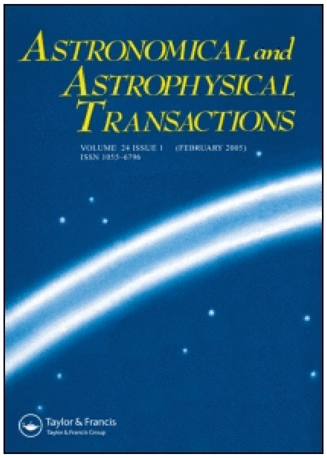

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TIDAL VARIATIONS OF THE INNER MANTLE POTENTIAL COEFFICIENTS

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Tidal variations of the inner gravitational field of the Earth's mantle coefficients of arbitrary order due to elastic deformations of the mantle caused by lunar and solar attraction have been found.

KEY WORDS Inner mantle potential, tidal variations, elastic deformations

1 INTRODUCTION

The Earth's mantle is a non-spherical, non-homogeneous cover with a quasi-concentric distribution of densities. Let R_0, be the mean radius of the Earth, and $\tilde{R}_0$, the radius of the bigger sphere which we can put in the mantle cavity (we assume that the centre of this sphere coincides with the Earth's centre of mass).

We will consider the mantle as a deformable elastic body subject to the attraction of external celestial bodies (the Moon and the Sun). The lunar and solar tidal deformations of the Earth will be described by the classical model (Takeuchi, 1950) which was studied in detail in the papers of Ferrandiz and Getino (1991–1994) for the construction of the rotation theory of the deformable Earth.

Let us introduce in to consideration the main Cartesian reference system $Cxyz$ with the origin at the Earth's centre of mass and with axes directed along its principal axes of inertia in the undeformed state. Let $\boldsymbol{r}_0$ and $\boldsymbol{r}$ be radius vectors of an arbitrary point (or a elementary volume dm) of the mantle in the absence of deformations and in the deformable state. As usual, we assume that particles of the deformable solid mantle deviate slightly from their positions which they occupy in the absence of deformation. The small displacement vector, $\boldsymbol{u}(\boldsymbol{r}, t)$, is defined in the following way

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{u}_0(\boldsymbol{r}_0, t) \Rightarrow (x, y, z)(t) = (x_0, y_0, z_0) + (u, v, w)(x_0, y_0, z_0; t), \qquad (1)$$

where (x, y, z) are the positional coordinates of the particle of the deformable body, and (x_0, y_0, z_0) are those that the same particle would have in the absence of deformations, (u, v, w) being the components of the displacement vector.

The components of the displacement vector during the deformation of the mantle under Newtonian attraction of the external bodies (the Moon, the Sun) are defined as (Takeuchi, 1950)

$$(u, v, w) = \sum_{n'=1}^{\infty} F_{n'}(r_0) \frac{\partial W_{n'}}{\partial(x, y, z)} + G_{n'}(r_0)(x, y, z) W_{n'}, \tag{2}$$

where $W_{n'}$, is the harmonic of the n' order of the tidal potential

$$W = \sum_{n'=1}^{\infty} W_{n'} = \frac{Gm^*}{r^*} \sum_{n'=1}^{\infty} \left(\frac{r}{r^*}\right)^{n'} P_{n'}(\cos S). \tag{3}$$

The functions $F_{n'}(r)$ and $G_{n'}(r)$ are defined by a set of ordinary differential equations that depend on the model of the radial distribution of density (Ferrandiz and Getino, 1993). In (3), G is the gravitational constant, r is the distance between the origin C and the point within the Earth where the potential is evaluated, m^*, r^* are the mass of and the distance to the perturbing body (the Moon, the Sun), and S is the angle between the vectors $\boldsymbol{r}$ and $\boldsymbol{r}^*$. $P_{n'}$ is Legendre's functions.

The main term in the full development of the tidal potential (3) is the second harmonic, since the factor r/r^* is of the order of 1/60 for the Moon and 1/23000 for the Sun.

Now we introduce spherical coordinates r, θ, φ for the elementary mass dm and r^*, δ^*, α^* for the perturbing body (here θ and δ^* are colatitude and latitude and φ, α^* are the longitudes). The expression of the $W_{n'}$ function can be presented in following form (Takeuchi, 1950):

$$W_{n'} = \sum_{m'=0}^{n'} q_{n'm'} r^{n'} P_{n'm'}(\cos\theta)(A^*_{n'm'} \cos m'\varphi + B^*_{n'm'} \sin m'\varphi, \tag{4}$$

where

$$q_{n'm'} = \frac{(n'-m')!}{(n'+m')!}(2 - \delta_{0m'}) = \frac{2(n'-m')!}{(n'+m')!\delta_{m'}}. \tag{5}$$

Here $\delta_{0m'}$ is a Kronecker symbol and $\delta_{m'} = \delta_{0m'} + 1$. For the coefficients $A^*_{n'm'}$, $B^*_{n'm'}$ we have the known formula:

$$\begin{Bmatrix} A^*_{n'm'} \\ B^*_{n'm'} \end{Bmatrix} = \frac{Gm^*}{a^{*n'+1}} \left(\frac{a^*}{r^*}\right)^{n'+1} P^{(m')}_{n'}(\sin\delta^*) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m'\alpha^* \tag{6}$$

defined as functions of time in the form of Fourier series (Kinoshita, 1977; Getino and Ferrandiz, 1991) with respect to the classical arguments of the Moon's orbital theory.

