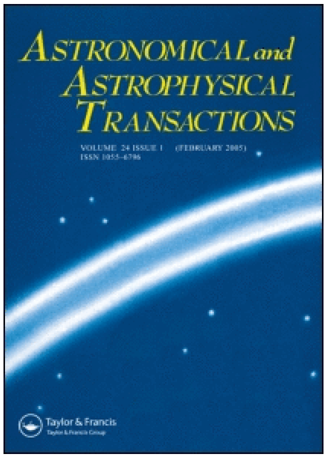

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ON INTEGRABLE CASES OF THE POINCARE PROBLEM

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The equations of motion for the classical Poincaré problem of the rotational motion of a rigid body with an ellipsoidal cavity containing liquid in canonical Andoyer variables have been obtain. Three integrable cases of this problem were established and their full systems of integrals and general solutions were constructed.

KEY WORDS Poincaré problem, integrable cases, Andoyer variables

1 INTRODUCTION

Oscillations of a rotating liquid core in the cavity of a rigid envelope have been studied by many authors, starting from the classical works of Poincaré (1910). The basis of these studies usually contains the equations of motion in quasi-coordinates, but in recent years a new approach to the classical problem has been suggested, based on the equations of motion in Andoyer variables (Sevilla and Romero, 1987; Getino and Ferrandiz, 1997; etc.). Andoyer variables and angle-action variables (for the Euler–Poinsot problem) were effectively used for studies of integrability of the native Kirgoff problem (Barkin and Borisov, 1989).

The papers of Sevilla and Romero (1987), Getino and Ferrandiz (1997) and others were directed to the problems of the Earth's rotation. The principal effects of the liquid core in the Earth's rotation were studied, and amplitudes of lunar and solar perturbations, corrected by the liquidcore influence, were constructed (Getino and Ferrandiz, 1997). The purpose of the above-mentioned papers was to give an analytical description of the Earth's rotation effects and the authors naturally used some simplifications of the equations of motion.

In this work we consider an exact treatment of the classical Poincaré problem in Andoyer variables to study integrable cases of this problem.

The canonical equations of the Poincaré problem in Andoyer variables were obtained. Three cases of the integrability of these equations have been identified and studied for which the equations of the problem are reduced to a one-degree-of-freedom system. The full system of first integrals and quadratures was obtained. One of them is the Poincaré case of an axysimmetric body and liquid core and the other two are generated cases, for which the ellipsoidal cavity is a thin disk.

2 CANONICAL EQUATIONS

Let us consider the rotational motion of a rigid body with an ellipsoidal cavity of a homogeneous incompressable liquid about its centre of mass in the field of an arbitrary external forces.

Let $CXYZ$ be a Cartesian reference system with the origin at the centre of mass C of the body, the axes of which are fixed in space. $Cxyz$ is a reference system the axes of which are directed along the principal central axes of inertia of the body. A, B and C are the principal moment's of inertia corresponding to the axes Cx, Cy and Cz.

The body is a composition of two bodies: P_m is a rigid envelope (mantle) and P_c is a liquid ellipsoidal body (core) situated in an ellipsoidal cavity of the mantle. We assume that that corresponding axes of the ellipsoid coincide with the coordinate axes Cx, Cy and Cz and their semiaxes are equal to a, b and c, respectively. We assume also that for the mutual positions of the mantle and core the coordinate axes Cx, Cy and Cz are principal central axes of inertia for the mantle, for the core and for the full mechanical system.

For variables and parameters of the full body we will not use any indexes, but for the mantle and core characteristics (parameters and variables) here we use notation with indexes m and c, respectively.

Let us denote the axial moment of inertia of the full body, of the mantle and the core as: A, B, C; A_m, B_m, C_m; A_c, B_c, C_c; and let F_c, E_c, D_c be the constant characteristics of the core similar to the products of inertia. Due to these assumptions we have the following simple relations (Poincaré, 1910):

$$A = A_m + A_c, \qquad B = B_m + B_c, \qquad C = C_m + C_c, \tag{1}$$

$$A_c = \frac{1}{5}m_c(b^2+c^2), \quad B_c = \frac{1}{5}m_c(a^2+c^2), \quad C_c = \frac{1}{5}m_c(a^2+b^2),$$
$$F_c = \frac{2}{5}m_c bc, \qquad E_c = \frac{2}{5}m_c ac, \qquad D_c = \frac{2}{5}m_c ab, \tag{2}$$

where m_c is the mass of the liquid core.

