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NATURAL RESOLUTION LIMIT

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Numerical simulations are used to study the well-known problem of a limiting detectable separation between two point-like components of a double source. Three versions of the problem are considered: (I) The Point Spread Function (PSF) is given a priori; (II) For a specified analytical form of the PSF, its parameters are to be estimated from observations of a single star; (III) The PSF is not known at all. The images of single and close binary stars are simulated by taking into account photon noise, the pixel structure of the detector and other factors that are common in real experiments. We generate large-sized samples of randomly blurred images of single and binary stars, and obtain maximum-likelihood estimates of 6 parameters for each binary image: the binary's total brightness, the relative brightness of its components, and 4 of their Cartesian coordinates. Monte Carlo simulations are consistent with the previously found analytical solution of the problem, according to which the limiting detectable separation between the components is $\theta_{\min} \approx \Delta_{80}/\sqrt{\psi}$, where Δ_{80} is the characteristic width of the PSF, and ψ is the signal-to-noise ratio within the image. Thus, the rough pixel structure of the detector and the presence of noise do not hinder the achievement of a resolution in the visual spectral range of order 0.001''-0.1'' with moderate-sized astronomical telescopes.

KEY WORDS Resolution, limiting resolving power, numerical simulations

1 INTRODUCTION

It is assumed, in the classical approach to the problem of limiting resolving power, that the observed blurred image of the object has been generated either by a single, or by a double source with incoherent point-like components. It is also suggested that the PSF is completely known from theoretical reasoning or special preliminary measurements. For example, one can use the Airy pattern, if a narrow spectral range with central wavelength λ has been used in observations, and the telescope is diffraction-limited. Then the characteristic angular size of a star image is determined by the radius of the first dark ring in the Airy pattern:

$$\theta_{\rm A} \approx 1.22 \frac{\lambda}{D} \text{ radians} \approx \frac{140''}{D_{\rm mm}}.$$
(1)



Figure 1 Three-dimensional (left) and two-dimensional intensity distributions in the diffraction image of a binary star with components of equal brightness, which correspond to the (a) Rayleigh, (b) Dawes-Danjon-Couder, (c) Sparrow, and (d) visual limits.

In the second equality, $\lambda = 5550$ Å is accepted, and the aperture D is measured in millimeters. If aberrations of the optical system are not too large, one can use Δ_{80} , the diameter of a circle that contains 80% of the energy in the image of a point source, instead of the Airy diameter $\Delta = 2\theta_A$.

Let case I corresponds to the version of the limiting resolution problem, when complete *a priori* information about the PSF is available. Obviously, the object should be considered as single if the uncertainty of the estimate (here and below, the term *estimate* is used in its exact statistical sense) of the components separation becomes of the order of its true value.

In practice, the investigator does not always have complete information about the PSF, so, alongside the classical, some other versions of the problem can be interesting. First of all, by leaving some free structural parameters, it is often still possible to consider an analytical representation of the PSF as known (case II). As one can see, the necessity of a tentative estimate of the free parameters from observations of a single star leads, to a certain degree, to the lower resolving power, in comparison with case I. Finally, we also consider the situation, when the PSF is completely unknown (case III). The information on this function should still be obtained from observations of a single comparison star. For the first time the problem under consideration was carefully studied by William Dawes (1865). As a result of lifelong observations with small refractors, Dawes came to the conclusion that the limiting resolution for a clear aperture, θ_{\min} , is approximately equal to $0.85\theta_A$ (Figure 1). Taking into account the inevitable uncertainty of the concept of limiting resolution, Lord Rayleigh proposed simply to accept $\theta_{\min} \approx \theta_A$ (Scientific Papers, 1964, p. 420). The subsequent propositions, deriving from the analytical representation of the diffraction PSF (for example, the criterion by Sparrow (1916), according to which $\theta_{\min} \approx 0.78\theta_A$), do not change the essence of the matter. As concerns experiment, Danjon and Couder (1935), proceeding from the criterion of minimal detectable contrast between images of the double components, again came to the Dawes's limit. Finally, Couteau (1978) believed that most experienced observers of double stars, using weak elongation of their images, are capable to achieve θ_{\min} of the order $0.5\theta_A$.

