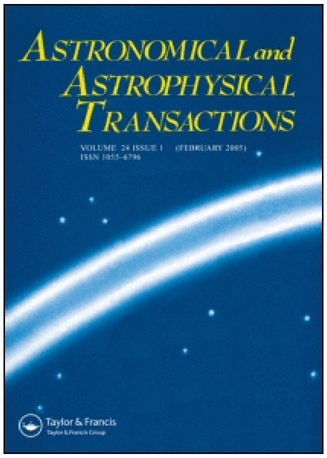

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GENERALIZED FORMS OF THE JACOBI EQUATION FOR STELLAR SYSTEMS

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An application of the Jacobi equation to the dynamics of gravitating systems is considered. Generalized forms of the Jacobi equation for open stellar systems and for systems with physical collisions of stars are obtained. Some consequences of the derived equations are given.

Keywords: Stellar dynamics; Open systems; Collisions of stars

1 INTRODUCTION

Poincaré (1913) and Eddington (1916) were the first to apply the following Jacobi (1884) equation to stellar systems:

$$\frac{1}{2}\frac{\mathrm{d}^2 J}{\mathrm{d}t^2} = 2K + W, \tag{1}$$

where J is the moment of inertia of the system, K is the kinetic energy, and W is the potential energy of the system. This equation constitutes the basis of an interesting approach to the dynamics of stellar systems in the subsequent works of Huang (1954), Linden-Bell (1967), Chandrasekhar and Elbert (1972) and Ferronsky *et al.* (1987). However, real stellar systems are, in essence, objects of non-stationary composition (Omarov, 1975). Equation (1) no longer holds for a variable number of members of a system. This question was considered by Ivanov and Omarov (1967). They found that, for gravitating systems that lose their members through a fairly clear spherical surface of radius $R(t)$, equation (1) can be generalized to the following equation:

$$\frac{1}{2}\frac{\mathrm{d}^2 J}{\mathrm{d}t^2} = \frac{1}{2}R^2\frac{\mathrm{d}^2 M}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}R}{\mathrm{d}t} + U_{\mathrm{R}}\right)R\frac{\mathrm{d}M}{\mathrm{d}t} + 2K + W, \tag{2}$$

where U_{R} is the mean value of the radial velocities of bodies, leaving the system through the external surface. In the work of Osipkov (1983) an application of equation (2) to the dynamics of stellar clusters was considered, taking into account the evaporation of stars.

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The matter exchange processes can occur not only through the boundary of the open system but also at each point of such a system (Klimontovich, 1995). For instance, we can refer to the model of the system dissipating over all the volume owing to asymmetric explosions of supernova stars (Omarov, 1991). A generalization of equation (2) is required for such open stellar systems. Variation in the composition of the gravitating system as a discrete set of point masses can occur not only because of its openness. In particular, physical collisions of stars play a significant role in the nuclei of galaxies and dense stellar clusters (Rees, 1982; Saslaw, 1989). One of the possible scenarios of the evolution of objects is a 'sticking' of its members and the formation of more massive stars. Such a case of a system of discrete point masses of non-stationary composition should be distinguished in the equation of gross dynamics of gravitating systems. Generalized forms of the Jacobi equation (1) which are given below will be the response to the questions asked above.

2 THE GENERALIZED JACOBI EQUATION FOR A GRAVITATIONAL SYSTEM OF FLUCTUATING COMPOSITION

We consider a subsystem of particles that continuously changes its composition, the particles being distinguished by a fairly arbitrary characteristic criterion (a system of fluctuating composition). For an arbitrary additive parameter A (not scalar necessarily) of such a system, one may deduce the following balance equation (Omarov, 1990):

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\bar{A}_\alpha\left(\frac{\partial n}{\partial t}\right)_\alpha + \bar{A}_\beta\left(\frac{\partial n}{\partial t}\right)_\beta + \sum_i \frac{\mathrm{d}A_i}{\mathrm{d}t}, \quad A = \sum_i A_i \tag{3}$$

where

$$\begin{gathered}\bar{A}_\alpha = \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_\alpha}\sum_j A_j(t+\Delta t)\right), \quad \bar{A}_\beta = \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_\beta}\sum_k A_k(t+\Delta t)\right), \\ \left(\frac{\partial n}{\partial t}\right)_{\alpha,\beta} = \lim_{\Delta t\to 0}\left(\frac{\Delta n_{\alpha,\beta}}{\Delta t}\right).\end{gathered} \tag{4}$$