In accordance with the Takeuchi solution the spherical components of the displacement vector are defined by

$$
\begin{aligned}
u_r &= \sum_{n'=1}^{\infty} \frac{1}{r} l_{n'} W_{n'}, \\
u_\theta &= \frac{1}{r} \sum_{n'=1}^{\infty} F_{n'} \frac{\partial W_{n'}}{\partial \theta}, \\
u_\varphi &= \frac{1}{r \sin\theta} \sum_{n'=1}^{\infty} F_{n'} \frac{\partial W_{n'}}{\partial \varphi},
\end{aligned} \tag{7}
$$

where

$$
l_{n'} = n' F_{n'} + r^2 G_{n'}. \tag{8}
$$

We will assume that linear displacements of the particles of the mantle are small and the small variation of density ρ is described by the formula of Takeuchi

$$
\rho = \rho_0 - u_r \frac{\mathrm{d}\rho_0}{\mathrm{d}r} - \rho_0 \Delta, \tag{9}
$$

where ρ_0 is the density corresponding to the undeformable state, $u_r = (xu + yv + zw)/r$ being the radial displacement, and $\Delta = \mathrm{div} \cdot \boldsymbol{u}$ is the volume divergence. Now we consider the gravitational force function $U = -V$ (V is the gravitational potential) of the non-spherical and deformable mantle. The equations of this force function for some external P and for internal $\bar{P}$ points are defined by (Douboshin, 1975)

$$
U(R, \Phi, \Lambda) = \frac{Gm}{R} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{R_0}{R}\right)^n P_n^{(k)}(\cos\Phi)[C_{nk}\cos k\Lambda + S_{nk}\sin k\Lambda], \tag{10}
$$

$$
U(\bar{R}, \bar{\Phi}, \bar{\Lambda}) = \frac{G\bar{m}}{\bar{R}_0} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{\bar{R}}{\bar{R}_0}\right)^n P_n^{(k)}(\cos\bar{\Phi})[\bar{C}_{nk}\cos k\bar{\Lambda} + \bar{S}_{nk}\sin k\bar{\Lambda}], \tag{11}
$$

where R, Φ, Λ and $\bar{R}$, $\bar{\Phi}$, $\bar{\Lambda}$ are spherical coordinates of the points P and $\bar{P}$ in the coordinate system $Cxyz$ (the angles Φ and $\bar{\Phi}$ are the colatitudes). $P_n^{(k)}$ are associated Legendre functions. Here m and $\bar{m}$ is the mass of the Earth and of its mantle. R_0 and $\bar{R}_0$ are the mean equatorial radius of the external and internal surfaces of the mantle.

Equations (10) and (11) converge, consequently, in the domains $R > R_0$ and $\bar{R} < \bar{R}_0$ (Douboshin, 1975).

The coefficients C_{nk}, S_{nk} and $\bar{C}_{nk}$, $\bar{S}_{nk}$ in (10),(11) are non-dimensional and are defined by the following volume integrals:

$$
C_{nk} = \frac{2(n-k)!}{m\delta_k(n+k)!} \iiint_\sigma \left(\frac{r}{R_0}\right)^n \rho P_n^{(k)}(\cos\theta)\cos k\varphi \,\mathrm{d}\sigma,
$$

$$S_{nk} = \frac{2(n-k)!}{m\delta_k(n+k)!} \iiint\limits_{\sigma} \left(\frac{r}{R_0}\right)^n \rho P_n^{(k)}(\cos\theta) \sin k\varphi \, d\sigma \tag{12}$$

and

$$\bar{C}_{nk} = \frac{2(n-k)!}{\bar{m}\delta_k(n+k)!} \iiint\limits_{\sigma} \left(\frac{\bar{R}_0}{r}\right)^{n+1} \rho P_n^{(k)}(\cos\theta) \cos k\varphi \, d\sigma,$$

$$\bar{S}_{nk} = \frac{2(n-k)!}{\bar{m}\delta_k(n+k)!} \iiint\limits_{\sigma} \left(\frac{\bar{R}_0}{r}\right)^{n+1} \rho P_n^{(k)}(\cos\theta) \sin k\varphi \, d\sigma, \tag{13}$$

Here r, θ, φ are spherical coordinates of elementary mass $\rho\, d\sigma$ in the deformable state. In (12), (13) we use special sine $\delta_m = 2$ for $m = 0$ and $\delta_m = 1$ for $m > 0$. Thus parameter is connected with the Kronecker symbol δ_{0m} by the formula

$$\delta_m = \delta_{0m} + 1$$

We can note that the expressions (12) and (13) are practically identical, if in the first case the following value of the density

$$\bar{\rho} = \rho \frac{\bar{R}_0^{n+1} R_0^n}{r^{2n+1}}$$

is implied. We will use this remark in the construction of the variations of the coefficients (13) due to tidal deformations. Here we follow Ferrandiz and Getino (1993), in which the variations of the coefficients (12) for arbitrary n were obtained.

The integrals (12), (13) are spread over the whole volume of the deformed mantle, so it is convenient to transform the domain of integration to the undeformable body. Taking the relations (1) into account, the Jacobian of this transformation, neglecting the quadratic terms, will have the form (Getino and Ferrandiz, 1990) $J \approx 1 + \Delta$.