Bearing in mind the possible change of the Airy diameter Δ by Δ_{80} for the nondiffraction PSF, it is convenient to introduce the dimensionless resolution parameter

$$\Re = \frac{\theta_{\min}}{\Delta}.$$
 (2)

As one can see, the results described above simply mean that $\Re \approx 1$ for visual inspection of images with a clear aperture. Since the classical approach is strongly based on properties of the human eye and brain, one may hope to achieve a deeper resolution by using a modern, nearly ideal, light detector and refined image analysis. Generally speaking, $\theta_{\min} \ll \theta_A$ is not excluded, even when the PSF width Δ_{80} for some reason is essentially larger than the diameter of the Airy circle (accessing the values $\Re \ll 1$ is described sometimes as the *super-resolution* phenomenon).

The starting point in the considered problem is the fact that only the presence of noise interferes with the achievement of an infinitely high resolving power (Toraldo di Francia, 1955; Wolter, 1961). Indeed, precisely knowing the PSF shape, one can easily reveal an infinitely small separation of the components. It is known also that Rayleigh's limit is surmountable even in the presence of moderate noise (Kozlov, 1964; Harris, 1964; Helstrom, 1968; Rushforth and Harris, 1968). The kinds of noise are diverse, however, only photon noise is really important, because it is unavoidable due to the quantum nature of light (Taking into account not only the visual spectral range, it is more correct to speak of radiation noise (Loudon, 1973)). Therefore, some *natural limit of resolving power* exists, and the non-trivial question consists only in the value of the natural resolution limit depending on experimental conditions.

The discussion below is based on the author's papers (Terebizh, 1990, 1993, 1995a, b) which consider the limiting resolution problem from the point of view of general theory by J. Neyman and E. Pearson of testing the statistical hypothesis (see, e.g., Kendall and Stuart, 1969). Terebizh and Cherbunina (1995), and Terebizh (1999) performed corresponding Monte Carlo simulations. Some recent investigations have been reviewed by den Dekker and van den Bos (1997).

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2 INVERSE PROBLEMS AND A PRIORI INFORMATION

The problem of limiting resolving power belongs to the extensive class of *inverse* problems of mathematical physics, where an object's true properties are to be learned on the basis of the observed picture. If James Jeans were right, and Great Architect is a mathematician ('The Great Architect of the Universe now begins to appear as a pure mathematician', Jeans (1948), p. 165), it should also be admitted that the Great Architect solved only the *direct* problem, leaving for us the vastly more complicated *inverse* one.

Practice shows that the quality of an inverse solution mainly depends on a priori information about the object, and only to a smaller degree on shape of the object, the PSF form, the statistical properties of noise etc. If we are dealing with an *image deconvolution* problem (Jansson, 1997) under scarce a priori information, it is appropriate to use term *image restoration*. In the opposite case, in which we are now interested, the a priori information is so extensive that the inverse problem in the Dawes-Rayleigh sense is reduced to the choice between two alternative objects – a single or binary star with point-like components. In fact, we are to find the *best decision* for given conditions, or to *test the statistical hypothesis* about the object on the basis of its blurred image (Kendall and Stuart, 1969). Another term that is used in the considered case is pattern recognition (Tou and Gonzalez, 1974).

In the main version of our problem (let us denote it as case 'B'), the *a priori* information is as follows. Two above mentioned alternative objects have the same total brightness; the angular separation of the binary's components, θ , is non-negative; their relative brightness is known; the same is true for the PSF; the photon noise and the background are subjected to the Poisson distribution. *Photon counting* (Mehta, 1970), that is the principally most effective method of image detecting, is applied. Strictly speaking, the statistics of photo-events is described by the Cox-Mandel distribution (see, e.g., Loudon, 1973; Terebizh, 1992), but its difference from the Poisson law is negligible for usual observational conditions.

Although the closed analytical solution of the binary decision problem has been found for any object shape, it is rather complicated, so we consider here, besides the main version (B), only two simple cases: (A) The alternative object to the binary star is an extended source with a Gaussian brightness distribution, (C) The alternative is the same object, but shifted as a whole. The purpose of considering cases (A) and (C) is to show the dependence of the results on the nature of the *a priori* information.

