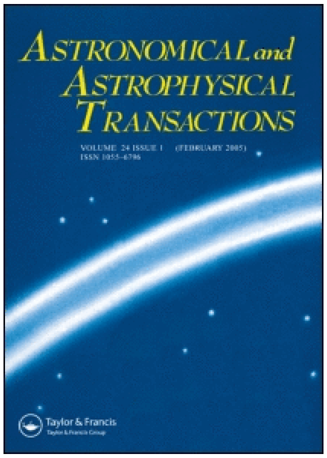

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ON THE CROSSING OF THIN SHELLS

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The crossing of two dust shells is considered as a simplified model for shell crossing during the spherically symmetric collapse of dust. Israel formalism is applied to study the gravitational collapse of two thin shells. The Schwarzschild coordinates are used for $r > 2m$ and Kruskal coordinates for $r < 2m$.

KEY WORDS General relativity, thin shells, models

1 INTRODUCTION

The dynamics of thin shells of matter in general relativity gas been discussed by many authors. Our approach is similar to that used by Israel (1966) and Kuchar (1968) for the study of collapse of spherical shells. Israel (1966) found invariant boundary conditions connecting the extrinsic curvature of a shell in space-times on both sides of it, shell with the matter of this shell. In this paper we study two thin shells of dust. In Section 2, we give the general formalism. In Section 3, this formalism is applied to two thin shells in the Schwarzschild space-time and the equations of motion for shells in Kruskal coordinates are given in Section 4, with the results concerning the shell crossing under the horizon in Section 5. The basic equation for the spherical shell in the different forms have been given by Lake (1979), also see Frauendiener (1995), Langer (1987) and Sato (1983).

2 THE FORMALISM

Let the time-like hypersurface Σ, which divides the Riemannian space-time $\mathcal{M}$. into two regions, $\mathcal{M}^-$ and $\mathcal{M}^+$, be the history of a thin spherical shell of matter. The regions $\mathcal{M}^-$ and $\mathcal{M}^+$ are covered by the mutually independent coordinate systems X^α_- and X^α_+. The hypersurface Σ represents the boundary of $\mathcal{M}^-$ and $\mathcal{M}^+$

respectively; consequently the intrinsic geometry of Σ induced by the metrics of $\mathcal{M}^-$ and $\mathcal{M}^+$ must be the same. Let Σ be parametrized by intrinsic coordinates ξ^a, given by

$$X^\alpha_\mp = X^\alpha_\mp(\xi^a). \tag{1}$$

(Greek indices refer to 4-dimensional indices, Latin indices refer to 3-dimensional indices on Σ). The metric has signature $+2$, and the Newtonian gravitational constant and light velocity are equal to unity as a consequence of the choice of units. The basic vectors $e_a = \frac{}{\xi^a}$ tangent to Σ have components

$$e^\alpha_{a\pm} = \frac{X^\alpha_\mp}{\xi^a} \tag{2}$$

with respect to the two four-dimensional coordinate systems in $\mathcal{M}^-$ and $\mathcal{M}^+$.

Their scalar products define the metric induced on the hypersurface Σ,

$$g_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b. \tag{3}$$

The metric induced by the metrics of both regions $\mathcal{M}^-$ and $\mathcal{M}^+$ must be identical, $g^+_{ab}(\xi) = g^-_{ab}(\xi) \equiv g_{ab}(\xi)$; this condition must be fulfilled when we want to join two regions of space-times on the hypersurface. The condition is stated independently of coordinate systems in $\mathcal{M}^-$ and $\mathcal{M}^+$. The unit normal vector n to Σ will have components n^+_α satisfying

$$n.n|_+ = 1. \tag{4}$$

We suppose n to be directed from $\mathcal{M}^-$ to $\mathcal{M}^+$. The manner in which Σ is bent in space $\mathcal{M}^-$ and $\mathcal{M}^+$ is characterized by the three-dimensional extrinsic curvature tensor

$$K^+_{ab} = -n^+_\alpha \frac{De^\alpha_{a\pm}}{\xi^b} = e_{a\alpha\pm}\frac{Dn^\alpha}{\xi^b}, \tag{5}$$

where D/ξ^b represents the absolute derivative with respect to ξ^b.