Here the index i identifies the members of the instant composition, and the index $j(k)$ identifies the particles leaving the system's composition in quantity Δn_α (adding to the composition in quantity Δn_β) for a fairly short positive time interval $[t, t+\Delta t]$. using an analogous equation for the quantity $\sum_i (\mathrm{d}A_i/\mathrm{d}t)$, we have

$$\frac{\mathrm{d}^2A}{\mathrm{d}t^2} = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\bar{A}_\alpha\left(\frac{\partial n}{\partial t}\right)_\alpha - \bar{A}_\beta\left(\frac{\partial n}{\partial t}\right)_\beta\right] - \overline{\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)}_\alpha\left(\frac{\partial n}{\partial t}\right)_\alpha + \overline{\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)}_\beta\left(\frac{\partial n}{\partial t}\right)_\beta + \sum_i \frac{\mathrm{d}^2A_i}{\mathrm{d}t^2}, \tag{5}$$

where

$$\overline{\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)}_\alpha = \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_\alpha}\sum_j \frac{\mathrm{d}A_j}{\mathrm{d}t}\right), \quad \overline{\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)}_\beta = \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_\beta}\sum_k \frac{\mathrm{d}A_k}{\mathrm{d}t}\right). \tag{6}$$

The constant mass m is ascribed to all particles. We denote by $(\partial M/\partial t)_\alpha$ the mass of particles taken from the composition of the system in unit time, and by $(\partial M/\partial t)_\beta$ the analogous quantity for particles added to the composition:

$$\left(\frac{\partial M}{\partial t}\right)_\alpha = m\left(\frac{\partial n}{\partial t}\right)_\alpha, \quad \left(\frac{\partial M}{\partial t}\right)_\beta = m\left(\frac{\partial n}{\partial t}\right)_\beta. \tag{7}$$

We introduce the moment of the system of fluctuating composition; $J = \sum_i m_i \boldsymbol{r}_i$. On the basis of equation (5) we obtain the equation

$$\frac{1}{2}\frac{\mathrm{d}^2 J}{\mathrm{d}t^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\overline{r_\alpha^2}\left(\frac{\partial M}{\partial t}\right)_\alpha - \overline{r_\beta^2}\left(\frac{\partial M}{\partial t}\right)_\beta\right] + \frac{1}{2}\overline{\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)_\alpha}\left(\frac{\partial M}{\partial t}\right)_\alpha - \frac{1}{2}\overline{\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)_\beta}\left(\frac{\partial M}{\partial t}\right)_\beta$$
$$= \tfrac{1}{2}\sum_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} m\boldsymbol{r}_i^2. \tag{8}$$

The quantities $\overline{(\mathrm{d}r^2/\mathrm{d}t)_\alpha}$ and $\overline{(\mathrm{d}r^2/\mathrm{d}t)_\beta}$ given by

$$\overline{\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)_\alpha} = \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_\alpha}\sum_j \frac{\mathrm{d}r_j^2}{\mathrm{d}t}\right), \quad \overline{\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)_\beta} = \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_\beta}\sum_k \frac{\mathrm{d}r_k^2}{\mathrm{d}t}\right), \tag{9}$$

are the mean values of the rates of change in the squares of the radius vectors of the particles that are assumed to be taken from the composition of the system at a given time and to be added to it respectively. The quantities $\overline{r_\alpha^2}$ and $\overline{r_\beta^2}$ defined by

$$\overline{r_\alpha^2} = \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_\alpha}\sum_j r_j^2\right), \quad \overline{r_\beta^2} = \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_\beta}\sum_k r_k^2\right), \tag{10}$$

are the mean values of the squares of the radius vectors of these particles.