3 ANALYTICAL RESULTS

When choosing the parent object on the basis of its blurred and noisy image, one can make errors of two types: (1) A single star is erroneously classified as a double star (first type of error); (2) A double star is classified as a single one (second type of error). Neyman and Pearson paid attention to the fact that in practice two types

of errors are almost always non-equivalent. For example, if single stars are often classified as double stars, there will be an illusion of too high a resolving power. In order to know the criterion's 'rigidity', it is reasonably to accept beforehand the probability of the first type of error α , and then find the decision rule in such a manner, that the probability of the second type of error, β , is minimal.

These features make essence of the Neyman-Pearson approach, which is now widely used in applications. In mathematical statistics, α is known as the *significance level* of the criterion, and $1 - \beta$ as its *power*. Neyman and Pearson proposed also the procedure that sometimes gives the *most powerful* criterion of testing the statistical hypothesis, and just such a criterion was found for the problem of limiting resolution. Therefore, we obtain the *theoretically unimprovable* rule of classifying the images with any given probability α . It is worth stressing that any particular method of distinguishing the objects, even quite attractive at first glance (see, e.g., Lucy, 1992), does not allow us to speak of reaching the *limiting* resolution.

Let F be the number of photo-events during the exposure time, and B be the number of background events within the image of a point source. Analytical formulae show that the resolution parameter \Re (2) mainly depends on the signal-to-noise ratio, which we define as

$$\psi = \frac{F}{\sqrt{F+B}}.$$
 (3)

Certainly, besides ψ , the limiting resolution depends on the PSF form, the statistical properties of the background, the accepted significance level of detection etc. However, just the above-specified dependence on ψ is dominant. Photon noise plays main role when $F \gg B$, and in that case we have $\psi \approx \sqrt{F}$.

The schematic relation between \Re and the signal-to-noise ratio ψ , which follows from the most powerful test, is depicted in Figure 2. The significance level α is equal to 0.20. In the main version of the problem, that is double star versus single one (case 'B'), we have

$$\Re \approx \psi^{-1/2}.\tag{4}$$

If a Gaussian spot is the alternative object to the double star (case 'A'), then

$$\Re \approx \psi^{-1/4}.\tag{5}$$

Finally, the limiting detectable shift of the whole object (case 'C') corresponds to

$$\Re \approx \psi^{-1}.\tag{6}$$

Equation (6) is a generalization of the long known theorem in mathematical statistics by E. Pitman.

4 NUMERICAL SIMULATIONS

Although the early, one-dimensional, Monte Carlo simulations do not take into account many real circumstances, the corresponding results are very indicative, and we consider them first.



Figure 2 Schematic representation of theoretical relation between resolution parameter \Re and signal-to-noise ratio ψ . Line A corresponds to the double star and the Gaussian spot as the alternative parent objects; line B – to the double and single stars as the alternative objects; line C – to shifting of the object as a whole.

Single simulation includes the following procedures. The double star with some components separation $\theta \geq 0$ (scaled by the PSF width Δ) has been randomly blurred 'photon by photon' according to the given shape of the PSF, and then a random background realization has been added to obtain the image pattern. A sample of nearly 10^6 random images was created by repeating the described procedures for the set of θ values, and for every image the maximum-likelihood estimate of separation $\hat{\theta}$ was calculated.

Figure 3 corresponds to the total object brightness $F = 10^4$ photo-events, equal brightness of the components, mean background level 10 events/pixel, and the diffraction PSF of width $\Delta = 100$ pixels. Theoretical formulae (3) and (4) predict $\Re \approx 0.10$ for these conditions. As one can see from Figure 3, the calculated estimates are tightly distributed near the true values, when the components are rather far from one another, say, $\theta = 0.30$ or $\theta = 0.15$. The variance of estimates increases, and the single point-like object with $\hat{\theta} = 0$ is preferred more frequently when the components are drawn together. After reaching some critical separation $\theta \approx 0.08$, the sample distribution density of estimates practically does not change, so it is impossible to recover the parent object on the basis of the observed image. Therefore, one can conclude that a relative separation $\theta \approx 0.08$ is to be considered as limiting for the given conditions. At the now accepted level of accuracy, this agrees with the predicted value $\Re \approx 0.10$. Perhaps, Figure 4 gives even more clear evidence of the existence of the resolution limit.