The orientation and rotation of the envelope P_m are given by the Euler angles and components of the angular velocity of its rotation ω_m, with respect to the reference system $CXYZ$:

$$\Psi_m, \Theta_m, \Phi_m;\ p_m, q_m, r_m. \tag{3}$$

For a description of the relative simple motion of the liquid (in Poincaré's sense) in the cavity P_c we introduce a new reference system $C_c x_c y_c z_c$, related to the core.

Its orientation and rotation we define by the Euler angles. To save the symmetrical notation of the variables let us suppose that the reference system $C_c x_c y_c z_c$ is the basic system and the other $Cxyz$ system is rotating with a definite angular velocity vector ω_c with respect to the first. The corresponding Euler angle and projections of the vector ω_c on the axes $Cxyz$ we denote as:

$$\Psi_c, \Theta_c, \Phi_c;\ p_c, q_c, r_c. \tag{4}$$

From a geometrical point of view the variables (3), (4) are similar. The components p_c, q_c, r_c, defining the vector ω_c, differ from the classical Poincaré notation by a sign (Poincaré, 1910). In (3), (4) Ψ is the angle of precession, Θ is the angle of nutation, Φ is the angle of intrinsic rotation, and the components of the angular velocities are defined by the following kinematical Euler equations:

$$\begin{aligned} p_s &= \sin\Phi_s \sin\Theta_s \dot{\Psi}_s + \cos\Phi_s \dot{\Theta}_s, \\ q_s &= \cos\Phi_s \sin\Theta_s \dot{\Psi}_s - \sin\Phi_s \dot{\Theta}_s, \\ r_s &= \cos\Theta_s \dot{\Psi}_s + \dot{\Phi}_s \qquad (s = m, c). \end{aligned} \tag{5}$$

The kinetic energy of the body with the liquid core in the variables (3), (4) is given by the known expression (Poincaré, 1910):

$$2T = Ap_m^2 + Bq_m^2 + Cr_m^2 + A_c p_c^2 + B_c q_c^2 + C_c r_c^2 - 2F_c p_m p_c - 2E_c q_m q_c - 2D_c r_m r_c. \tag{6}$$

Let us assume that the body moves under the action of potential forces with force function

$$U = U(\Psi_m, \Theta_m, \Phi_m, \Psi_c, \Theta_c, \Phi_c, t). \tag{7}$$

The canonical momentum conjugated to the introduced generalized coordinates (to Euler angles) are defined:

$$\begin{aligned} p_{\Psi_s} &= \frac{\partial T}{\partial \dot{\Psi}_s} = \lambda_s \sin\Theta_s \sin\Phi_s + \mu_s \sin\Theta_s \cos\Phi_s + \nu_s \cos\Theta_s, \\ p_{\Theta_s} &= \frac{\partial T}{\partial \dot{\Theta}_s} = \lambda_s \cos\Phi_s - \mu_s \sin\Phi_s, \\ p_{\Phi_s} &= \frac{\partial T}{\partial \dot{\Phi}_s} = \nu_s \qquad (s = \ , c) \end{aligned} \tag{8}$$

For simplicity here and below we omit the first index (m) of variables, putting $s = (\ , c)$. In this case:

$$\lambda = Ap_m - F_c p_c, \quad \mu = Bq_m - E_c q_c, \quad \nu = Cr_m - D_c r_c \tag{9}$$

and

$$\lambda_c = A_c p_c - F_c p_m, \quad \mu_c = B_c q_c - E_c q_m, \quad \nu_c = C_c r_c - D_c r_m \tag{10}$$

are the projections of the vector of the full angular momentum of the body with liquid core (9) (with respect to its centre of mass) and of the vector angular momentum of the liquid core (10) (with respect to the centre of mass of the liquid core) on the axes of the Cartesian reference system $Cxyz$ and $C_c x_c y_c z_c$, respectively. In the canonical variables (4), (5), (8), the equations of motion of the Poincaré problem have the following form:

$$\frac{d(\Psi_s, \Theta_s, \Phi_s)}{dt} = \frac{\partial K}{\partial(p_{\Psi_s}, p_{\Theta_s}, p_{\Phi_s})},$$
$$\frac{d(p_{\Psi_s}, p_{\Theta_s}, p_{\Phi_s})}{dt} = -\frac{\partial K}{\partial(\Psi_s, \Theta_s, \Phi_s)}. \tag{11}$$

