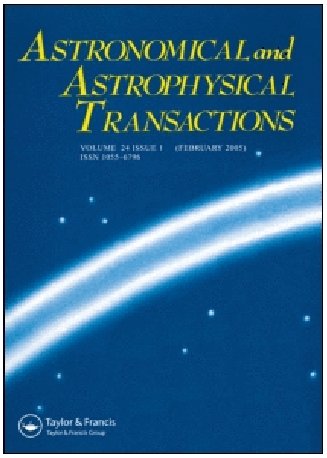

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E. A. Malkov [a]; T. N. Nuzhnova [a]
[a] Fessenkov Astrophysical Institute, National Academy of Sciences, 480068 Almaty, Kazakhstan.

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COLLISIONLESS COLLAPSE OF STELLAR SYSTEMS: ANALYTICAL VISION

E. A. MALKOV* and T. N. NUZHNOVA

Fessenkov Astrophysical Institute, National Academy of Sciences, 480068 Almaty, Kazakhstan

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The present paper surveys some results on the collisionless system evolution that were obtained by the Gravitating System Dynamics Laboratory team from the Fessenkov Astrophysical Institute.

Keywords: Collisionless gravitating systems; dissipationless collapse; analytical models of stellar systems

The first self-consistent phase models of collisionless gravitating systems were described in the monographs by Chandrasekhar (1942) and Ogorodnikov (1958). The main features of the models determined by a particular characteristic of the media, namely that the size of the system is much less than the free path of particles that it consists of, were displayed but not yet realized by scientists at that time. In fact, a phase model of the stellar systems was considered to be rather exotic and artificial until the 1960s when Antonov (1960, 1961, 1962) investigated the stability of such models. Antonov was probably the first to study the stability of collisionless gravitating systems as a whole, unlike others who studied the stability of individual orbits. Such a 'collective' approach to collisionless systems were developed and popularized also by Fridman and co-workers; the work by A. M. Fridman, V. L. Polyachenko, A. G. Morozov and I. G. Schuhman was devoted to the investigating of the collisionless systems with non-spherical geometry (see for example the monograph by Polyachenko and Fridman (1976)).

Since the beginning of the 1980s, computer simulations have provided the main results in studies of the collisionless systems (Aarseth and Binney, 1978; Klypin, 1980) and pure analytical methods were abandoned. However, it should be said that a long series of numerical experiments always produces a long series of pictures but very seldom does it produce conclusions that one can accept as new information: It is certainly very difficult to describe the collisionless collapse in terms of fundamental principles, and not only as a sequence of pictures. However, one should use not only a keyboard to obtain results but also a pen from time to time. On the other hand, numerical experiments that have not been planned properly cost

*Correspondence: E. A. Malkov, Fessenkov Astrophysical Institute, National Academy of Science, 480068, Almaty, Kazakhstan. E-mail: mlkv@softroad.kz

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much time and money and it seems reasonable to use some qualitative analytical results to plan them.

In the present paper, we recall some results on the collisionless system evolution that have been obtained by analytical methods.

It is possible to build exact analytical phase models of non-stationary stellar systems. This possibility is provided by the existence of the non-conservative integrals of the movement equations of motion of a particle in a variable gravitating field. Chandrasekhar (1942) was the first to present the integral like that in his book. Later, Schürer (1943) showed that Chandrasekhar's integral can be obtained by space–time transformations from Jacobi's integral, which is the energy integral in non-rotating systems. Kurth (1949) established that the potential permitting such an integral must be quadratic in the self-consistent case when the phase density satisfies the Poisson equation. With the help of Schürer transformations, Kurth investigated spherical oscillations of homogeneous stellar systems. He found that the radius of such a system is changed like the radius vector in Kepler's problem. Schürer's transformations and the corresponding integrals were rediscovered by physicists later (1949). The existence of Schürer's transformations is connected with the invariant properties of Langrangian and non-conservative integrals and can be obtained directly with the help of the Nother theorem. Let the potential of a gravitating system explicitly depend on time. What is the form of the potential $\Phi(t, X)$ when there is a group of transformations that conserve the form of the action operator? What is the form of the first integrals in that case? The connection between the transformation groups permitted by the Lagrange function (or the action operator when the Lagrange function depends on time) and the first integrals of the Euler equations is set by the Nother (1959) theorem which can be formulated as follows. Let a group of transformations $X(x_1, \ldots, x_n)$ be determined in the space of variables t:

$$\begin{aligned} X' &= X'(t, X, s), \qquad X'(t, X, 0) \\ t' &= t'(t, X, s), t'(t, X, 0) \end{aligned} \tag{1}$$