For a rigid non-spherical mantle (with quasi-spherical distribution of density) the integrals (12), (13) provide a constant value. And in case of the deformable body they will be defined as functions of time as the position vector $\boldsymbol{r}$ (1)–(3) will be some function of time due to the tidal gravitational influence from the Moon and Sun on the mantle as an elastic body. In our approach it is sufficient to assume that the particles of the deformable solid deviate slightly from the position they would occupy in the absence of the deformation. We can neglect the products of the deformations terms (theory of linear elasticity).

The full description of the tidal potential, the definition of the components of the displacement vector, and of linear transformations for the volume integral and other questions are given in detail by Getino and Ferrandiz (1994).

2 TIDAL VARIATION OF $\bar{C}_{nm}$

In this section let us undertake the general expression for the tidal variations of the coefficients $\bar{C}_{nm}$ ($m \neq 0$) which will be applied later to get the expressions of $\bar{S}_{nm}$.

According with (13) the variation of $\tilde{C}_{nm}$ is given by the integral

$$\bar{I}_{nm} = \iiint_{\sigma} \rho \bar{K}_{nm} \,\mathrm{d}\sigma, \tag{14}$$

where we have introduced the notation

$$\bar{K}_{nm} = r^{-n-1} P_{nm}(\cos\theta) \cos m\varphi. \tag{15}$$

Here $P_{nm} = P_n^{(m)}$ are associated Legendre functions.

The tidal variation of the coefficient (or integral (13)) in our linear theory (or approximation) will be defined by the formula

$$\delta \bar{I}_{nm} = \bar{Z}_1 + \bar{Z}_2, \tag{16}$$

where

$$\bar{Z}_1 = \iiint_{\sigma_0} \rho_0 \delta \bar{K}_{nm} \,\mathrm{d}\sigma_0, \tag{17}$$

$$\bar{Z}_2 = -\iiint_{\sigma_0} u_r \frac{\mathrm{d}\rho_0}{\mathrm{d}r} \delta \bar{K}_{nm} \,\mathrm{d}\sigma_0, \tag{18}$$

where u_r is the radial displacement (7), and the integral is extended to the 'unperturbed' volume (in our case on some mantle cover with external and internal radii R_0 and $\bar{R}_0$ and having a concentric distribution of density).

In reality we have

$$\begin{aligned} \bar{I}_{nm} &= \bar{I}^0_{nm} + \delta \bar{I}_{nm} \\ &= \iiint_{\sigma_0} (\rho_0 + \delta\rho_0)(K^0_{nm} + \delta K_{nm})(1 + \Delta) \,\mathrm{d}\sigma_0 \\ &= \iiint_{\sigma_0} \rho_0 K^0_{nm} \,\mathrm{d}\sigma_0 + \iiint_{\sigma_0} (\delta\rho_0 K^0_{nm} + \rho_0 \delta K_{nm} + \rho_0 \Delta K^0_{nm}) \,\mathrm{d}\sigma_0. \end{aligned} \tag{19}$$

So by (9)

$$\delta\rho_0 = -u_r \frac{\mathrm{d}\rho_0}{\mathrm{d}r} - \rho_0 \Delta,$$
$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

and for the first and third terms in the second integral (17) we will have

$$\delta\rho_0 + \rho_0 \Delta = -u_r \frac{\mathrm{d}\rho_0}{\mathrm{d}r}$$

and for the variation $\bar{I}_{nm}$ we have a representation in the form (15)–(18).

3 ANALYTICAL EXPRESSION FOR $\tilde{Z}_1$

First let us calculate the variation $\delta\bar{K}_{nm}$. We have

$$\delta\bar{K}_{nm} = \sum_{n'}^{\infty} \operatorname{grad}\bar{K}_{nm}\cdot u_{n'}, \quad u_{n'} = (u_{n'}, v_{n'}, w_{n'}) \tag{20}$$

or, in coordinate form,

$$\delta\bar{K}_{nm} = \sum_{n'=1}^{\infty}\left(\frac{\partial\bar{K}_{nm}}{\partial r}u_{n'r} + \frac{1}{r}\cdot\frac{\partial\bar{K}_{nm}}{\partial\theta}u_{n'\theta} + \frac{1}{r\sin\theta}\cdot\frac{\partial\bar{K}_{nm}}{\partial\varphi}u_{n'\varphi}\right) \tag{21}$$

where the components of the displacement vector in accordance with Takeuchi (1950) can be represented as:

$$\begin{aligned} u_{n'r} &= \frac{1}{r}l_{n'}W_{n'}, \\ u_{n'\theta} &= \frac{1}{r}F_{n'}\frac{\partial W_{n'}}{\partial\theta}, \\ u_{n'\varphi} &= \frac{1}{r\sin\theta}F_{n'}\frac{\partial W_{n'}}{\partial\varphi}, \end{aligned} \tag{22}$$

where

$$l_{n'} = n'F_{n'} + r^2G_{n'} \tag{23}$$

and $W_{n'}$ is a n^{th} harmonic of the tidal potential (3), (4).