On the basis of formulae (5)–(10) for the Hamiltoman K we obtain the following expression:

$$K = T - U = \frac{1}{2}[\Lambda\lambda^2 + M\mu^2 + N\nu^2 + \Lambda_c\lambda_c^2 + M_c\mu_c^2 + N_c\nu_c^2]$$
$$+ P_c\lambda\lambda_c + Q_c\mu\mu_c + R_c\nu\nu_c - U(\Psi, \Theta, \Phi, \Psi_c, \Theta_c, \Phi_c, t), \tag{12}$$

where

$$\lambda_s = p_{\Theta_s}\cos\Phi_s + \frac{\sin\Phi_s}{\sin\Theta_s}(p_{\Psi_s} - p_{\Phi_s}\cos\Theta_s),$$
$$\mu_s = -p_{\Theta_s}\sin\Phi_s + \frac{\cos\Phi_s}{\sin\Theta_s}(p_{\Psi_s} - p_{\Phi_s}\cos\Theta_s),$$
$$\nu_s = p_{\Phi_s},$$
$$G_s = \sqrt{p_{\Theta_s}^2 + p_{\Phi_s}^2 + \cos ec^2\Theta_s(p_{\Psi_s} - p_{\Phi_s}\cos\Theta_s)^2}. \tag{13}$$

The constant coefficients in the Hamiltonian (12), (13) are defined by:

$$\Lambda = \frac{A_c}{\Delta_1}, \qquad M = \frac{B_c}{\Delta_2}, \qquad N = \frac{C_c}{\Delta_3},$$
$$\Lambda_c = \frac{A}{\Delta_1}, \qquad M_c = \frac{B}{\Delta_2}, \qquad N_c = \frac{C}{\Delta_3},$$
$$P_c = \frac{F_c}{\Delta_1}, \qquad Q_c = \frac{E_c}{\Delta_2}, \qquad P_c = \frac{D_c}{\Delta_3},$$
$$\Delta_1 = A_cA - F_c^2, \quad \Delta_2 = B_cB - E_c^2, \quad \Delta_3 = C_cC - D_c^2. \tag{14}$$

Remark. From the differential equations (11)–(14) the classical equations of the Poincaré problem, including the Helmgolz equations, are obtained directly.

3 CANONICAL EQUATIONS IN ANDOYER VARIABLES

Let us fullfil the canonical transformation from variables

$$\Psi_s, \Theta_s, \Phi_s; p_{\Psi_s}, p_{\Theta_s}, p_{\Phi_s} \quad (s = , c) \tag{15}$$

to Andoyer variables (Andoyer, 1923)

$$L_s, G_s, H_s, l_s, g_s, h_s \quad (s = , c). \tag{16}$$

This canonical transformation of variables is discussed in many papers and more details are given by Barkin (in press). Here we will not give a description of these variables, referring the reader to the above paper, or many other. Here we give only the main and simple form of this canonical transformation:

$$\lambda_s = G_s \sin\theta_s \sin l_s, \quad \mu_s = G_s \sin\theta_s \cos l_s, \quad \nu_s = G_s \cos\theta_s \qquad (s = , c),$$

where

$$\cos\theta_s = \frac{L_s}{G_s}, \quad \sin\theta_s = \frac{\sqrt{G_s^2 - L_s^2}}{G_s}.$$

The Euler angles Ψ_s, Θ_s, Φ_s and the direction cosines of the axes of the body $Cxyz$ are expressed in Andoyer variables by known formulae (Barkin, in press).