For the set of i particles, subject to the force $\boldsymbol{F}_i$, we have

$$\sum_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} m\boldsymbol{r}_i^2 = \sum_i 2m\left[\left(\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t}\right)^2 + \boldsymbol{r}_i\cdot\boldsymbol{F}_i\right] = 4K + 2\sum_i \boldsymbol{r}_i\cdot\boldsymbol{F}_i, \tag{11}$$

where T is the kinetic energy of the system: $K = \frac{1}{2}\sum_i (\mathrm{d}\boldsymbol{r}_i/\mathrm{d}t)^2$. Thus, we have the following equation for a system of fluctuating composition with a fairly arbitrary characteristic used to distinguish in space its members, namely the i particles:

$$\frac{1}{2}\frac{\mathrm{d}^2 J}{\mathrm{d}t^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{r}_\alpha^2\left(\frac{\partial M}{\partial t}\right)_\alpha - \boldsymbol{r}_\beta^2\left(\frac{\partial M}{\partial t}\right)_\beta\right] + \frac{1}{2}\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)_\alpha\left(\frac{\partial M}{\partial t}\right)_\alpha - \frac{1}{2}\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)_\beta\left(\frac{\partial M}{\partial t}\right)_\beta$$
$$= 4K + \sum_i \boldsymbol{r}_i\cdot\boldsymbol{F}_i. \tag{12}$$

One can expand somewhat the form of equation (12) for the case of an open system distinguished by a volume V with boundary S (Omarov, 1986). We denote by $\sigma_\alpha(\boldsymbol{r}, t)$ and $\sigma_\beta(\boldsymbol{r}, t)$ the surface densities on S of the particles taken from the system and added to it respectively and we assume that, in unit time, $(\partial n/\partial t)_\alpha$ particles altogether are taken from the system and $(\partial n/\partial t)_\beta$ added to it. The expressions

$$\tilde{\sigma}_\alpha(\boldsymbol{r},t) = \sigma_\alpha(\boldsymbol{r},t)\frac{(\partial M/\partial t)_\alpha}{(\partial n/\partial t)_\alpha}, \quad \tilde{\sigma}_\beta(\boldsymbol{r},t) = \sigma_\beta(\boldsymbol{r},t)\frac{(\partial M/\partial t)_\beta}{(\partial n/\partial t)_\beta}, \tag{13}$$

give the masses that pass in unit time through unit area of the boundary S into the volume V and out of it respectively. Let the functions $\boldsymbol{u}_\alpha(\boldsymbol{r},t)$ and $\boldsymbol{u}_\beta(\boldsymbol{r},t)$ describe the distributions on S of the velocities of particles taken from the system and added to it respectively. Accordingly, in case of the open system considered here, the general equation (12) becomes

$$\frac{1}{2}\frac{\mathrm{d}^2 J}{\mathrm{d}t^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\oint_S r^2[\tilde{\sigma}_\alpha(\boldsymbol{r},t) - \tilde{\sigma}_\beta(\boldsymbol{r},t)]\,\mathrm{d}S\right] + \oint_S \boldsymbol{r}[\boldsymbol{u}_\alpha(\boldsymbol{r},t)\tilde{\sigma}_\alpha(\boldsymbol{r},t) - \boldsymbol{u}_\beta(\boldsymbol{r},t)\tilde{\sigma}_\beta(\boldsymbol{r},t)]\,\mathrm{d}S = 2K + \sum_i \boldsymbol{r}_i \cdot \boldsymbol{F}_i. \tag{14}$$

One can show that the average of equation (14) over time (Omarov, 1986) gives the virial theorem obtained by Schweitz (1977) for the open system of particles that occupies a volume V with unchanged boundary S.

