.33 (e)	
T_{g} (years)	2.858	16.087 (e)	2.213	2.155
	2.871 (e)	15.847 (RamBo)	2.216 (e)	2.157 (e)
T_{h} (years)	22.157		1272.1	1067.2
	20.113 (e)	18.657 (e)		
T_{Eul} (years)	147.70	1447.6	498.53	498.53
		964.88 (RamBo)		

(The values of angles $\rho = \rho_0$ and i are given in degrees)

solutions of the problem (in the neighbourhood of the Cassini motions). In this section we shall evaluate the periods of these librations. It is worth remarking that the amplitudes and phases of these librations can be determined only on the basis of observational data. Only for the Moon were the above-mentioned characteristics determined by laser ranging. In the case of synchronous satellites, neglecting small eccentricity effects we can present analytical expressions for the periods of resonant librations in the following forms (Barkin, 1979a, 1983):

$$
\begin{aligned}
T_{\text{g}} &= \frac{2\pi}{3^{1/2} n (1+\cos\rho)\,(C_{22}/I)^{1/2}}, \\
T_{\text{l}} &= \frac{16\pi I}{\left[\begin{array}{l}[(C_{20}-2C_{22})(C_{20}+2C_{22})]^{1/2} \\ \qquad \times n[(15\cos^2\rho - 6\cos\rho - 1)(5+6\cos\rho+21\cos^2\rho)]^{1/2}\end{array}\right]}, \\
T_{\text{h}} &= \frac{8\pi I}{(-C_{20}-2C_{22}) n \{-\chi \sin i/\sin\rho[\chi\cos(\rho-i) - 2(4-3\delta)\cos(2\rho) + 2\delta\cos\rho]\}^{1/2}}.
\end{aligned}
\tag{17}
$$

The values of the periods (17) for resonant librations of the Moon and Titan are presented in Table 5. The periods of the Moon are in good agreement with observational data (periods given in square brackets). An important phenomenon is observed in the free and forced pole motion of resonant satellites. In reality, because $\rho_0 = 0$ the period of pole perturbed motion consists of only half the period of Euler pole motion (in the absence of perturbing bodies):

$$
T_{\text{l}} = \frac{2\pi I}{2[(C_{20}-2C_{22})(C_{20}+2C_{22})]^{1/2} n} = \frac{T_{\text{Eul}}}{2}. \tag{18}
$$

Because of the small values of the angle ρ_0 this tendency will be safe.

If we use another approach (on the basis of averaging methods) for periods of resonant librations, we can use the formulae obtained in Barkin's (1983) dissertation see also Barkin (1979c):

$$T_1 = \frac{T_0}{\kappa[(C_{20} - 2C_{22})(C_{20} + 2C_{22})]^{1/2}[\Lambda_1(\rho)\Lambda_2(\rho)]^{1/2}}, \kappa = \frac{3}{4N}\left(1 + \frac{m}{M}\right),$$

$$\begin{aligned} \Lambda_{1,2}(\rho) = {} & 2(2 - 3\sin^2\rho)X_0^{-3.0} + \frac{2}{3}N^2\left(1 + \frac{m}{M}\right) \pm \left[\sin^2\rho\, X_N^{-3.0}\right. \\ & \left. + (X_N^{-3.2} + X_{-N}^{-3.2})(1 + \cos^2\rho) + 2\cos\rho\,(X_N^{-3.2} - X_{-N}^{-3.2})\right], \end{aligned} \tag{19}$$

$$T_g = \frac{T_0}{[3(C_{22}/I)\varepsilon_2(1 + \cos\rho)^2 X_N^{-3.2}]^{1/2}}. \tag{20}$$

$$T_h = \frac{T_0}{\left[\begin{array}{c}\{\varepsilon_1(n_\Omega/n)\sin i\,\sin\rho_0[\varepsilon_1(n_\Omega/n)\sin i\cos e\, c^3\rho_0 \\ + (1/I)(-C_{20}X_0^{-3.0} + C_{22}X_N^{-3.2}\varepsilon_2)]\}^{1/2}\end{array}\right]}. \tag{21}$$

In the general case of resonance (equation (12)) and for small values of inclination ρ from equation (19) we obtain the relation between the Eulerian period and the resonant period similar to that in equation (18):

$$\begin{aligned} T_1 &= \frac{T_{\mathrm{Eul}}}{\kappa[\Lambda_1(0)\Lambda_2(0)]^{1/2}}, \quad \kappa = \frac{3}{4N}\left(1 + \frac{m}{M}\right), \\ \Lambda_{1,2}(0) &= 4(X_0^{-3.0} \pm X_N^{-3.2}) + \frac{2}{3}N^2\left(1 + \frac{m}{M}\right). \end{aligned} \tag{22}$$

Equation (19) defines the period of perturbed pole motion, equation (20) the period of librations in longitude and equation (21) the period of librations of the angular momentum vector. Let us remark that Euler periods in the case of small amplitudes of the pole motion of synchronous satellites as given by equation (18) and (22) are approximately twice the corresponding perturbed periods (19) (see Table 5). The values for the periods of resonant librations calculated from, equations (19)–(21) are indicated by (e). The values for the periods of the Moon's resonant librations are in good agreement with their values obtained on the base of laser ranging data. The values for the periods of Mercury's resonant librations are in good agreement with their values obtained by numerical simulations of the equations for the translatory–rotary motion of Mercury (Rambaux and Bois, 2004) (indicated in Table 4 as (RamBo)).

3 CONCLUSIONS

The results obtained describe some new aspects and particularities of the rotation of Titan in comparison with similar characteristics of the resonant rotations of the Moon and Mercury. In particular, these results are of important interest for effective realization of future missions to the Moon, Mercury and Titan (Anselmi and Spohn, 2001; Milani *et al.*, 2001; Spohn *et al.*, 2001) and for study of the inner structure of these celestial bodies. A prediction of the high endogenous activity of Titan has been given earlier on the base of the shell dynamics mechanism (Barkin, 2002). For the considered model, Titan occupies the third position with respect to the value of endogenous energy (power) after the very active satellites Io and Europa. This means that sufficiently high tectonic activity of Titan can be observed in reality. This conclusion is very important and some confirmation of this be obtained from the data of the Cassini–Huygens expedition to Saturn in 2005.

Acknowledgements

Barkin's work was partially funded by Grant SAB2000-0235 from the Secretaria de Estado de Educacion y Universidades and by Grant 02-05-64176, from the Russian Foundation for Basic Research.

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