As result we obtain the following canonical form of the differential equations of the Poincaré problem:

$$\begin{aligned} \frac{\mathrm{d}(l_s, g_s, h_s)}{\mathrm{d}t} &= \frac{\partial K}{\partial(L_s, G_s, H_s)}, \\ \frac{\mathrm{d}(L_s, G_s, H_s)}{\mathrm{d}t} &= -\frac{\partial K}{\partial(l_s, g_s, h_s)} \qquad (s = , c) \end{aligned} \tag{17}$$

with Hamiltonian

$$\begin{aligned} K = {} & \frac{1}{2}G^2 \left[\left(\frac{A_c}{\Delta_1}\sin^2 l + \frac{B_c}{\Delta_2}\cos^2 l\right)\sin^2\theta + \frac{C_c}{\Delta_3}\cos^2\theta\right] \\ & + \frac{1}{2}G_c^2 \left[\left(\frac{A}{\Delta_1}\sin^2 l_c + \frac{B}{\Delta_2}\cos^2 l_c\right)\sin^2\theta_c + \frac{C}{\Delta_3}\cos^2\theta_c\right] \\ & + GG_c \left[\left(\frac{F_c}{\Delta_1}\sin l \sin l_c + \frac{E_c}{\Delta_2}\cos l \cos l_c\right)\sin\theta\sin\theta_c + \frac{D_c}{\Delta_3}\cos\theta\cos\theta_c\right] \\ & - U(l, g, h, L, G, H; l_c, g_c, h_c, L_c, G_c, H_c; t). \end{aligned} \tag{18}$$

Here θ and θ_s are the angles between the vectors of the angular momenta of the body and of the liquid core G and G_c and the corresponding coordinate axes Cz and Cz_c:

$$\cos\theta_c = \frac{L}{G}, \quad \cos\theta = \frac{L_c}{G_c}, \quad \sin\theta = \frac{\sqrt{G^2 - L^2}}{G}, \quad \sin\theta_c = \frac{\sqrt{G_c^2 - L_c^2}}{G_c}.$$

Equations similar to (17)–(18) were effectively used in studies of unperturbed and perturbed Earth rotation (Getino and Ferrandiz, 1997). For these studies some simplification and reduction of the equations of the motion were used. Here we consider prelimanary studies of the integrability of these equations in an exact treatment of the problem, but we will be restricted to the case of the free motion of a body in the absence any force action (in this case $U = 0$).

4 CASES OF REDUCTION OF THE PROBLEM TO ONE DEGREE OF FREEDOM

The problem (17), (18) is obviously reduced to a one-degree-of-freedom system if the following conditions are satisfied:

$$\frac{A_c}{\Delta_1} = \frac{B_c}{\Delta_2}, \quad \frac{A}{\Delta_1} = \frac{B}{\Delta_2}, \quad \frac{F_c}{\Delta_1} = \frac{E_c}{\Delta_2}, \tag{19}$$

$$\Delta_1 = AA_c - F_c^2, \quad \Delta_2 = BB_c - E_c^2. \tag{20}$$

If we use the simple canonical transformation

$$l - l_c = \lambda, \qquad L = \Lambda,$$
$$l_c = \lambda_c, \quad L + L_c = \Lambda_c$$

the Hamiltonian of the Poincaré problem will be

$$K = \frac{A_c}{2\Delta_1}G^2 + \frac{1}{2}\left(\frac{C_c}{\Delta_3} - \frac{A_c}{\Delta_1}\right)\Lambda^2 + \frac{A}{2\Delta_1}G_c^2 + \frac{1}{2}\left(\frac{C}{\Delta_3} - \frac{A}{\Delta_1}\right)(\Lambda_c - \Lambda)^2$$
$$+ \frac{F_c}{\Delta_1}\sqrt{G^2 - \Lambda^2}\sqrt{G_c^2 - (\Lambda_c - \Lambda)^2}\cos\lambda + \frac{D_c}{\Delta_3}\Lambda(\Lambda_c - \Lambda). \tag{21}$$

Here for more generality we will consider the integrable cases of the problem (21) from formal point of view, assuming that the problem parameters $(a, b, c; A, B, C)$ admit arbitrary values independently from their mechanical sence.

The following integrable cases of the Poincaré problem can be established directly as a result of analysis of the expressions of the Hamiltonians (18), (21).

Integrable cases (parameters $a, b, c; A, B, C$).