The infinitesimal vector $\Xi(\xi, \theta_1, \ldots, \theta_n)$ has the components

$$\xi = \left.\frac{\partial t'(t, X, s)}{\partial s}\right|_{s=0}, \qquad \theta_i = \left.\frac{\partial x_i'(t, X, s)}{\partial s}\right|_{s=0}. \tag{2}$$

Let also a function $L = L(t, X, \mathrm{d}X/\mathrm{d}t)$ be determined. We say that the operator

$$l[X] = \int_{t_1}^{t_2} L\left(t, X, \frac{\mathrm{d}X}{\mathrm{d}t}\right) \mathrm{d}t \tag{3}$$

is invariant with respect to the group (1) if

$$\int_{t_1}^{t_2} L\left(t, X, \frac{\mathrm{d}X}{\mathrm{d}t}\right) \mathrm{d}t = \int_{t_1'}^{t_2'} L\left(t', X', \frac{\mathrm{d}X'}{\mathrm{d}t'}\right) \mathrm{d}t', \tag{4}$$

and it does not depend on the integration path $[t_1, t_2]$. The equality (4) under the condition that it does not depend on the integration path is equivalent to the following equality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial X}\left(\Theta - \frac{\mathrm{d}X}{\mathrm{d}t}\xi\right) + L\xi\right] = 0. \tag{5}$$

So, the following theorem (the Nother theorem) is true: if the operation (3) is invariant under the transformation group with the infinitesimal vector (2), then the Euler equation has the first integral

$$J = \frac{\partial L}{\partial X}\left(\Theta - \frac{\mathrm{d}X}{\mathrm{d}t}\right) + L\xi. \tag{6}$$

As is known, the Lagrange function is determined accurately up to the the full derivative (Landau and Lifshitz, 1973). Thus leads to some generalization of the Nother theorem; the first integral of the equations of motion of the form

$$J = \frac{\partial L}{\partial X}\left(\Theta - \frac{\mathrm{d}X}{\mathrm{d}t}\right) + L\xi + C, \tag{7}$$

exists when there is the full derivative of the function $C(t, X)$ in the right-hand side of equation (5) (Ibragimov, 1972).

In the case of the equation of motion of a particle in a field with potential $\Phi(t, X)$ the Lagrange function has the form

$$L = \frac{1}{2}\sum_i x_i^2 - \Phi(t, X). \tag{8}$$

The condition for the correspondence operator to be invariant under the group transformation (1) is expressed as the following equalities:

$$\begin{gathered}
\frac{\partial \xi}{\partial x_i} = 0, \\
\frac{1}{2}\frac{\partial \xi}{\partial t} - \frac{\partial \theta_i}{\partial x_i} = 0, \\
\frac{\partial \theta_i}{\partial x_j} + \frac{\partial \theta_j}{\partial x_i} = 0, \qquad i \neq j \\
\frac{\partial \theta_i}{\partial t} - \frac{\partial \xi}{\partial x_i}\Phi = \frac{\partial C}{\partial x_i}, \\
\xi\frac{\partial \Phi}{\partial t} + \sum_i \theta_i \frac{\partial \Phi}{\partial x_i} + \Phi\frac{\partial \xi}{\partial t} = \frac{\partial C}{\partial t}.
\end{gathered} \tag{9}$$

After the analysis of equations (9), one can make the following conclusion: the group transformation exists in only two essentially distinct cases.