We now consider another particular type of system of fluctuating composition in the form of a set of gravitating particles i distinguished by the condition that they are self-consistent. In this case we introduce the potential energy of the system:

$$W = -\tfrac{1}{2}G\sum_{\mu\neq\nu}\frac{m_\mu m_\nu}{|\boldsymbol{r}_\nu - \boldsymbol{r}_\mu|}, \tag{15}$$

where G is the gravitational constant, and the indexes μ and ν identify the members of pairs formed from the set of i particles. In an inertial coordinate system we must have

$$\boldsymbol{F}_i = -\mathrm{grad}_{r_i} W, \quad 2K + \sum_i \boldsymbol{r}_i \cdot \boldsymbol{F}_i = 2K + W, \tag{16}$$

so that equation (12) can be put in the form

$$\frac{1}{2}\frac{\mathrm{d}^2 J}{\mathrm{d}t^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\overline{r_\alpha^2}\left(\frac{\partial M}{\mathrm{d}t}\right)_\alpha - \overline{r_\beta^2}\left(\frac{\partial M}{\partial t}\right)_\beta\right] + \frac{1}{2}\overline{\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)_\alpha}\left(\frac{\partial M}{\partial t}\right)_\alpha - \frac{1}{2}\overline{\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)_\beta}\left(\frac{\partial M}{\partial t}\right)_\beta = 2K + W. \tag{17}$$

Equation (17) is a generalization of the Jacobi equation (1) for a gravitating system of fluctuating composition.

Averaging equation (17) over the time, we find that (Omarov, 1986)

$$\frac{1}{2}\left[\overline{\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)_\alpha}\left(\frac{\partial M}{\partial t}\right)_\alpha - \overline{\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)_\beta}\left(\frac{\partial M}{\partial t}\right)_\beta\right]_{\mathrm{av}} = (2T + W)_{\mathrm{av}}. \tag{18}$$

Equation (18) is a generalized form of the virial theorem in stellar dynamics.

3 THE GENERALIZED JACOBI EQUATION FOR A MODEL OF A STELLAR SYSTEM WITH PHYSICAL COLLISIONS OF STARS

As we have noted above, physical collisions of stars play a significant role in the nuclei of a galaxies (Rees, 1982; Saslaw, 1989). One of the possible scenarios of evolution of such objects is a 'sticking' of its members and the formation of more massive stars. Accordingly, one can imagine a multiparticle system in which a background particles of constant mass m are continuously 'sticking' to the given l members ($l = 1, 2, \ldots, N$). These members with continuously increasing masses $m_l(t)$ we shall call l bodies, and they are distinguished by the condition $m_l(t) \gg m$ for times that are interesting to us. Let us ascribe the index i to those members of instantaneous composition of the system whose instantaneous spatial positions are not overlapping with l bodies of mass $m_l(t)$. Thus, our model of a stellar system with continuous physical collisions of its members consists of a subsystem of l bodies with mass $m_l(t)$ and subsystem of i particles of variable composition on the whole, subject to the analytical condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_l m_l(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\sum_i m_i(=m). \tag{19}$$

The members of this system (l bodies and i particles) are connected at each moment of time by regular (in a mathematical sense) gravitational forces, and l bodies can additionally be under the action of reactive forces, conditioned by possible non-isotropic addition of masses. We now deduce for this the analogue of the dynamic equation (1).

In our case we have

$$J = \sum_l m_l r_l^2 + \sum_i m_i r_i^2. \tag{20}$$

For a subsystem of l bodies with variable masses $m_l(t)$ we find that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} m_l \boldsymbol{r}_l^2 = 2m_l\left(\frac{\mathrm{d}\boldsymbol{r}_l}{\mathrm{d}t}\right)^2 + 2m_l\boldsymbol{r}_l\cdot\frac{\mathrm{d}^2\boldsymbol{r}_l}{\mathrm{d}t^2} + 4\boldsymbol{r}_l\cdot\frac{\mathrm{d}\boldsymbol{r}_l}{\mathrm{d}t}\frac{\mathrm{d}m_l}{\mathrm{d}t} + \boldsymbol{r}_l^2\frac{\mathrm{d}^2 m_l}{\mathrm{d}t^2}. \tag{21}$$