I. Relations: $a = b$, $A = B$.
Values of other parameters:

$$A_c = B_c = \frac{1}{5}m_c(a^2 + c^2), \quad C_c = \frac{2}{5}m_c a^2,$$

$$F_c = E_c = \frac{2}{5}m_c ac, \quad D_c = \frac{2}{5}m_c a^2, \quad A_m = B_m.$$

Arbitrary parameters: a, c, A, C.

II. Relations: $c = 0$, $a = b$.
Values of other parameters:

$$A_c = B_c = \frac{C_c}{2} = \frac{1}{5}m_c a^2, \quad F_c = E_c = 0, \quad D_c = \frac{2}{5}m_c a^2$$

Arbitrary parameters: a, A, B, C.

III. Relations: $c = 0$, $A = B$.

Values of other parameters:

$$A_c = \frac{1}{5} m_c b^2, \quad B_c = \frac{1}{5} m_c a^2, \quad C_c = \frac{1}{5} m_c (a^2 + b^2),$$

$$F_c = E_c = 0, \quad D_c = \frac{2}{5} m_c a^2, \quad A_m = B_m = \frac{1}{5} m_c (a^2 - b^2).$$

Arbitrary parameters: a, b, A, C.

So for all these cases of integrability of the Poincaré problem we have four arbitrary parameters.

5 POINCARE CASE

Case I of the integrability of the problem was noted by Poincaré. Here we obtain quadratures of this problem using our approach to the problem on the basis of the equations in Andoyer variables.

First we note that the parameter relations for case I present the general case when conditions (19) are satisfied. In fact from equations (19) we have:

$$\frac{A}{B} = \frac{A_m}{B_m} = \frac{A_c}{B_c} = \frac{F_c}{E_c} = \frac{b}{a} = \chi. \tag{22}$$

Let us show that the constant χ is equal to 1.

In fact from equations (19) we have $\Delta_1/\Delta_2 = \chi$, but relations (22) for expressions (20) give

$$\frac{\Delta_1}{\Delta_2} = \frac{AA_c - F_c^2}{BB_c - E_c^2} = \chi^2.$$

This means that all equations (19), (20) are satisfied only for one case $\chi = 1$ and we obtain the conditions for the first case of the integrability of the Hamiltonian problem (18).

Now we obtain the quadratures of the first case (for Poincaré case).

Introduce a new notation for the parameters of the problem and present the Hamiltonian of the problem (21) in the following compact form:

$$K = \frac{1}{2}[\delta G^2 + \delta_c G_c^2] + \frac{1}{2} k[\varphi(\Lambda, \Lambda_c) + \psi(\Lambda, \Lambda_c, G, G_c) \cos \lambda], \tag{23}$$

where

$$\begin{aligned} \varphi(\Lambda, \Lambda_c) &= \Lambda^2 + \alpha(\Lambda_c - \Lambda)^2 + \beta \Lambda (\Lambda_c - \Lambda), \\ \psi(\Lambda, \Lambda_c, G, G_c) &= \gamma \sqrt{G^2 - \Lambda^2} \sqrt{G_c^2 - (\Lambda_c - \Lambda)^2} \end{aligned} \tag{24}$$

and δ, δ_c, k, α, β, γ are constant coefficients:

$$\delta = \frac{A_c}{AA_c - F_c^2}, \qquad \delta_c = \frac{A}{AA_c - F_c^2}, \qquad k = \frac{C_c}{CC_c - D_c^2} - \frac{A_c}{AA_c - F_c^2}$$
$$\alpha = \frac{C\Delta_1 - A\Delta_3}{C_c\Delta_1 - A_c\Delta_3}, \quad \beta = \frac{D_c\Delta_1}{C_c\Delta_1 - A_c\Delta_3}, \quad \gamma = \frac{F_c\Delta_3}{C_c\Delta_1 - A_c\Delta_3}, \tag{25}$$
$$\Delta_1 = AA_c - F_c^2, \quad \Delta_3 = CC_c - D_c^2 = C_m C_c.$$

Canonical equations corresponding to the Hamiltonian (23)–(25)can now be described in the following form:

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \delta G + \frac{1}{2}k\frac{\partial\psi}{\partial G}\cos\lambda, \qquad \frac{\mathrm{d}G}{\mathrm{d}t} = 0,$$
$$\frac{\mathrm{d}g_c}{\mathrm{d}t} = \delta_c G_c + \frac{1}{2}k\frac{\partial\psi}{\partial G_c}\cos\lambda, \qquad \frac{\mathrm{d}G_c}{\mathrm{d}t} = 0,$$
$$\frac{\mathrm{d}\lambda_c}{\mathrm{d}t} = \frac{1}{2}k\left[\frac{\partial\varphi}{\partial\Lambda_c} + \frac{\partial\psi}{\partial\Lambda_c}\cos\lambda\right], \quad \frac{\mathrm{d}\Lambda_c}{\mathrm{d}t} = 0,$$
$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{1}{2}k\left[\frac{\partial\varphi}{\partial\Lambda} + \frac{\partial\psi}{\partial\Lambda}\cos\lambda\right], \qquad \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \frac{1}{2}k\psi\sin\lambda. \tag{26}$$

Equations (26) are integrated very easily. They admit the following first integrals:

$$\Lambda_c = \Lambda_c^0, \quad G = G^0, \quad G_c = G_c^0, \quad K = K^0. \tag{27}$$

Here (and below) the index '0' denotes the initial (constant) values of the corresponding variables and Hamiltonian.

The fourth integral in (27) can be described in the reduced form:

$$\varphi(\Lambda, \Lambda_c) + \psi(\Lambda, \Lambda_c, G, G_c)\cos\lambda = c_0, \tag{28}$$

where c_0 is a reduced constant of energy

$$c_0 = \frac{2K^0 - \delta {G^0}^2 - \delta_c {G_c^0}^2}{k}. \tag{29}$$

From equation (28), (29) we obtain the simple relations:

$$\cos\lambda = \frac{c_0 - \varphi(\Lambda)}{\psi(\Lambda)}, \quad \sin\lambda = \pm\frac{\sqrt{\psi(\Lambda)^2 + 2c_0\varphi(\Lambda) - \varphi(\Lambda)^2 - c_0^2}}{\psi(\Lambda)}.$$

Using these equations we can write the equation for the variable Λ in the form

$$\pm\frac{\mathrm{d}\Lambda}{\Delta} = \mathrm{d}\tau, \tag{30}$$

where $\tau = (1/2)kt$ is the new independent variable and

$$\Delta = a_4\Lambda^4 + a_3\Lambda^3 + a_2\Lambda^2 + a_1\Lambda^1 + a_0 \tag{31}$$

with constant coefficients:

$$\begin{aligned}
a_4 &= -(1+\alpha-\beta)^2+\gamma^2,\\
a_3 &= -2\Lambda_c[(1+\alpha-\beta)(2\alpha-\beta)+\gamma^2],\\
a_2 &= \gamma^2(L_c^2-G^2-G_c^2)-\Lambda_c^2[(2\alpha-\beta)^2+2(1+\alpha-\beta)(\alpha-c_0/\Lambda_c^2)],\\
a_1 &= -2\Lambda_c[\gamma^2G^2+(\alpha\Lambda_c^2-c_0)(2\alpha-\beta)],\\
a_0 &= \gamma^2G^2(G_c^2-\Lambda_c^2)-(c_0-\alpha\Lambda_c^2)^2.
\end{aligned} \tag{32}$$

Equation's (31), (32) define the dependence of the variable Λ on time. From the other equations of the system it is easy to define the dependencies of the variables g, g_c, λ, λ_c on the variable Λ. After some algebra the final quadratures of the Poincaré problem are given by the formulae:

$$\begin{aligned}
g &= g_0+\delta G(t-t_0)\pm\int_{\Lambda_0}^{\Lambda}\frac{F(\Lambda)\,\mathrm{d}\Lambda}{(G^2-\Lambda^2)\Delta(\Lambda)},\\
g_c &= g_c^0+\delta_c G_c(t-t_0)\pm G_c\int_{\Lambda_0}^{\Lambda}\frac{F(\Lambda)\,\mathrm{d}\Lambda}{[G^2-(\Lambda_c-\Lambda)^2)]\Delta(\Lambda)},\\
\lambda_c &= \lambda_c^0\mp\int_{\Lambda_0}^{\Lambda}\frac{(\alpha_1\Lambda-2\alpha_0\Lambda_c^2)\,\mathrm{d}\Lambda}{\Lambda_c\Delta(\Lambda)}\pm\int_{\Lambda_0}^{\Lambda}\frac{(\Lambda-\Lambda_c)F(\Lambda)\,\mathrm{d}\Lambda}{[G^2-(\Lambda_c-\Lambda)^2)]\Delta(\Lambda)},\\
\lambda &= \lambda^0\mp\int_{\Lambda_0}^{\Lambda}\frac{(2\alpha_2\Lambda+\alpha_1)\,\mathrm{d}\Lambda}{\Delta(\Lambda)}\pm\int_{\Lambda_0}^{\Lambda}\left[\frac{\Lambda}{\Lambda^2-G^2}+\frac{\Lambda_c-\Lambda}{G_c^2-(\Lambda_c-\Lambda)^2}\right]\frac{F(\Lambda)\,\mathrm{d}\Lambda}{\Delta(\Lambda)},\\
&\pm\int_{\Lambda_0}^{\Lambda}\frac{\mathrm{d}\Lambda}{\Delta(\Lambda)}=\frac{1}{2}k(t-t_0),
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
F(\Lambda) &= \alpha_2\Lambda^2+\alpha_1\Lambda+\alpha_0,\\
\alpha_2 &= \beta-\alpha-1,\\
\alpha_1 &= -(2\alpha+\beta)\Lambda_c,\\
\alpha_0 &= c_0-\alpha\Lambda_c^2.
\end{aligned} \tag{34}$$

6 CASE II

In Andoyer variables (16) the Hamiltonian of this case of integrability is described as:

$$K = K(L, L_c, G, G_c, l, -, -, -)$$

$$= \frac{1}{2}(G^2 - L^2)\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right) + \frac{1}{2C_m}L^2 + \frac{1}{2A_c}(G_c^2 - L_c^2)$$
$$+ \frac{C}{2C_m C_c}L_c^2 - \frac{1}{C_m}LL_c. \tag{35}$$

The general solution of equations (17) with Hamiltonian (35) are given by the following first integrals and quadratures:

$$\pm \int_{l_0}^{l} \frac{\mathrm{d}l}{\Delta(l)} = t - t_0,$$
$$g = g_0 \pm G \int_{l_0}^{l} \left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right) \frac{\mathrm{d}l}{\Delta(l)},$$
$$l_c = l_c^0 + L_c\left(\frac{C}{C_m C_c} - \frac{1}{A_c}\right)(t - t_0)$$
$$\mp \frac{L_c}{C_m^2} \int_{l_0}^{l} \left(\frac{1}{C_m} - \frac{\sin^2 l}{A} - \frac{\cos^2 l}{B}\right)^{-1} \frac{\mathrm{d}l}{\Delta(l)}$$
$$\mp \frac{1}{C_m} \int_{l_0}^{l} \left(\frac{1}{C_m} - \frac{\sin^2 l}{A} - \frac{\cos^2 l}{B}\right)^{-1} \mathrm{d}l,$$
$$g_c = g_c^0 + \frac{G_c}{A_c}(t - t_0),$$
$$L = L_0 \mp \left(\frac{1}{A} - \frac{1}{B}\right) \int_{l_0}^{l} \frac{(G^2 - L^2)\sin l \cos l \,\mathrm{d}l}{\Delta(l)},$$
$$G = G_0,$$
$$L_c = L_c^0,$$
$$G_c = G_c^0, \tag{36}$$

where

$$\Delta = \sqrt{\frac{L_c^2}{C_m^2} - \left[G^2\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right) + c_0\right]\left(\frac{1}{C_m} - \frac{\sin^2 l}{A} - \frac{\cos^2 l}{B}\right)} \tag{37}$$

and

$$c_0 = (G_c^2 - L_c^2)\frac{1}{A_c} + \left(\frac{1}{C_c} + \frac{1}{C_m}\right)L_c^2 - 2K_0. \tag{38}$$

In (36), (37) $l_0, g_0, l_c^0, g_c^0; L_0, G_0, L_c^0, G_c^0$ are a full set of arbitrary constans of integration.