The first is when

$$\Phi(t, X) = \frac{1}{\xi(t)}\tilde{\Phi}\left(\frac{X}{[\xi(t)]^{1/2}}\right). \tag{10}$$

Then the infinitesimal vector has the form

$$\boldsymbol{\Xi} = \left(\xi(t), \frac{1}{2}\frac{\mathrm{d}\xi}{\mathrm{d}t}x_1, \frac{1}{2}\frac{\mathrm{d}\xi}{\mathrm{d}t}x_2, \frac{1}{2}\frac{\mathrm{d}\xi}{\mathrm{d}t}x_3\right), \tag{11}$$

where $\mathrm{d}^3\xi/\mathrm{d}t^3 = 0$ and the first integral,

$$J^{(1)} = \sum\left(\frac{1}{2}\frac{\mathrm{d}\xi}{\mathrm{d}t}x_i - \frac{\mathrm{d}x_i}{\mathrm{d}t}\xi\right)\frac{\mathrm{d}x_i}{\mathrm{d}t} + \left[\frac{1}{2}\sum\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)^2 - \Phi\right]\xi - \frac{1}{4}\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}\sum x_i^2. \tag{12}$$

It should be noted that it is always possible to transform to the variable $\tilde{t}(t, X)$ and $\tilde{X}(t, X)$ where the transformation group is simply the parallel shifts group along $\tilde{t}$ in the case of one-parameter group. The infinitesimal vector has the form $\tilde{\boldsymbol{\Xi}} = (1, 0, 0, 0)$ in this case. It is obvious that the transformation equations for the system where the equatiosn of motion are stationary have the form

$$\begin{aligned}
&\xi\frac{\partial\tilde{t}}{\partial t} + \frac{1}{2}\frac{\mathrm{d}\xi}{\mathrm{d}t}\sum x_i\frac{\partial\tilde{t}}{\partial x_i} = 1,\\
&\xi\frac{\partial\tilde{x}_i}{\partial t} + \frac{1}{2}\frac{\mathrm{d}\xi}{\mathrm{d}t}\sum x_j\frac{\partial\tilde{x}_i}{\partial x_j} = 0.
\end{aligned} \tag{13}$$

So, the transformation functions are independent functions of the first integrals of the following system:

$$\frac{\mathrm{d}t}{\xi} = \frac{\mathrm{d}x_i}{\frac{1}{2}(\mathrm{d}x/\mathrm{d}t)x_i} = \mathrm{d}\tilde{t}. \tag{14}$$

Let us choose those functions as follows:

$$\tilde{t} = \int\frac{\mathrm{d}t}{\xi}, \qquad \tilde{x}_i = \frac{x_i}{\xi^{1/2}}. \tag{15}$$

The transformations (15) are called the Schürer transformations in stellar dynamics. The integral (12) has the form of the energy integral in new variables.

In the second case when the transformation group exists, the potential has the form

$$\Phi(t, X) = \frac{1}{2}\sum A_i(t)x_i^2. \tag{16}$$

The three-dimensional transformation group is determined by the following vectors:

$$\begin{aligned}
\boldsymbol{\Xi}_1 &= \left(\xi_1(t), \frac{1}{2}\frac{\mathrm{d}\xi_1}{\mathrm{d}t}x_1, 0, 0\right),\\
\boldsymbol{\Xi}_1 &= \left(\xi_2(t), 0, \frac{1}{2}\frac{\mathrm{d}\xi_2}{\mathrm{d}t}x_2, 0\right),\\
\boldsymbol{\Xi}_1 &= \left(\xi_3(t), 0, 0, \frac{1}{2}\frac{\mathrm{d}\xi_3}{\mathrm{d}t}x_3\right).
\end{aligned} \tag{17}$$

The functions $\xi_i(t)$ obey the equality

$$\frac{\mathrm{d}^3\xi_i}{\mathrm{d}t^3} + 4\frac{\mathrm{d}\xi_i}{\mathrm{d}t} + 2\xi_i\frac{\mathrm{d}A_i}{\mathrm{d}t} = 0, \tag{18}$$

where $\mathrm{d}A_i/\mathrm{d}t \neq 0$.