Now we apply some algebra to the expression $\delta\bar{K}_{nm}$. First we obtain the next derivatives of the function (18):

$$\begin{aligned} \frac{\partial\bar{K}_{nm}}{\partial r} &= -(n+1)r^{-n-2}P_n^{(m)}(\cos\theta)\cos m\varphi = -\frac{(n+1)}{r}\bar{K}_{nm}, \\ \frac{\partial\bar{K}_{nm}}{\partial\theta} &= r^{-n-1}\frac{\partial P_n^{(m)}(\cos\theta)}{\partial\theta}\cos m\varphi, \\ \frac{\partial\bar{K}_{nm}}{\partial\varphi} &= r^{-n-1}P_n^{(m)}(\cos\theta)(-m\sin m\varphi). \end{aligned} \tag{24}$$

Now from (21) we find

$$\begin{aligned} \delta\bar{K}_{nm} = \sum_{n'=2}^{\infty}\Bigg[&\frac{(n+1)}{r^2}\bar{K}_{nm}l_{n'}W_{n'} + r^{-n-3}\frac{\partial P_n^{(m)}(\cos\theta)}{\partial\theta}F_{n'}\frac{\partial W_{n'}}{\partial\theta}\cos m\varphi \\ &- \frac{1}{\sin^2\theta}mr^{-n-3}F_{n'}P_n^{(m)}(\cos\theta)\frac{\partial W_{n'}}{\partial\varphi}\sin m\varphi\Bigg], \end{aligned} \tag{25}$$

where (4)

$$W_{n'} = \sum_{m'=0}^{n'} q_{n'm'}r^{n'}P_{n'm'}(\cos\theta)(A^*_{n'm'}\cos m'\varphi + B^*_{n'm'}\sin m'\varphi), \tag{26}$$

$$q_{n'm'} = \frac{(n'-m')!}{(n'+m')!}(2-\delta_{0m'}) = \frac{2(n'-m')!}{(n'+m')!\delta_{m'}}. \tag{27}$$

For the coefficients $A^*_{n'm'}$, $B^*_{n'm'}$ we have the following known formula:

$$\begin{Bmatrix} A^*_{n'm'} \\ B^*_{n'm'} \end{Bmatrix} = \frac{Gm^*}{a^{*n'+1}} \left(\frac{a^*}{r^*}\right)^{n'+1} P^{(m')}_{n'}(\sin\delta^*) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m'\alpha^*. \tag{28}$$

For the derivatives $\partial W_{n'}/\partial\theta$ and $\partial W_{n'}/\partial\varphi$ we have:

$$\frac{\partial W_{n'}}{\partial\theta} = \sum_{m'=0}^{n'} q_{n'm'} r^{n'} \frac{\partial P_{n'm'}(\cos\theta)}{\partial\theta}(A^*_{n'm'}\cos m'\varphi + B^*_{n'm'}\sin m'\varphi),$$
$$\frac{\partial W_{n'}}{\partial\varphi} = \sum_{m'=0}^{n'} m' q_{n'm'} r^{n'} P_{n'm'}(-A^*_{n'm'}\sin m'\varphi + B^*_{n'm'}\cos m'\varphi). \tag{29}$$

Now let us substitute formulae (26)–(29) into (25) and reduce the expression $\delta\bar{K}_{nm}$:

$$\begin{aligned}\delta\bar{K}_{nm} = \sum_{n'=1}^{\infty}\sum_{m'=0}^{n'} & -(n+1)r^{-n-3} l_{n'} P_{nm}(\cos\theta)\cos m\varphi\{q_{n'm'}r^{n'}P_{n'm'}(\cos\theta) \\ & \times [A^*_{n'm'}\cos m'\varphi + B^*_{n'm'}\sin m'\varphi]\} \\ & + r^{-n-3}F_{n'}\frac{\partial P_{nm}}{\partial\theta}\cos m\varphi\left\{q_{n'm'}r^{n'}\frac{\partial P_{n'm'}}{\partial\theta}[A^*_{n'm'}\cos m'\varphi + B^*_{n'm'}\sin m'\varphi]\right\} \\ & - r^{-n-3}F_{n'}\frac{1}{\sin^2\theta}P_{nm}(\cos\theta)m\sin m\varphi\{m'q_{n'm'}r^{n'}P_{n'm'}(\cos\theta) \\ & \times [-A^*_{n'm'}\sin m'\varphi + B^*_{n'm'}\cos m'\varphi]\}\end{aligned}$$

Finally we have:

$$\begin{aligned}\delta K_{nm} = \sum_{n'=1}^{\infty}\sum_{m'=0}^{n'} q_{n'm'} r^{n'-n-3} \Bigg\{ & A^*_{n'm'}\Bigg[-(n+1)l_{n'}P_{nm}P_{n'm'}\cos m\varphi\cos m'\varphi \\ & + F_{n'}\left(\frac{\partial P_{nm}}{\partial\theta}\frac{\partial P_{n'm'}}{\partial\theta}\cos m\varphi\cos m'\varphi + \frac{P_{nm}P_{n'm'}}{\sin^2\theta}mm'\sin m\varphi\sin m'\varphi\right)\Bigg] \\ & + B^*_{n'm'}\Bigg[-(n+1)l_{n'}P_{nm}P_{n'm'}\cos m\varphi\sin m'\varphi \\ & + F_{n'}\left(\frac{\partial P_{nm}}{\partial\theta}\frac{\partial P_{n'm'}}{\partial\theta}\cos m\varphi\sin m'\varphi - \frac{P_{nm}P_{n'm'}}{\sin^2\theta}mm'\sin m\varphi\cos m'\varphi\right)\Bigg]\Bigg\}.\end{aligned} \tag{30}$$

Let us calculate the volume integral

$$Z_1 = \iiint_{\sigma_0} \rho_0\delta K_{nm}\,\mathrm{d}\sigma_0 = \int_r\int_\theta\int_\varphi \rho_0\delta K_{nm} r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi. \tag{31}$$

