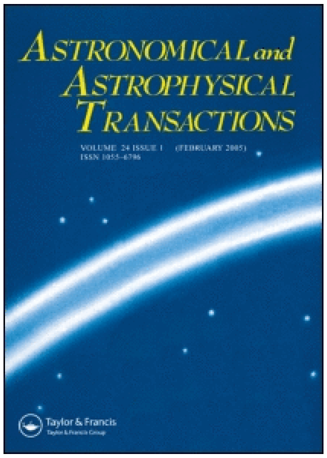

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UNIVERSAL SOLUTION FOR MOTIONS IN A CENTRAL FORCE FIELD

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Central forces are very interesting in Mechanics, since they are simple and model several physical problems quite well. Here we consider a potential function of the type $U(r) = -A/r - B/(2r^2)$. For this potential by means of the regularizing Sundman's transformation, and making use of Stumpff's functions, we obtain a solution that is regular and valid for all possible types of motion, bounded or not.

KEY WORDS Central forces, universal variables, Sundman transformation

1 INTRODUCTION

Central force fields are among the first forces one meet in Mechanics. Indeed, they represent integrable systems, for they are systems of only one degree of freedom, and besides, fundamental motions like the Keplerian one, harmonic oscillators and diffusors belong to this class of forces. Besides, more complex problems are represented by adding perturbations to these more simple problems mentioned above.

In this paper, we deal with potentials of the type

$$U(r) = -\frac{A}{r} - \frac{B}{2r^2}, \tag{1}$$

where A and B are parameters, independent of the radial distance r. This potential represents the class of what Deprit (1981) dubbed *quasi-Keplerian* systems.

Deprit (1981) obtained analytical solutions by means of a canonical transformation – the *torsion* – which converts the quasi-Keplerian Hamiltonian into a pure Keplerian one.

For this type of potentials, Rodriguez and Brun (1998) studied under what conditions particles moving under a similar potential have closed orbits. In this paper, we are dealing exclusively with the analytical integration.

We meet several physical problems with this potential, for it represents the intermediary problem in celestial mechanics (Deprit, 1981; Aparicio and Floría, 1996), that is to say, the resulting problem from a perturbed Keplerian problem (like the satellite problem), after some simplification based on Lie transformations is performed. Another problem of this type is the so called Maneff's potential (Maneff, 1924)

$$U(r) = -\frac{Gm_1m_2}{r}\left(1 + \frac{3G(m_1+m_2)}{2c^2r}\right),$$

where G is the Gaussian constant, m_1 and m_2 the masses of two particles, and c is the speed of light. This simple post-Newtonian non-relativistic potential may be used for describing the secular motions of the pehihelia of the inner planets. Recently, some authors (Mioc, 1995; Aparicio and Floría, 1996) addressed their attention to this problem.

In particular, in the work of Mioc (1995), a solution valid even for collisions is presented; however, the solution is split into three cases, elliptic, parabolic and hyperbolic (which actually means energy < 0, $= 0$ or > 0, respectively). Here we present a solution – in a similar way as in (Viñuales, Cid, and Elipe, 1995) – that is valid for whatever value of the energy by means of a universal formulation, the Stumpff functions.

2 CLASSICAL SOLUTION

Since the motion of a particle in the gravitational field given by (1) is planar, we can use polar coordinates (r, θ) to describe the motion on its plane. The Lagrangian function is

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{A}{r} + \frac{B}{2r^2},$$

and its Lagrangian equations are

$$\ddot{r} - r\dot{\theta}^2 + \frac{A}{r^2} + \frac{B}{r^3} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0. \tag{2}$$

The second equation just tells us that the angular momentum $\Theta = r^2\dot{\theta}$ is an integral.

The energy h is

$$h = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{A}{r} - \frac{B}{2r^2},$$

and by virtue of the second equation (2), it is

$$h = \frac{1}{2}\dot{r}^2 - \frac{A}{r} - \frac{C}{2r^2}, \tag{3}$$

where now $C = B - \Theta^2$.

From this last equation, and choosing as the initial instant t_0 ($r(t_0) = r_0$) we have

$$t - t_0 = \int_{r_0}^{r} \frac{r\,\mathrm{d}r}{\sqrt{2hr^2 + 2Ar + C}}.$$

To solve this quadrature, it is necessary to analyze three cases separately, depending on the sign of the energy integral h. It is an elementary task to find the solution for the three cases:

a. $h > 0$

$$t - T = \frac{1}{2h}\left[\sqrt{2hr^2 + 2Ar + C} - \frac{A}{\sqrt{2h}}\log\left(r + \frac{A}{2h} + \sqrt{\frac{2hr^2 + 2Ar + C}{2h}}\right)\right]_{r_0}^{r}$$

b. $h = 0$

$$t - T = \left[\frac{Ar - C}{3A^2}\sqrt{2Ar + C}\right]_{r_0}^{r}$$

c. $h < 0$

$$t - T = \frac{1}{2h}\left[\sqrt{2hr^2 + 2Ar + C} - \frac{A}{\sqrt{-2h}}\arcsin\frac{2hr + A}{\sqrt{A^2 - 2hC}}\right]_{r_0}^{r}$$

Note that equations (2) are singular in the case of collisions ($r = 0$), and that, depending on the different values of the energy, it is necessary to split the set of solutions. A more compact solution, regular and valid for whatever initial conditions would be desirable. This is achieved in the next section.