The integrals have the form

$$J_i = \left(\frac{1}{2}\frac{\partial \xi_i}{\partial t}x_i - \frac{\mathrm{d}x_i}{\mathrm{d}t}\xi_i\right)\frac{\mathrm{d}x_i}{\mathrm{d}t} + \left[\frac{1}{2}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)^2 - \frac{A}{2}x_i^2\right]\xi_i - \frac{1}{4}\frac{\mathrm{d}^2\xi_i}{\mathrm{d}t^2}x_i^2. \tag{19}$$

After the substitution $a_i = \xi_i^{1/2}$ the integrals take the more convenient form

$$J_i = \frac{1}{2}a_i^2\left(\frac{\mathrm{d}x_i}{\mathrm{d}t} - \frac{1}{a_i}\frac{\mathrm{d}a_i}{\mathrm{d}t}\right)^2 + \frac{\alpha_i^2}{2a_i^2}x_i^2, \tag{20}$$

where a_i obey the equations

$$\frac{\mathrm{d}^2 a_i}{\mathrm{d}t^2} = \frac{\alpha_i^2}{a_i^3} - A_i(t)a_i, \qquad \alpha_i = \text{constant}. \tag{21}$$

There is an additional condition in the case of the self-gravitating field: $A_i(t) \propto 1/a_i^3$.

It is rather simple to find the non-stationary distribution functions with the help of the stationary functions and the non-conservative integrals (12). Let us present some examples.

The distribution function of the non-stationary Camm sphere has the form

$$f \propto \left(\frac{L^2}{r_0^2 +} 2\alpha^2 r_0^2 - 2\sum_{i=1}^{3} J_i\right)^{-1/2}, \tag{22}$$

where $\alpha = \alpha_1 = \alpha_2 = \alpha_3$ and r_0 is some constant.

The Freeman spheriod is described by the function

$$f \propto \frac{B_+\delta(J_1 + J_2 + \alpha_1 L_3) + \beta_-\delta(J_1 + J_2 - \alpha_1 L_3)}{(\alpha_3^2 - 2J_3 + \alpha_3 L_3)^{1/2}}, \tag{23}$$

where β_+ and β_- are constants ($\beta_+ + \beta_- = 1$) and L_3 is the angular momentum with respect to the x_3 axis.

The elliptical discs studied by Bisnovatyi-Kogan and Zeldovich (1970) with the boundary equation $x_i^2 a_1^2(t) + x_2^2/a_2^2(t) = 1$ are described by the function

$$f \propto \left(1 - \frac{2J_1}{\alpha_1^2} - \frac{2J_2}{\alpha_2^2}\right)^{-1/2}. \tag{24}$$

So, it is necessary to substitute the conservative integrals by the non-conservative integrals in the known distribution functions to obtain new non-stationary analytical models. It is rather easy to obtain the non-stationary models of the Freeman (1966) ellipsoids, the turned-over Kondratyev (1986) discs and the ellipsoid with oblique rotation (Kondratyev, 1984) in that way.

There arises the question: what is the use of models with such an exotic and non-realistic distribution in phase space? We think that the non-stationary models with homogeneous space density as well as their stationary analogies have to be considered as approximate models with three-dimensional geometry which still permit analytical investigations and estimations. So, these models are intended for studies of the origin of the shape of the stellar system. Their existence proves the possibility of three-dimensional models with positive phase density. As for the behaviour of large-scale oscillations which form the system geometry as a whole, it needs to be borne in mind that it depends only on the second-order momenta, and the results are valid for any distribution function with the same second-order momenta. Investigations of this kind of evolution can be made with the help of the nonlinear equations of ellipsoidal oscillations that have been obtained by Kondratyev and Malkov (1986). In the explicit form these equations are as follows:

$$\frac{\mathrm{d}^2 a_i}{\mathrm{d}t^2} = -A_i a_i + \frac{1}{2}\left(\frac{(L_j + C_j)^2}{(a_i - a_k)^3} + \frac{(L_j - C_j)^2}{(a_i + a_k)^3} + \frac{(L_k + C_k)^2}{(a_i - a_j)^3} + \frac{(L_k - C_k)^2}{(a_i + a_j)^3}\right) + \frac{P_{ii}}{a_i^3}, \quad (25)$$