For a subsystem of i particles of non-stationary composition we shall use equation (3). We shall distribute the particles with index j over l gravitating bodies with increasing masses (Omarov, 1995): $j = \sum_l j(l)$ and $\Delta n_\alpha = \sum_l \Delta n_{\alpha,l}$. Accordingly we suppose that

$$\begin{aligned}\left(\frac{\partial n}{\partial t}\right)_{\alpha,l} &= \lim_{\Delta t\to 0}\left(\frac{\Delta n_{\alpha,l}}{\Delta t}\right), \quad \overline{r^2_{\alpha,l}} = \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_{\alpha,l}}\sum_j (l) r^2_{j(l)}\right),\\ \overline{\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)_{\alpha,l}} &= \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_{\alpha,l}}\sum_{jl}\frac{\mathrm{d}}{\mathrm{d}t} r^2_{j(l)}\right).\end{aligned} \tag{22}$$

Now, equation (3) takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_i m r_i^2 = -\frac{\mathrm{d}}{\mathrm{d}t}\sum_l m \boldsymbol{r}^2_{\alpha,l}\left(\frac{\partial n}{\partial t}\right)_{\alpha,l} - \sum_l m\overline{\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)_{\alpha,l}}\left(\frac{\partial n}{\partial t}\right)_{\alpha,l} + \sum_i \frac{\mathrm{d}r^2}{\mathrm{d}t} m r_i^2. \tag{23}$$

Since j particles of soft s are 'sticking' to the lth body with increasing mass $m_l(t)$, then we have

$$\left(\frac{\partial n}{\partial t}\right)_{\alpha,l}=\frac{1}{m}\frac{\mathrm{d}m_l}{\mathrm{d}t},\quad \overline{r_{\alpha,l}^2}=r_l^2,\quad \overline{\left(\frac{\mathrm{d}r^2}{\mathrm{d}t}\right)}_{\alpha,l}=2\lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_{\alpha,l}}\sum_{jl}\boldsymbol{r}_{j(l)}\cdot\frac{\mathrm{d}}{\mathrm{d}t}r_{j(l)}\right)=2\boldsymbol{r}_l\cdot\boldsymbol{u}_{\alpha,l},\tag{24}$$

where $\boldsymbol{u}_{\alpha,l}$ is the absolute velocity of the mass that is added to the body. From equations (23) and (24) it follows that

$$\sum_l 2\boldsymbol{r}_l\cdot\frac{\mathrm{d}\boldsymbol{r}_l}{\mathrm{d}t}\frac{\mathrm{d}m_l}{\mathrm{d}t}+\frac{1}{2}\boldsymbol{r}_l^2\frac{\mathrm{d}^2m_l}{\mathrm{d}t^2}+\boldsymbol{r}_l\cdot\boldsymbol{v}_{\alpha,l}\frac{\mathrm{d}m_l}{\mathrm{d}t}-\frac{1}{2}\sum_i\frac{\mathrm{d}^2}{\mathrm{d}t^2}mr_i^2=-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_i mr_i^2,\tag{25}$$

where $\boldsymbol{v}_{\alpha,l}$ is relative velocity of the added mass: $\boldsymbol{v}_{\alpha,l}=\boldsymbol{u}_{\alpha,l}-\mathrm{d}\boldsymbol{r}/\mathrm{d}t$. On the basis of equations (20), (21) and (25) we find that

$$\frac{1}{2}\frac{\mathrm{d}^2J}{\mathrm{d}t^2}=2K'-\sum_l\boldsymbol{r}_l\cdot\boldsymbol{v}_{\alpha,l}\frac{\mathrm{d}m_l}{\mathrm{d}t}+m_l\boldsymbol{r}_l\cdot\frac{\mathrm{d}^2\boldsymbol{r}_l}{\mathrm{d}t^2}+\sum_i m\boldsymbol{r}_i\cdot\frac{\mathrm{d}^2\boldsymbol{r}_i}{\mathrm{d}t^2},\tag{26}$$

where K' is the kinetic energy of the system given by $K'=\frac{1}{2}\sum_l m_l(\mathrm{d}\boldsymbol{r}_l/\mathrm{d}t)^2+\frac{1}{2}\sum_i m_i(\mathrm{d}\boldsymbol{r}_i/\mathrm{d}t)^2$.