3 UNIVERSAL FORMULATION

The Newtonian equations of motion are

$$\ddot{\boldsymbol{x}} = -\left(\frac{A}{r^2} + \frac{B}{r^3}\right)\frac{\boldsymbol{x}}{r}. \tag{4}$$

Hence,

$$\dot{\boldsymbol{x}}^2 + \boldsymbol{x}\cdot\ddot{\boldsymbol{x}} = (\dot{r}^2 + r^2\dot{\theta}^2) - \frac{A}{r} - \frac{B}{r^2},$$

and taking into account the first equation of (2), there results

$$\dot{\boldsymbol{x}}^2 + \boldsymbol{x}\cdot\ddot{\boldsymbol{x}} = \dot{r}^2 + r\ddot{r}, \tag{5}$$

but, on the other hand,

$$\dot{\boldsymbol{x}}^2 = 2h + \frac{2A}{r} + \frac{B}{r^2},$$

hence

$$\dot{\boldsymbol{x}}^2 + \boldsymbol{x}\cdot\ddot{\boldsymbol{x}} = 2h + \frac{A}{r},$$

which, after replacing it into (5), gives the formula

$$\dot{r}^2 + r\ddot{r} = 2h + \frac{A}{r}. \tag{6}$$

At this point, we make a change of the independent variable $\mathrm{d}t = r\,\mathrm{d}s$. This is a classical change, dating from Sundman (Sundman, 1995) for regularizing the equations of motion in the three-body problem. After this change, and putting $(') = \mathrm{d}/\mathrm{d}s$, the equation (6) is converted into

$$r'' - 2hr = A, \tag{7}$$

that is always regular, since there is no denominator in it, and besides, the equation is linear with constant coefficients, which integration is immediate. Indeed, let us denote by

$$x_1(s;h) = \frac{1}{2}[\exp(\sqrt{2h}s) + \exp(-\sqrt{2h}s)],$$
$$x_2(s;h) = \frac{1}{2}[\exp(\sqrt{2h}s) - \exp(-\sqrt{2h}s)](2h)^{-1/2},$$

two particular independent solutions – its Wronskian is the unit – of the second order homogeneous equation. Note that although h appears in the denominator of x_2, this solution is regular for whatever value of h since $\lim_{h\to 0} x_2(s;h) = s$.

Let us take

$$x_3(s;h) = \frac{A(x_1(s;h) - 1}{2h},$$

to be the particular solution of the complete equation (7), which again is regular for whatever value of h. The general solution of (7) is the linear combination

$$r(s;h;\alpha_1,\alpha_2) = \alpha_1 x_1(s;h) + \alpha_2 x_2(s;h) + x_3(s;h), \tag{8}$$

where α_1 and α_2 are arbitrary constants, obtained from the initial conditions.

If r_0 and r_0' denote the values of r and r' at the initial instant $t_0 = s_0$, by differentiating the expression (8),

$$r' = (2h\alpha_1 + A)x_2(s;h) + \alpha_2(s;h),$$

one has the result

$$r_0 = r(s_0;h) = \alpha_1, \quad r_0' = r'(s_0;h) = \alpha_2,$$

and hence

$$r(s;h) = r_0 x_1(s;h) + r_0' x_2(s;h) + \frac{A(x_1(s;h) - 1)}{2h}. \tag{9}$$

The two independent variables t and s are related through the quadrature

$$t = \int_{s_0}^{s} r(s,h)\,\mathrm{d}s,$$

which solution is

$$t = r_0 x_2(s;h) + \frac{r_0'(x_1(s;h) - 1)}{2h} + \frac{A(x_2(s;h) - s)}{2h}. \tag{10}$$

an expression that is valid, too, for all h, for $\lim_{h\to 0}(x_2(s;h) - s)/(2h) = s^3/6$. Equation (10) is a one-to-one map from $\mathbb{R}$ onto $\mathbb{R}$, and the same happens with its inverse. Note that the new variable s is a generalized eccentric anomaly (Danby, 1988, pp. 169)

So far, we did not choose any particular situation for the initial conditions. By choosing the initial instant t_0 to be the instant of passage for the periastrum T (which corresponds to the shortest radial distance), we have $r'_0 = 0$, and equations (9) and (10), simply read

$$
\begin{aligned}
r(s;h) &= r_0 x_1(s;h) + \frac{A(x_1(s;h) - 1)}{2h}, \\
t - T &= r_0 x_2(s;h) + \frac{A(x_2(s;h) - s)}{2h}.
\end{aligned} \tag{11}
$$

At this point, it is time to introduce the Stumpff functions (Stumpff, 1947; Stumpff, 1965). These functions are denned by the absolutely convergent expansions

$$
c_n(z) = \sum_{k\geq 0}(-1)^k \frac{z^k}{(2k+n)!}, \quad \text{with } n \geq 0 \text{ and } z \in \mathbb{C}.
$$

If we define the associated Stumpff's functions as

$$
\nu_n(s;h) = s^n c_n(-2hs^2), \tag{12}
$$

there results that

$$
\begin{aligned}
\nu_0(s;h) &= x_1(s;h), \quad \nu_2(s;h) = \frac{x_1(s;h) - 1}{2h}, \\
\nu_1(s;h) &= x_2(s;h), \quad \nu_3(s;h) = \frac{x_2(s;h) - s}{2h}.
\end{aligned}
$$

Thus, we may express the solution (11) in terms of these associated Stumpff functions as

$$
\begin{aligned}
r &= r_0\nu_0(s;h) + A\nu_2(s;h), \\
t - T &= r_0\nu_0(s;h) + A\nu_3(s;h).
\end{aligned} \tag{13}
$$

Note that the functions $\nu_2(s;h)$, $\nu_3(s;h)$, in despite of their appearance, are well defined for all values of h because of the definition (12).

In summary, we conclude that after some manipulation of the equations, and by means of the regularizing Sundman's transformation, we obtain the equations of motion as a regular system made of a second-order differential equations with constant coefficients. By means of the Stumpff functions, the same solution is valid for all possible types of motion.

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