Integrating with respect to the variable φ can reduce the integral (31). For this we take into account the trigonometric integrals:

$$\int_0^{2\pi} \cos m\varphi \cos m'\varphi \, \mathrm{d}\varphi = \pi \delta_{mm'} \delta_m,$$

$$\int_0^{2\pi} \sin m\varphi \sin m'\varphi \, \mathrm{d}\varphi = \pi \delta_{mm'},$$

$$\int_0^{2\pi} \cos m\varphi \sin m'\varphi \, \mathrm{d}\varphi = 0, \tag{32}$$

where

$$\delta_{mm'} = \begin{cases} 1 & \text{for } m = m' \\ 0 & \text{for } m \neq m' \end{cases} \qquad \delta_m = \begin{cases} 2 & \text{for } m = 0 \\ 1 & \text{for } m > 0 \end{cases}$$

As a result for integral (31) we obtain the more compact representation:

$$\bar{Z}_1 = \sum_{n'=1}^{\infty} \int_r \int_\theta q_{n'm'} \rho_0 r^{n'-n-1} A^*_{n'm} \pi$$
$$\times \left[-(n+1) l_{n'} P_{nm} P_{n'm} + F_{n'} \left(\frac{\partial P_{nm}}{\partial \theta} \frac{\partial P_{n'm}}{\partial \theta} + \frac{m^2}{\sin^2 \theta} P_{nm} P_{n'm} \right) \right] \sin \theta \, \mathrm{d}r \, \mathrm{d}\theta. \tag{33}$$

As a result of integration with respect to θ we have the following integrals (Getino, 1992):

$$\int_0^{\pi} P_{nm} P_{n'm} \sin \theta \, \mathrm{d}\theta = \frac{2}{(2n+1)} \frac{(n+m)!}{(n-m)!} \delta_{nn'}, \tag{34}$$

$$\int_0^{\pi} \left(\frac{\partial P_{nm}}{\partial \theta} \frac{\partial P_{n'm}}{\partial \theta} + \frac{m^2}{\sin^2 \theta} P_{nm} P_{n'm} \right) \sin \theta \, \mathrm{d}\theta = \frac{2n(n+1)}{(2n+1)} \frac{(n+m)!}{(n-m)!} \delta_{nn'}. \tag{35}$$

On the basis of these formulae we find:

$$\bar{Z}_1 = q_{nm} \frac{2\pi}{(2n+1)} A^*_{nm} \frac{(n+m)!}{(n-m)!} \int_r r^{-1} (n+1)(-l_n + nF_n) \rho_0 \, \mathrm{d}r. \tag{36}$$

In accordance with (23)

$$l_n = nF_n + r^2 G_n \tag{37}$$

and we obtain

$$-l_n + nF_n = -r^2 G_n, \tag{38}$$

and as a result (using formula (27)) we obtain

$$\tilde{Z}_1 = -\frac{4\pi(n+1)}{\delta_m(2n+1)} \cdot \frac{Gm^*}{a^{*n+1}} \left[\left(\frac{a^*}{r^*}\right)^{n+1} P_{nm}(\sin\delta^*)\cos m\alpha^* \right] \int rG_n\rho_0 \,\mathrm{d}r. \tag{39}$$

It is interesting to note that in the case of an internal mantle potential, the coefficient function $\tilde{Z}_1$, does not depend on the function F_n compared to the case of external geopotential coefficients.

Let us consider the expression for $\tilde{Z}_2$:

$$\tilde{Z}_2 = -\int\limits_{\sigma_0} u_r \frac{\mathrm{d}\rho_0}{\mathrm{d}r} \tilde{K}_{nm} \,\mathrm{d}\sigma_0, \tag{40}$$

where (see (15), (22), (26))

$$\begin{aligned} \tilde{K}_{nm} &= r^{-n-1} P_{nm}(\cos\theta)\cos m\varphi, \\ u_r &= \sum_{n'=1}^{\infty} \frac{1}{r} l_{n'} W_{n'}, \\ W_{n'} &= \sum_{m'=0}^{n'} q_{n'm'} r^{n'} P_{n'm'}(\cos\theta)(A^*_{n'm'}\cos m'\varphi + B^*_{n'm'}\sin m'\varphi). \end{aligned} \tag{41}$$

Substituting (41) into (40) we have

$$\begin{aligned} \tilde{Z}_2 = \sum_{n'=1}^{\infty} \sum_{m'=0}^{n'} \int\limits_r \int\limits_\theta \int\limits_\varphi \Big\{ & -q_{n'm'} r^{n'-n+1} P_{nm} P_{n'm'} \frac{\mathrm{d}\rho_0}{\mathrm{d}r} l_{n'} \\ & \times \cos m\varphi (A^*_{n'm'}\cos m'\varphi + B^*_{n'm'}\sin m'\varphi) \Big\} \sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\varphi. \end{aligned} \tag{42}$$