The surface energy-momentum tensor t_{ab} is determined by the jump $[K_{ab}] = K^+_{ab} - K^-_{ab}$. The Σ represents the history of a surface layer (a singular hypersurface of order one) if $K^+_{ab} \neq K^-_{ab}$.

The Einstein equations determine the relation between the extrinsic curvature K^+_{ab} and three-dimensional intrinsic energy-momentum tensor ($t_{ab} = t_{\alpha\beta} e^\alpha_a e^\beta_b$)

$$[K_{ab}] = -8\pi(t_{ab} - \frac{1}{2} t g_{ab}), \tag{6}$$

where $t = t^a_a$. We can write this relation in the form

$$t_{ab} = -\frac{1}{8\pi}([K_{ab}] - g_{ab}[K]), \tag{7}$$

where $[K] = g^{ab}[K_{ab}]$. These are the field equations for the shell.

3 THE MOTION OF COLLAPSING TWO CONCENTRIC SHELLS

3.1 The Motion Of One-Shell In Schwarzschild Space

The shell is spherically symmetric. Therefore, the space-time outside the shell can be described by the line element,

$$ds^2 = -f\,\mathrm{d}t_+^2 + f^{-1}\,\mathrm{d}r_+^2 + r_+^2\,\mathrm{d}\Omega^2, \tag{8}$$

where

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2,$$

is the line element on the unit sphere and

$$f = 1 - \frac{2m}{r},$$

where m is the gravitational mass in the exterior space. Inside the shell space-time is flat, i.e. $f = 1$. As exterior and interior coordinates, we use $X_+^\alpha = (t_+, r_+, \theta, \phi)$ and $X_-^\alpha = (t_-, r_-, \theta, \phi)$, respectively.

The intrinsic coordinates on Σ are the proper time τ measured by the comoving observer on the shell and the spherical angles θ, ϕ: $\xi^a = (\tau, \theta, \phi)$. Let the equation of the shell be (the condition (3) implies the continuity of r on the shell),

$$r_\pm = R(\tau).$$

We get from (2) and (4)

$$\begin{aligned} e^\alpha_{\tau\pm} &= (\dot{t}_+, \dot{R}, 0, 0), \\ e^\alpha_{\theta\pm} &= (0, 0, 1, 0), \\ e^\alpha_{\phi\pm} &= (0, 0, 0, 1), \end{aligned}$$

and

$$n_{\alpha\pm} = (-\dot{R}, \dot{t}_+, 0, 0), \tag{9}$$

where a dot represents the derivative with respect to proper time, prime the derivative with respect to R, and R is the radius of the shell.

Thus the three-dimensional metric tensor on Σ is

$$g_{ab} = (-1, R^2, R^2 \sin^2\theta).$$

From (5) we get the extrinsic curvature K_{ab}^{+} in $\mathcal{M}^-$ and $\mathcal{M}^+$,

$$\begin{aligned} K^-_{\tau\tau} &= \dot{R}\ddot{t}_- - \ddot{R}\dot{t}_-, \\ K^-_{\theta\theta} &= R\dot{t}_-, \\ K^-_{\phi\phi} &= R\dot{t}_- \sin^2\theta, \end{aligned}$$

and

$$K^{+}_{\tau\tau} = -\dot{t}_{+}\ddot{R} + \dot{R}\ddot{t}_{+} - \frac{1}{2}\dot{t}_{+}[\dot{t}^{2}_{+}ff' - 3f^{-1}f'\dot{R}^{2}],$$
$$K^{+}_{\theta\theta} = fR\dot{t}_{+},$$
$$K^{+}_{\phi\phi} = fR\dot{t}_{+}\sin^{2}\theta. \tag{10}$$

We suppose that the three-dimensional energy-momentum tensor has the form

$$t_{ab} = Pg_{ab} + (P+\sigma)U_{a}U_{b},$$

where σ is the surface density and P is the surface pressure.