In the course of two-dimensional simulations the images of single and close binary stars were created by taking into account photon noise, random background, stray light, the pixel structure of the detector, dark current, read-out noise and spatial



Figure 3 Sample distribution densities of separation estimates $\hat{\theta}$ between components of a double star for various values of true separation θ . Both parameters are scaled by the PSF width.

variations in detector sensitivity (Figure 5). According to the known sampling theorem by V. Kotel'nikov and C. Shannon, no fewer than 2 pixels were situated at the PSF radius, in order to save the small-scale structure of the image. We generate large-sized samples of randomly blurred images of single and binary stars, and obtain maximum-likelihood estimates of 6 parameters for each binary image: the binary's total brightness, the relative brightness of its components, and their 4 Cartesian coordinates. The method of Nelder and Mead (1965) was applied for multidimensional optimization.

It is convenient to discuss the experimental conditions in the context of astronomical observations. Table 1 shows one of the considered sets of such conditions. Note that the pixel size, 248 milli-arcseconds (mas), is larger than all three values of the true separation between the components. Nevertheless, the 'wings' of the components' images allow us to estimate their separation in all three versions down to approximately 50 mas (Figure 6), whereas formulae (3) and (4) give $\psi \approx 700$ and $\Re \approx 60$ mas for the data in Table 1.

A satisfactory agreement between theory and numerical simulations has also been found for other values of signal-to-noise ratios. The accompanying simulations of single-star images allow the limiting photometric and positional accuracy to be estimated for real light detectors. These results are also in agreement with the theoretical conclusion (6) that the relative accuracy of the above estimates is inversely proportional to the signal-to-noise ratio.



Figure 4 Relation between the mean value of the separation estimates $\langle \hat{\theta} \rangle$ and the true separation θ of the double star components.



Figure 5 Intensity distribution in the image of the double star with component separation of 100 mas (above) and corresponding contour map (below). Fragment is depicted of size 14 by 14 pixels; the pixel size is equal to 248 mas.

5 CONCLUDING REMARKS

We have considered above the case of a clear or slightly obscured aperture, when the diffraction image is the classical Airy pattern or a somewhat changed one. Both for theory and practice, it would be of interest to answer the question: is it possible



Figure 6 Coordinate estimates of bright (crosses) and weak (points) components of a double star under observational conditions, specified in Table 1. Milliarcseconds (mas) are shown along both axes; the size of squares corresponds to the pixel size. Rows correspond to three values of the true components separation: 200, 100 and 50 mas; columns correspond to the cases: (I) the PSF is completely known, (II) the analytical form of the PSF is given; some free parameters are estimated; (III) information about the PSF is lacking. In each case, a sample of 100 independent image realizations was processed.

to reach a deeper resolution by using an arbitrary aperture shape, in other words, with the help of suitable *apodization*?

The interferometric observations of A. Michelson (1920) with two slits, and subsequent similar investigations show that some gain in resolving power can be obtained, even in visual analysis of an image. In order to use more interferometric fringes, one should make the slits as narrow as possible. On the other hand, since the resolution parameter \Re strongly depends on the signal-to-noise ratio, it seems quite probable that some optimal apodization exists given *a priori* information about the object under investigation. The corresponding study is now in progress.

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	Table	1.	Model	observational	conditions
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Characteristics	Value
Magnitudes of the binary's components	9.0 ^m , 10.0 ^m
Angular separation between components	200, 100, 50 mas
Polar angle of the weak component	60°
Magnitudes of the comparison stars	7.0, 4.0
Telescope aperture	60 cm
Linear central obscuration	0.25
Exposure time	10 s
Overall transparency	0.50
Central wavelength	6500Å
Spectral bandwidth	400Å
Sky background	$21.0^{\rm m} \rm \ arcsec^{-2}$
Stray light	$23.0^{\mathrm{m}} \mathrm{arcsec}^{-2}$
Pixel size of the CCD	$9 \times 9 \ \mu m$
Mean quantum efficiency of the CCD	$0.33 \text{ events photon}^{-1}$
Standard deviation of quantum efficiency	3%
Dark current	$0.2 \text{ events s}^{-1} \text{ pixel}^{-1}$
Read-out noise (rms)	15 events $pixel^{-1}$

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