Taking into account relations (32) we obtain:

$$\tilde{Z}_2 = \sum_{n'=1}^{\infty} \int\limits_r \int\limits_\theta \left\{ -q_{n'm'} \pi r^{n'-n+1} P_{nm} P_{n'm} \frac{\mathrm{d}\rho_0}{\mathrm{d}r} l_{n'} A^*_{n'm} \right\} \sin\theta \,\mathrm{d}\theta \,\mathrm{d}r \tag{43}$$

and using formula (34) we find

$$\tilde{Z}_2 = -\int\limits_r q_{nm} \pi r \frac{\mathrm{d}\rho_0}{\mathrm{d}r} l_n A^*_{nm} \left[\frac{2}{(2n+1)} \cdot \frac{(n+m)!}{(n-m)!} \right] \mathrm{d}r, \tag{44}$$

where

$$l_n = nF_n + r^2 G_n \tag{45}$$

or

$$\tilde{Z}_2 = -\frac{4\pi}{(2n+1)\delta_m} \frac{Gm^*}{a^{*n+1}} \left[\left(\frac{a^*}{r^*}\right)^{n+1} P_{nm}(\sin\delta^*)\cos m\alpha^* \right]$$

$$\times \int_r \frac{d\rho_0}{dr}(nF_n + r^2 G_n)\, dr. \tag{46}$$

Returning to our initial formula (13), (14) for variations of the coefficients $\tilde{C}_{nm}$ we have:

$$\delta\tilde{C}_{nm} = \frac{2(n-m)!\tilde{R}_0^{n+1}}{\tilde{m}\delta_m(n+m)!}(\tilde{Z}_1 + \tilde{Z}_2)$$
$$= \frac{2(n-m)!\tilde{R}_0^{n+1}}{\tilde{m}\delta_m(n+m)!}\left[\frac{Gm^*}{a^{*n+1}}\left(\frac{a^*}{r^*}\right)^{n+1} P_{nm}(\sin\delta^*)\right]\cos m\alpha^* \left(\frac{-4\pi}{2n+1}\right)$$
$$\times \int_r \left[(n+1)rG_n\rho_0 + \frac{d\rho_0}{dr}(nF_n + r^2G_n)\right] dr. \tag{47}$$

Formula (47) defines the tidal variations of the coefficients $\tilde{C}_{nm}$, of the inner gravitational potential of the mantle. A similar formula holds for the tidal variations of the other coefficients $\tilde{S}_{nm}$. The final results can be presented in the form of the following formulae for the variations of the coefficients of the inner gravitational mantle potential:

$$\delta\tilde{C}_{nm} = \frac{2(n-m)!}{\delta_m(n+m)!}\tilde{D}_{nt}\left[\left(\frac{a^*}{r^*}\right)^{n+1} P_{nm}(\sin\delta^*)\cos m\alpha^*\right],$$
$$\delta\tilde{S}_{nm} = \frac{2(n-m)!}{\delta_m(n+m)!}\tilde{D}_{nt}\left[\left(\frac{a^*}{r^*}\right)^{n+1} P_{nm}(\sin\delta^*)\sin m\alpha^*\right], \tag{48}$$

where

$$\tilde{D}_{nt} = -\frac{4\pi G}{2n+1}\left(\frac{\tilde{R}_0}{a^*}\right)^{n+1}\frac{m^*}{\tilde{m}}\tilde{I}_n,$$
$$\tilde{I}_n = \int_r \left[(n+1)rG_n\rho_0 + \frac{d\rho_0}{dr}(nF_n + r^2G_n)\right] dr. \tag{49}$$

Here we also present similar formulae which define the tidal variations of the coefficients C_{nm}, S_{nm} of the external gravitational potential of the mantle (12) (Getino and Ferrandiz, 1994):

$$\delta C_{nm} = \frac{2(n-m)!}{\delta_m(n+m)!}D_{nt}\left[\left(\frac{a^*}{r^*}\right)^{n+1} P_{nm}(\sin\delta^*)\cos m\alpha^*\right],$$
$$\delta S_{nm} = \frac{2(n-m)!}{\delta_m(n+m)!}D_{nt}\left[\left(\frac{a^*}{r^*}\right)^{n+1} P_{nm}(\sin\delta^*)\sin m\alpha^*\right], \tag{50}$$

where

$$D_{nt} = -\frac{4\pi G}{2n+1}\left(\frac{1}{a^*}\right)^{n+1}\frac{m^*}{mR_0^n}I_n,$$

$$I_n = \int\limits_r \left(nr^{2n}\rho_0[(2n+1)F_n + r^2G_n] - \frac{\mathrm{d}\rho_0}{\mathrm{d}r} r^{2n+1}(nF_n + r^2G_n) \right) \mathrm{d}r. \quad (51)$$

From formulae (48), (49) and (50), (51) we find the simple relations:

$$\delta\tilde{C}_{nm} = k_n \delta C_{nm}, \quad \delta\tilde{S}_{nm} = k_n \delta S_{nm}, \quad (52)$$

where

$$k_n = \frac{m}{\bar{m}} \tilde{R}_0^{n+1} \bar{R}_0^n \frac{\bar{I}_n}{I_n}.$$

For $n = 2$ the numerical value I_2, was obtained as a result of integration over the mantle volume on the basis of the Takeuchi model 2 (Getino and Ferrandiz, 1991):

$$I_2 = 1.917290 \times 10^{50} \text{ c.g.s.} \quad (53)$$

with the corresponding numerical values of the parameter

$$D_{2t} = \begin{cases} 6.953379 \times 10^{36} & \text{(the Moon)} \\ 3.185508 \times 10^{36} & \text{(the Sun)} \end{cases} \quad (54)$$