$$\frac{\mathrm{d}L_i}{\mathrm{d}t} = \frac{1}{2}\left(\frac{(L_k + C_k)L_j}{(a_j - a_i)^2} - \frac{(L_j - C_j)^2 L_k}{(a_k + a_i)^2} + \frac{(L_k + C_k)L_j}{(a_j - a_i)^2} - \frac{(L_j - C_j)L_k}{(a_k + a_i)^2}\right), \quad (26)$$

$$\frac{\mathrm{d}C_i}{\mathrm{d}t} = P_{jk} - \frac{1}{2}\left(\frac{(L_j + C_j)L_k}{(a_k - a_i)^2} - \frac{(L_k - C_k)L_j}{(a_j + a_i)^2} - \frac{(L_j + C_j)L_k}{(a_k - a_i)^2} + \frac{(L_k - C_k)L_j}{(a_j + a_i)^2}\right), \quad (27)$$

$$\begin{aligned}\frac{\mathrm{d}P_{ii}}{\mathrm{d}t} = {} & \left[(L_k - C_k)\left(\frac{1}{a_j^2} - \frac{1}{(a_i + a_j)^2}\right) - (L_k + C_k)\left(\frac{1}{a_j^2} - \frac{1}{(a_i - a_j)^2}\right)\right]P_{ij} \\ & + \left[(L_j + C_j)\left(\frac{1}{a_k^2} - \frac{1}{(a_i - a_k)^2}\right) - (L_j - C_j)\left(\frac{1}{a_k^2} - \frac{1}{(a_i + a_k)^2}\right)\right]P_{ik}, \end{aligned} \quad (28)$$

$$\begin{aligned}\frac{\mathrm{d}P_{ij}}{\mathrm{d}t} = {} & \frac{1}{2}\left[(L_k + C_k)\left(\frac{1}{a_i^2} - \frac{1}{(a_i - a_j)^2}\right) - (L_k - C_k)\left(\frac{1}{a_i^2} - \frac{1}{(a_i + a_j)^2}\right)\right]P_{ii} \\ & + \frac{1}{2}\left[(L_k - C_k)\left(\frac{1}{a_j^2} - \frac{1}{(a_i + a_j)^2}\right) - (L_k + C_k)\left(\frac{1}{a_j^2} - \frac{1}{(a_i - a_j)^2}\right)\right]P_{jj} \\ & + \frac{1}{2}\left[(L_j + C_j)\left(\frac{1}{a_k^2} - \frac{1}{(a_k - a_i)^2}\right) - (L_j - C_j)\left(\frac{1}{a_k^2} - \frac{1}{(a_k + a_i)^2}\right)\right]P_{jk} \\ & + \frac{1}{2}(L_i - C_i)\left(\frac{1}{a_k^2} - \frac{1}{(a_j + a_k)^2}\right) - (L_i + C_i)\left(\frac{1}{a_k^2} - \frac{1}{(a_j - a_k)^2}\right)]P_{ik}. \end{aligned} \quad (29)$$

Here a_i are the semi-axes, $A_i = \frac{3}{2}GM\int_0^\infty \mathrm{d}s/(a_i^2 + s)[(a_1^2 + s)(a_2^2 + s)(a_3^2 + s)]^{1/2}$, M is the ellipsoid mass, L is the angular momentum, C is the rotation, $P_{ij} = a_i a_j \Pi_{ij}$ and Π_{ij} is the double kinetic energy tensor (all of these values are divided by $M/5$).

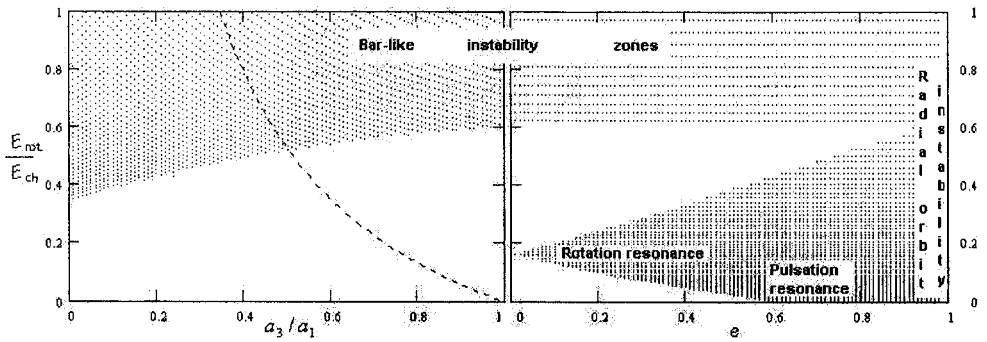

FIGURE 1 Shape evolution during collisionless collapse.