Therefore the components of t_{ab} are

$$t_{\tau\tau} = \frac{1}{4\pi R}(\dot{t}_{+}f - \dot{t}_{-}),$$
$$t_{\theta\theta} = -\frac{R^{2}}{8\pi}\left[H + f'\dot{t}_{+}\left(\frac{3\dot{R}^{2}}{2f} - \frac{1}{2}f\dot{t}^{2}_{+}\right) + \frac{1}{R}(\dot{t}_{-} - f\dot{t}_{+})\right],$$
$$t_{\phi\phi} = t_{\theta\theta}\sin^{2}\theta, \tag{11}$$

where

$$H = \ddot{R}(\dot{t}_{-} - \dot{t}_{+}) + \dot{R}(\ddot{t}_{+} - \ddot{t}_{-}).$$

Because τ is a proper time on the shell the conditions

$$\dot{t}_{-} = \sqrt{1+\dot{R}^{2}},$$
$$\dot{t}_{+} = f^{-1}\sqrt{f+\dot{R}^{2}}, \tag{12}$$

must be fulfilled. Supposing $t_{\tau\tau} = \sigma$ and $t_{\theta\theta} = t_{\phi\phi} = P$, we get from (12) and (11),

$$4\pi R\sigma = \sqrt{1+\dot{R}^{2}} - \sqrt{f+\dot{R}^{2}}, \tag{13a}$$
$$8\pi R^{2}P = \frac{1}{\sqrt{1+\dot{R}^{2}}\sqrt{f+\dot{R}^{2}}}[m\sqrt{1+\dot{R}^{2}} + E], \tag{13b}$$

where

$$E = R(R\ddot{R} - \sqrt{1+\dot{R}^{2}}\sqrt{f+\dot{R}^{2}})(\sqrt{1+\dot{R}^{2}} - \sqrt{f+\dot{R}^{2}}).$$

We put (13a) into (13b) to get the relation between the surface pressure and the surface density in the form

$$P = \frac{m}{8\pi R^{2}\sqrt{f+\dot{R}^{2}}} + \left(\frac{R\ddot{R} - \sqrt{1+\dot{R}^{2}}\sqrt{f+\dot{R}^{2}}}{2\sqrt{1+\dot{R}^{2}}\sqrt{f+\dot{R}^{2}}}\right)\sigma. \tag{14}$$

In the case of dust $P = 0$, and $U^a = (1, 0, 0)$; then the equation of motion for an infinitely thin shell of dust which interacts only gravitationally is given by (13a) as

$$\sqrt{1 + \dot{R}^2} - \sqrt{f + \dot{R}^2} = \frac{M}{R}, \tag{15}$$

where $M = 4\pi R^2 \sigma$ is the rest mass of the shell.

3.2 The Motion Of Two-Shells

The shell which is the 'inner shell' in the initial moment is denoted by A. The metric of the space-time between the shells is

$$\mathrm{d}s^2 = -f_1\,\mathrm{d}t_1^2 + f_1^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2, \tag{16}$$

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$$

and

$$f_1 = 1 - \frac{2m_1}{r},$$

where m_1 is the gravitational mass in the space between shells. Inside the shell is the flat space-time, and according to (15), the equation of motion of this shell is

$$\sqrt{1 + \dot{R}_A^2} - \sqrt{f_{1A} + \dot{R}_A^2} = \frac{M_A}{R_A}, \tag{17}$$

where R_A is the radius of the shell A and $M_A = 4\pi R_A^2 \sigma$ is the total rest mass of dust particles of the shell.

Coordinate times in spaces inside the inner shell, between shells and outside are t_-, t_1 and t respectively. Similarly, the equation of motion of the outer shell B is

$$\sqrt{f_{1B} + \dot{R}_B^2} - \sqrt{f_B + \dot{R}_B^2} = \frac{M_B}{R_B}, \tag{18}$$

where M_B is the rest mass of shell B. Finally $\dot{R}_A = \mathrm{d}R_A/\,\mathrm{d}\tau_A$, $\dot{R}_B = \mathrm{d}R_B/\,\mathrm{d}\tau_B$.

The relations (17) and (18) represent the equations of motion of shells before their intersection. From (17) and (18) we can write the velocities as a function of masses and radii:

$$\dot{R}_A = \pm\sqrt{\left(\frac{m_1}{M_A} + \frac{M_A}{2R_A}\right)^2 - 1}, \tag{19}$$

$$\dot{R}_B = \pm\sqrt{\left(\frac{m - m_1}{M_B} + \frac{M_B}{2R_B}\right)^2 + \frac{2m_1}{R_B} - 1}. \tag{20}$$

(The choice of sign is given by the direction of motion; the sign $(-)$ means the case of a collapse and $(+)$ to the case of an expansion).