We can produce analogous transformations for the variations of the inner potential coefficients (18). As a result, we have:

$$\begin{aligned}
\delta\bar{J}_2 &= -\frac{6\bar{R}_0^3}{\bar{m}} \bar{D}_t \left(\frac{a^\bullet}{r^\bullet}\right)^3 P_2(\sin\delta^\bullet), \\
\delta\bar{C}_{22} &= \frac{\bar{R}_0^3}{2\bar{m}} \bar{D}_t \left(\frac{a^\bullet}{r^\bullet}\right)^3 P_2^{(2)}(\sin\delta^\bullet)\cos 2\alpha^\bullet, \\
\delta\bar{S}_{22} &= \frac{\bar{R}_0^3}{2\bar{m}} \bar{D}_t \left(\frac{a^\bullet}{r^\bullet}\right)^3 P_2^{(2)}(\sin\delta^\bullet)\sin 2\alpha^\bullet, \\
\delta\bar{C}_{21} &= \frac{2\bar{R}_0^3}{\bar{m}} \bar{D}_t \left(\frac{a^\bullet}{r^\bullet}\right)^3 P_2^{(1)}(\sin\delta^\bullet)\cos \alpha^\bullet, \\
\delta\bar{S}_{21} &= \frac{2\bar{R}_0^3}{\bar{m}} \bar{D}_t \left(\frac{a^\bullet}{r^\bullet}\right)^3 P_2^{(1)}(\sin\delta^\bullet)\sin \alpha^\bullet,
\end{aligned} \quad (55)$$

where

$$\begin{aligned}
\bar{D}_t &= \frac{Gm^\bullet}{a^{\bullet 3}} \frac{2\pi}{15} \bar{I}_2, \\
\bar{I}_2 &= \int\limits_r \left[\frac{2\rho}{r}(5F_2(r) + r^2G_2(r)) - \left(\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{5\rho}{r}\right)(2F_2(r) + r^2G_2(r)) \right] \mathrm{d}r.
\end{aligned} \quad (56)$$

Integral (56) as well as integral (51) is spread over the elastic mantle and depends on its internal structure. Directly from formulae (48),(50),(52) we obtain

$$\delta\bar{J}_2 = k_2\delta J_2, \quad \delta\bar{C}_{22} = k_2\delta C_{22}, \quad \delta\bar{S}_{22} = k_2\delta S_{22},$$

$$\delta\tilde{C}_{21} = k_2\delta C_{21}, \quad \delta\tilde{S}_{21} = k_2\delta S_{21}, \tag{57}$$

where the coefficient of proportionality is defined as

$$k_2 = \bar{R}_0^3 R_0^2 \frac{\tilde{I}_2 m}{I_2 \tilde{m}}. \tag{58}$$

Using the variations of coefficients (50), (51), which were studied earlier (Ferraiidiz and Getino, 1993), and the numerical value of the k-parameter (54), (58), we can calculate the values of the variations of the other coefficients (57). Variations of the coefficients (53) were obtained on the basis of known trigonometric relations of the functions of time which appear in formulae (50) for the corresponding perturbing bodies (Kinoshita, 1977; Getino and Ferrandiz, 1991):

$$\begin{aligned}
\left(\frac{a^\bullet}{r^\bullet}\right)^3 P_2(\sin\delta^\bullet) &\cong 3\sum_i B_i \cos\Theta_i, \\
\left(\frac{a^\bullet}{r^\bullet}\right)^3 P_2^{(2)}(\sin\delta^\bullet) \begin{Bmatrix}\cos 2\alpha^\bullet \\ \sin 2\alpha^\bullet\end{Bmatrix} &\cong 3\sum_i\sum_\tau D_i(\tau) \begin{Bmatrix}-\cos \\ \sin\end{Bmatrix} (2\mu + 2\nu - \tau\Theta_i), \\
\left(\frac{a^\bullet}{r^\bullet}\right)^3 P_1^{(2)}(\sin\delta^\bullet) \begin{Bmatrix}\cos \alpha^\bullet \\ \sin \alpha^\bullet\end{Bmatrix} &\cong 3\sum_i\sum_\tau C_i(\tau) \begin{Bmatrix}\sin \\ \cos\end{Bmatrix} (\mu + \nu - \tau\Theta_i),
\end{aligned} \tag{59}$$

where

$$\begin{aligned}
B_i &= -\frac{1}{6}(3\cos^2\varepsilon - 1)A_i^{(0)} - \frac{1}{2}\sin 2\varepsilon A_i^{(1)} - \frac{1}{4}\sin^2\varepsilon A_i^{(2)}, \\
C_i(\tau) &= -\frac{1}{4}\sin 2\varepsilon A_i^{(0)} + \frac{1}{2}(1 + \tau\cos\varepsilon)(-1 + 2\tau\cos\varepsilon)A_i^{(1)} \\
&\quad + \frac{\tau}{4}\sin\varepsilon(1 + \tau\cos\varepsilon)A_i^{(2)}, \\
D_i(\tau) &= -\frac{1}{2}\sin^2\varepsilon A_i^{(0)} + \tau\sin\varepsilon(1 + \tau\cos\varepsilon)A_i^{(1)} - \frac{1}{4}(1 + \tau\cos\varepsilon)^2 A_i^{(2)}, \\
\tau &= \pm 1.
\end{aligned} \tag{60}$$

Numerical values of the coefficients $A_i^{(j)}$ were obtained by Kinoshita (1977). In the construction of these relations, it was supposed that the angle σ between the vector of the angular moment of the Earth $\boldsymbol{G}$ and polar axis Cz is small (so $\sin\sigma \approx 0$, $\cos\sigma \approx 1$). ε is the angle between ecliptic plane and the intermediate plane, which is orthogonal to the vector $\boldsymbol{G}$ ($\varepsilon = 23.45°$).