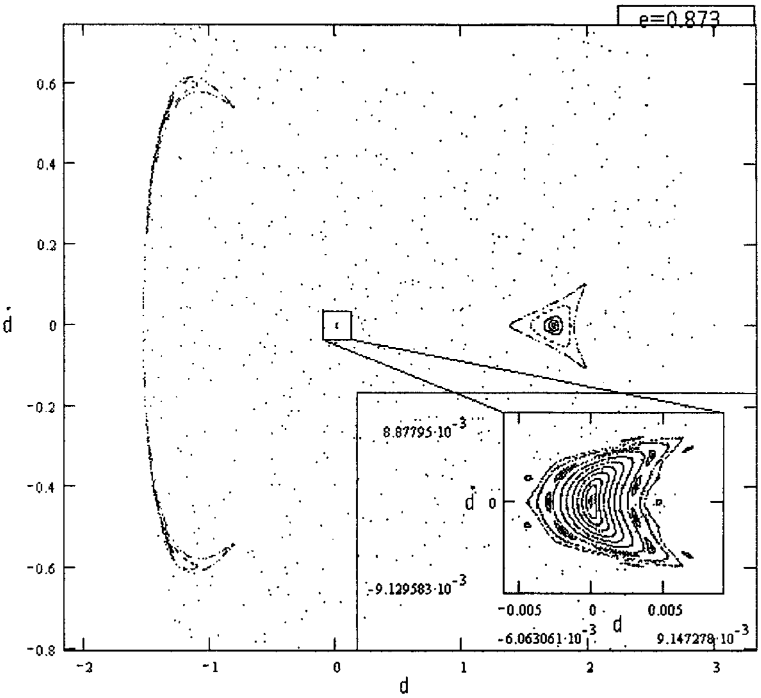

FIGURE 2 Development of instability in the vicinity of a point.

The results on the shape evolution during collisionless collapse obtained by different methods with the help of non-stationary analytical models are summarized in Figure 1. The unstable zone of the stationary spheroids is presented in the left-hand diagram (shaded area); $E_{\rm rot}/E_{\rm ch}$ is the ratio of the rotation energy to the heat (chaotic) energy. The dashed curve presents Maclaurin's spheroids. The right-hand diagram shows the unstable zone of the pulsating rotating spheres (shaded area). The parameter e can be expressed via the virial ratio $V = 2T/|U|$ (where T and U are the kinetic and potential energies respectively) in the initial state: $e = 1-V$. The instability due to the parametric resonance 4 : 1 when the period of the ellipsoidal mode oscillation is exactly four times the rotation period is labelled Rotation resonance. It occurs when $E_{\rm rot}/E_{\rm ch} \approx 0.165$. The bar-like instability exists when $E_{\rm rot}/E_{\rm ch}$ varies from approximately 0.35 ($E_{\rm rot}/|U| \approx 0.13$) (disc) to approximately 0.62 ($E_{\rm rot}/|U| \approx 0.19$) (sphere). It should be noted that the bar-like instability of the pulsating sphere occurs at the same value of the $E_{\rm rot}/E_{\rm ch}$ and, so does not depend on the virial ratio.

Equations (25)–(29) can be used to investigate the nonlinear evolution of ellipsoidal models in a more detailed way. The sample picture in Figure 2 shows some details of the development of an instability in the vicinity of a point (Omarov, 1999).

As it has been mentioned above, this paper is intended to remind researchers whose work is based on computer simulation about some analytical results concerning the dissipationless collapse investigations. We have tried to present a list of references (which is not totally comprehensive naturally) that could be useful for studying that topic.

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