Using these we can calculate m_1 and m,

$$m_1 = M_A\sqrt{1+\dot{R}_A^2} - \frac{M_A^2}{2R_A}, \tag{21}$$

$$m = m_1 + M_B\sqrt{f_{1B}+\dot{R}_B^2} - \frac{M_B^2}{2R_B}. \tag{22}$$

The intersection of shells. The aim is to find velocities of shells after the intersection $\dot{R}_{A+}$, $\dot{R}_{B-}$ as a function of velocities before the intersection $\dot{R}_{A-}$, $\dot{R}_{B+}$ and parameters M_A, M_B and $R_A = R_B = R$, where R is the radius at the point of intersection.

The external Schwarzschild mass m is constant during the collapse of shells (Birkhoff's theorem).

However, the Schwarzschild mass m_1 between shells will be changed after the intersection. The mass after the intersection we call m_2, and we use the notation f_2 for the corresponding factor f in the Schwarzschild metric.

The equations of motion (17) and (18) after intersection is

$$\sqrt{1+\dot{R}_{B-}^2} - \sqrt{f_2+\dot{R}_{B-}^2} = \frac{M_B}{R}, \tag{23}$$

$$\sqrt{f_2+\dot{R}_{A+}^2} - \sqrt{f+\dot{R}_{A+}^2} = \frac{M_A}{R}. \tag{24}$$

(After the intersection, shell A will be the external and B the internal shell).

These are two equations in three unknown: velocities after the intersection and the Schwarzschild mass between shells. However, for the shells interacting only gravitationally, we have one condition in addition:

the scalar product of four-velocities before intersection $U_A^\alpha = (\dot{t}_1, \dot{R}_{A-}, 0, 0)$, $U_B^\alpha = (\dot{t}_1, \dot{R}_{B+}, 0, 0)$ has to be equal the scalar product of four-velocities after intersection $U_A^\alpha(\dot{t}_2, \dot{R}_{A+}, 0, 0)$, $U_B^\alpha = (\dot{t}_2, \dot{R}_{B-}, 0, 0)$,

$$U_A^\alpha U_B^\beta g_{1\alpha\beta}|_{(\text{before})} = U_A^\alpha U_B^\beta g_{2\alpha\beta}|_{(\text{after})}, \tag{25}$$

therefore

$$-\dot{t}_{1A}\dot{t}_{1B}f_1 + \dot{R}_{A-}\dot{R}_{B+}f_1^{-1} = -\dot{t}_{2A}\dot{t}_{2B}f_2 + \dot{R}_{A+}\dot{R}_{B-}f_2^{-1}. \tag{26}$$

We insert (23) and (24) into (26) and we get one equation in one unknown quantity m_2.

To simplify the solution of the equations of motion of shells, we use the Schwarzschild time t_1 between shells as the independent variable instead of the proper time. The method used does not allow us to follow the shells beyond the Schwarzschild radius. This case we shall solve in the Kruskal coordinates.

4 THE COLLAPSE OF TWO SHELLS IN KRUSKAL COORDINATES

4.1 The Motion Of One-Shell

In order to follow the motion of the shell under the horizon we shall use the Kruskal coordinates in the Schwarzschild space-time. We write the metric in this coordinates

$$\mathrm{d}s^2 = \psi^2(-\,\mathrm{d}v^2 + \mathrm{d}u^2) + r^2\,\mathrm{d}\Omega^2, \quad \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2, \tag{27}$$

and

$$\psi^2 = \frac{32m^3}{r}\exp\left(-\frac{r}{2m}\right).$$

Here r is a function of u and v defined implicitly by

$$u^2 - v^2 = \left(\frac{r}{2m} - 1\right)\exp\left(\frac{r}{2m}\right). \tag{28}$$

As the intrinsic coordinates on the shell we take the proper time measured by the comoving observer on the shell τ and the spherical angles θ, ϕ: $\xi^a = (\tau, \theta, \phi)$.