The argument Θ, is a linear combination with numerical coefficients of the arguments of the Moon's orbital theory:

$$\begin{aligned}
\Theta_i &= m_1 l_M + m_2 l_S + m_3 F + m_4 D + m_5 \Omega, \\
F &= l_M + g_M, \\
D &= l_M + g_M + h_M - l_S - g_S - h_S, \\
i &= (m_1, m_2, m_3, m_4, m_5).
\end{aligned}$$

Here l_M, g_M, h_M and l_S, g_S, h_S are the Delaunay variables for the Moon and the Sun. $\mu + \nu$ is the angle of the Earth's rotation.

Variations of the coefficients (50) were presented in the following form (Ferrandiz and Getino, 1993):

$$\begin{aligned}
\delta J_2 &= \sum_i K_2(i) \cos \Theta_i, \\
\delta C_{22} &= \sum_i K_{22a}(i) \cos(2\mu + 2\nu - \Theta_i) + \sum_i K_{22b}(i) \cos(2\mu + 2\nu + \Theta_i), \\
\delta S_{22} &= -\sum_i K_{22a}(i) \sin(2\mu + 2\nu - \Theta_i) - \sum_i K_{22b}(i) \sin(2\mu + 2\nu + \Theta_i), \\
\delta C_{21} &= \sum_i K_{21a}(i) \sin(\mu + \nu - \Theta_i) + \sum_i K_{21b}(i) \sin(\mu + \nu + \Theta_i), \\
\delta S_{21} &= \sum_i K_{21a}(i) \cos(\mu + \nu - \Theta_i) + \sum_i K_{21b}(i) \cos(\mu + \nu + \Theta_i).
\end{aligned} \tag{61}$$

In Table 1 of Ferrandiz and Getino (1993), the values of the main coefficients K_2, K_{21a}, K_{21b}, K_{22a} and K_{22b} are listed as well as their respective arguments Θ_i.

Variations of the coefficients (57) are defined by analogous formulae:

$$\begin{aligned}
\delta \tilde{J}_2 &= k_2 \sum_i K_2(i) \cos \Theta_i, \\
\delta \tilde{C}_{22} &= k_2 \sum_i K_{22a}(i) \cos(2\mu + 2\nu - \Theta_i) + k_2 \sum_i K_{22b}(i) \cos(2\mu + 2\nu + \Theta_i), \\
\delta \tilde{S}_{22} &= -k_2 \sum_i K_{22a}(i) \sin(2\mu + 2\nu - \Theta_i) - k_2 \sum_i K_{22b}(i) \sin(2\mu + 2\nu + \Theta_i), \\
\delta \tilde{C}_{21} &= k_2 \sum_i K_{21a}(i) \sin(\mu + \nu - \Theta_i) + k_2 \sum_i K_{21b}(i) \sin(\mu + \nu + \Theta_i), \\
\delta \tilde{S}_{21} &= k_2 \sum_i K_{21a}(i) \cos(\mu + \nu - \Theta_i) + k_2 \sum_i K_{21b}(i) \cos(\mu + \nu + \Theta_i).
\end{aligned} \tag{62}$$

In Table 1 the main coefficients of the variations (34) are given; they are listed together with their respective arguments Θ_i (1 unit $= 10^{-9}$). These values were obtained for the value $k_2 = 0.10743$ which was found from formulae (23), (28), (30) for the well-known Earth model 1066A (Gilbert and Dziewonski, 1975).

By an analogous method we can obtain variations of the coefficients of the third and higher harmonics of the mantle external and internal potentials due to its tidal deformations.

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Table 1.

l_M, l_S, F, D, Ω	$k_2K_2(i)$	$k_2K_{22a}(i)$	k_2K_{22b}	k_2K_{21a}	k_2K_{21b}
1 0 0 −2 0	0.2742	−0.0071	−0.0071	−0.0656	−0.0656
1 0 0 0 0	1.4337	−0.0372	−0.0372	−0.3429	−0.3429
0 0 0 2 0	0.2380	−0.0062	−0.0062	−0.0569	−0.0569
1 0 2 0 1	0.2154	0.0375	−0.0016	0.1574	−0.0231
0 0 2 0 1	1.1251	0.1959	−0.0084	0.8226	−0.1202
0 0 0 0 1	−1.1372	−0.1981	0.0084	−0.8317	0.1216
−1 0 2 2 2	0.0988	−0.1912	−0.0004	0.1586	−0.0070
−1 0 2 0 2	−0.0767	0.1487	0.0003	−0.1232	0.0053
1 0 2 0 2	0.5198	−1.0063	−0.0019	0.8350	−0.0358
0 0 2 2 2	0.0829	−0.1608	−0.0003	0.1334	−0.0058
0 0 2 0 2	2.7139	−5.2563	−0.0098	4, 3613	−0.1876
0 1 0 0 0	0.2005	−0.0053	−0.0053	−0.0480	−0.0480
0 1 2 −2 2	0.0740	−0.1429	−0.0003	0.1188	0.0012
0 0 2 −2 2	1.2598	−2.4397	−0.0046	2.0244	0.0216

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