Let us denote by $\boldsymbol{F}_{\mathrm{g}l}$ ($\boldsymbol{F}_{\mathrm{g}i}$) the resultant of gravitational forces acting on the lth body (ith particle). The equation of motion for a body of variable mass $m_l(t)$ has the form

$$m_l\frac{\mathrm{d}^2\boldsymbol{r}_l}{\mathrm{d}t^2}=\boldsymbol{v}_{\alpha,l}\frac{\mathrm{d}m_l}{\mathrm{d}t}+\boldsymbol{F}_{\mathrm{g}l}.\tag{27}$$

For a particle of instantaneous composition of the background medium we have

$$m\frac{\mathrm{d}^2\boldsymbol{r}_i}{\mathrm{d}t^2}=\boldsymbol{F}_{\mathrm{g}i}.\tag{28}$$

Accordingly, equation (26) takes the form

$$\frac{1}{2}\frac{\mathrm{d}^2J}{\mathrm{d}t^2}=2K'+\sum_l\boldsymbol{r}_l\cdot\boldsymbol{F}_{\mathrm{g}l}+\sum_i\boldsymbol{r}_i\cdot\boldsymbol{F}_{\mathrm{g}i}.\tag{29}$$

Here only virials of gravitational forces are considered. Let us ascribe the index p to the members of current composition of a system that consists of a subsystem of l bodies with mass $m_S(t)$ and i particles of mass m:

$$\sum_l\boldsymbol{r}_l\cdot\boldsymbol{F}_{\mathrm{g}l}+\sum_i\boldsymbol{r}_i\cdot\boldsymbol{F}_{\mathrm{g}i}=\sum_p\boldsymbol{r}_p\cdot\boldsymbol{F}_{\mathrm{g}p},\tag{30}$$

where $\boldsymbol{F}_{gp}$ is the corresponding gravitational force. Let us introduce a potential energy of the gravitational interaction:

$$W'=-\tfrac{1}{2}G\sum_{\mu\neq\nu}\frac{m_\mu m_\nu}{|\boldsymbol{r}_\nu-\boldsymbol{r}_\mu|},\tag{31}$$

where the indexed μ and v are ascribed to the masses and radius vectors of the members of pairs composed from a set of p particles. Since

$$\sum_p \boldsymbol{r}_p \cdot \boldsymbol{F}_p = -\sum_p \boldsymbol{r}_p \operatorname{grad}_{r_p} W' = W', \tag{32}$$

then equation (26) can be given the following final form

$$\frac{1}{2}\frac{\mathrm{d}^2 J}{\mathrm{d}t^2} = 2K' + W'. \tag{33}$$

Thus, the classical Jacobi equation (1) remains in its formal external form under transmission to the model of stellar system with 'sticking' members. The possibility of such a generalization of equation (1) should be expected from qualitative considerations; in the case of the 'sticking' of stars, only a redistribution of the constant mass of a system between its members occurs. In our derivation of equation (33), and equation (25) is of interest, which gives us a connection between the dynamic conditions of a subsystem of l bodies and i particles:

$$\sum_l 2\boldsymbol{r}_l \cdot \frac{\mathrm{d}\boldsymbol{r}_l}{\mathrm{d}t}\frac{\mathrm{d}m_l}{\mathrm{d}t} + \frac{1}{2}\boldsymbol{r}_l^2 \frac{\mathrm{d}^2 m_l}{\mathrm{d}t^2} + \boldsymbol{r}_l \cdot \boldsymbol{v}_{\alpha,l}\frac{\mathrm{d}m_l}{\mathrm{d}t} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_i m\boldsymbol{r}_i^2 + m\left(\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t}\right)^2 + \boldsymbol{r}_i \cdot \boldsymbol{F}_{gi}. \tag{34}$$

In particular, the behaviour of a subsystem in the gravitational field of the whole system can be such that

$$-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_i m\boldsymbol{r}_i^2 + m\left(\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t}\right)^2 + \boldsymbol{r}_i \cdot \boldsymbol{F}_{gi} = 0. \tag{35}$$