7 CASE III

In this case we have the following expression for the Hamiltonian:

$$\begin{aligned} K &= K(L, L_c, G, G_c, l_c, -, -, -) \\ &= \frac{1}{2}(G^2 - L^2)\frac{1}{A} + \frac{C_c}{2\Delta_3}L^2 + \frac{1}{2}(G_c^2 - L_c^2)\left(\frac{\sin^2 l_c}{A_c} + \frac{\cos^2 l_c}{B_c}\right) \\ &+ \frac{C}{2\Delta_3}L_c^2 - \frac{D_c}{\Delta_3}LL_c. \end{aligned}$$

Solving the integral of energy $K = K_0$ we obtain an expression for the variable L_c as a function of the variable l_c:

$$L_c(l_c) = \left(\frac{LD_c}{\Delta_3} \pm \Delta\right)\left(\frac{C}{\Delta_3} - \frac{\sin^2 l_c}{A_c} - \frac{\cos^2 l_c}{B_c}\right)^{-1},$$

where

$$\Delta = \sqrt{\frac{L^2 D_c^2}{\Delta_3^2} - \left(\frac{C}{\Delta_3} - \frac{\sin^2 l_c}{A_c} - \frac{\cos^2 l_c}{B_c}\right)\left[\left(\frac{\sin^2 l_c}{A_c} + \frac{\cos^2 l_c}{B_c}\right)G_c^2 + c_0\right]}$$

with a constant

$$c_0 = (G^2 - L^2)\frac{1}{A} + \frac{C_c}{\Delta_3}L^2 - 2K_0.$$

The general solution of this case of integrability is given by the following integrals:

$$\begin{aligned} l &= l_0 + L_0\left(\frac{C_c}{\Delta_3} - \frac{1}{A}\right)(t - t_0) \mp \frac{D_c}{\Delta_3}\int_{l_c^0}^{l_c} \frac{L_c(l_c)\,\mathrm{d}l_c}{\Delta(l_c)}, \\ g &= g_0 + \frac{G_0}{A}(t - t_0), \\ l_c &= l_c^0 - \frac{LD_c}{\Delta_3}(t - t_0) \pm G_c\int_{l_c^0}^{l_c} L_c(l_c)\left(\frac{C}{\Delta_3} - \frac{\sin^2 l_c}{A_c} - \frac{\cos^2 l_c}{B_c}\right)\frac{\mathrm{d}l_c}{\Delta(l_c)}, \\ g_c &= g_c^0 \pm G_c\int_{l_c^0}^{l_c}\left(\frac{\sin^2 l_c}{A_c} + \frac{\cos^2 l_c}{B_c}\right)\frac{\mathrm{d}l_c}{\Delta(l_c)}, \\ L &= L_0, \\ G &= G_0, \\ G_c &= G_c^0, \\ L_c &= L_c^0 \pm \left(\frac{1}{B_c} - \frac{1}{A_c}\right)\int_{l_c^0}^{l_c}\frac{(G_c^2 - L_c^2(l_c))\sin l_c \cos l_c\,\mathrm{d}l_c}{\Delta(l_c)}. \end{aligned} \tag{39}$$

Together with the full set of arbitrary constants of integration: l_0, g_0, l_c^0, g_c^0, L_0, G_0, L_c^0, G_c^0.

Inversions of the full systems of integrals for all integrable cases (33), (34), (36)–(38), (39) can be given as elliptical functions.

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References

Andoyer, H. (1923) *Cours de Mécanique Céleste*, Vol. 7, Gauthier-Villars, Paris.

Barkin, Yu. V. and Borisov A. V. (1989) *Nonintegrability of Kirgoffs Equations and Another Problems of Dynamics of Rigid Body*, Dep. in VINITI AN SSSR, N 5037-V-89, 103 p. (in Russian).

Barkin, Yu. V. *Astron. Astrophys. Trans.* (in press).

Getino, J. (1995) *Geophys. J. Int.* **122**, 803.

Getino, J. and Ferrandiz, J. M. (1997) *Geophys. J. Int.* **130**, 326.

Poincaré, H. (1910) *Bull. Astron.* **27**, 321.

Sevilla, M. and Romero, P. (1987) *Bull. Geod.* **61**, 1.