Accordingly, in this case the subsystem of l bodies will possess a configuration peculiarity:

$$\sum_l 2\boldsymbol{r}_l \cdot \frac{\mathrm{d}\boldsymbol{r}_l}{\mathrm{d}t}\frac{\mathrm{d}m_l}{\mathrm{d}t} + \frac{1}{2}\boldsymbol{r}_l^2 \frac{\mathrm{d}^2 m_l}{\mathrm{d}t^2} + \boldsymbol{r}_l \cdot \boldsymbol{v}_{\alpha,l}\frac{\mathrm{d}m_l}{\mathrm{d}t} = 0. \tag{36}$$

Equation (34) can be considered to be a consequence of a generalized Jacobi equation (33) by taking into account the condition (19), the notation (20) and the equations of motion (27) and (28).

4 CONCLUSION

Equation (12) for a system of fluctuating composition is a consequence of equation (5) under $A = \sum_i m\boldsymbol{r}_i^2$ and the classical equation of motion for the ith particle. Let us write equation (5) for a quantity $\sum_i m\boldsymbol{r}_i$ under the analogous dynamic condition:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_i m\boldsymbol{r}_i^2 = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{r}_\alpha\left(\frac{\partial M}{\partial t}\right)_\alpha - \boldsymbol{r}_\beta\left(\frac{\partial M}{\partial t}\right)_\beta\right] - \boldsymbol{u}_\alpha\left(\frac{\partial M}{\partial t}\right)_\alpha - \boldsymbol{u}_\beta\left(\frac{\partial M}{\partial t}\right)_\beta + \sum_i \boldsymbol{F}_i, \tag{37}$$

where

$$\begin{aligned} \boldsymbol{r}_\alpha &= \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_\alpha}\sum_j \boldsymbol{r}_j\right), \quad \boldsymbol{r}_\beta = \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_\beta}\sum_k \boldsymbol{r}_k\right), \\ \boldsymbol{u}_\alpha &= \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_\alpha}\sum_j \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_j\right), \quad \boldsymbol{u}_\beta = \lim_{\Delta t\to 0}\left(\frac{1}{\Delta n_\beta}\sum_k \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_k\right). \end{aligned} \tag{38}$$

Let us introduce the radius vector $\boldsymbol{r}$ of the centre of masses of the system of fluctuating composition:

$$\sum_i m_i \boldsymbol{r}_i = M\boldsymbol{r}, \quad M = \sum_i m_i, \quad \frac{\mathrm{d}M}{\mathrm{d}t} = \left(\frac{\partial M}{\partial t}\right)_\beta - \left(\frac{\partial M}{\partial t}\right)_\alpha. \tag{39}$$

Accordingly, equation (37) gives the equation of motion of the centre of masses of the system:

$$M\frac{\mathrm{d}^2\boldsymbol{r}_l}{\mathrm{d}t^2} = -\boldsymbol{v}_\alpha\left(\frac{\partial M}{\partial t}\right)_\alpha - \boldsymbol{v}_\beta\left(\frac{\partial M}{\partial t}\right)_\beta - \frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{\rho}_\alpha\left(\frac{\partial M}{\partial t}\right)_\alpha - \boldsymbol{\rho}_\beta\left(\frac{\partial M}{\partial t}\right)_\beta\right] + \sum_i \boldsymbol{F}_i, \tag{40}$$

where

$$\boldsymbol{\rho}_\alpha = \boldsymbol{r}_\alpha - \boldsymbol{r}, \quad \boldsymbol{\rho}_\beta = \boldsymbol{r}_\beta - \boldsymbol{r}, \quad \boldsymbol{v}_\alpha = \boldsymbol{u}_\alpha - \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}, \quad \boldsymbol{v}_\beta = \boldsymbol{u}_\beta - \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}. \tag{41}$$

Thus, equations (12) and (40) are obtained on the basis of the basic equation (5) with use of the same dynamic condition. Hence, one may expect an analogy between the behaviour of the moment of inertia of a gravitating system of fluctuating composition and the motion of the centre of masses of such a system. This question will be the subject of a future study.

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