From (2) and (4) we get

$$\begin{aligned} e^{\alpha}_{\tau+} &= (\dot{v}_+, \dot{u}_+, 0, 0), \\ e^{\alpha}_{\theta+} &= (0,0,1,0), \\ e^{\alpha}_{\phi+} &= (0,0,0,1), \end{aligned}$$

and

$$n_{\alpha+} = (-\dot{u}_+, \dot{v}_+, 0, 0). \tag{29}$$

The non-vanishing components of the extrinsic curvature K^+_{ab}, in $\mathcal{M}^+$ are

$$\begin{aligned} K^+_{\tau\tau} &= \psi^2(\dot{u}\ddot{v} - \dot{v}\ddot{u}) - \frac{1}{2\psi^2}\frac{\partial\psi^2}{\partial R}\left(\dot{v}\frac{\partial R}{\partial u} + \dot{u}\frac{\partial R}{\partial v}\right), \\ K^+_{\theta\theta} &= R(u,v)\left[\dot{u}\frac{\partial R}{\partial v} + \dot{v}\frac{\partial R}{\partial u}\right], \\ K^+_{\phi\phi} &= K^+_{\theta\theta}\sin^2\theta, \end{aligned} \tag{30}$$

The flat space-time is supposed inside the shell. From (7), (10), (30), the components of t_{ab} are

$$\begin{aligned} t_{\tau\tau} &= -\frac{1}{4\pi R}\left(\dot{u}\frac{\partial R}{\partial v} + \dot{v}\frac{\partial R}{\partial u} - \dot{t}_-\right), \\ t_{\theta\theta} &= -\frac{R^2}{8\pi}\left[\psi^2(\dot{u}\ddot{v} - \dot{v}\ddot{u}) - \frac{1}{2\psi^2}\frac{\partial\psi^2}{\partial R}\left(\dot{v}\frac{\partial R}{\partial u} + \dot{u}\frac{\partial R}{\partial v}\right) + Y\right], \\ t_{\phi\phi} &= t_{\theta\theta}\sin^2\theta, \end{aligned} \tag{31}$$

where

$$Y = \ddot{R}\dot{t}_- - \dot{R}\ddot{t}_- + \frac{1}{R}\dot{t}_- - \frac{1}{R}\left(\dot{u}\frac{\partial R}{\partial v} + \dot{v}\frac{\partial R}{\partial u}\right),$$

as τ is a proper time on the shell the conditions

$$\dot{t}_- = \sqrt{1+\dot{R}^2}, \tag{32a}$$

$$\dot{v} = \sqrt{\dot{u}^2 + \psi^{-2}}, \tag{32b}$$

have to be fulfilled. If we suppose that $t_{\tau\tau} = \sigma$ and $t_{\theta\theta} = t_{\phi\phi} = P$ and insert the conditions (32) into (31) we get

$$\sqrt{1+\dot{R}^2} - \left(\dot{u}\frac{\partial R}{\partial v} + \dot{v}\frac{\partial R}{\partial u}\right) = \frac{M}{R}, \tag{33a}$$

$$\psi^2(\dot{u}\ddot{v} - \dot{v}\ddot{u}) - H + \frac{(R\ddot{R} - \dot{R}^2 - 1)}{R\sqrt{1+\dot{R}^2}} = -8\pi P, \tag{33b}$$

where

$$H = \left(\dot{v}\frac{\partial R}{\partial u} + \dot{u}\frac{\partial R}{\partial v}\right)\left(\frac{1}{R} + \frac{1}{2\psi^2}\frac{\partial \psi^2}{\partial R}\right),$$

and v, $\dot{v}$ are determined by (28) and (32b) as the function of u. Inserting (33a) into (33b) we get the relation between the surface pressure and the surface density in the form

$$P = N_1 + N_2\sigma, \tag{34}$$

where

$$\sigma = \frac{1}{4\pi R}\left(\sqrt{1+\dot{R}^2} + \frac{8m^2}{R}(v\dot{u} - u\dot{v})\exp\left(-\frac{R}{2m}\right)\right), \tag{35}$$

and

$$\begin{aligned} N_1 &= \frac{\sqrt{\dot{v}^2 - \psi^{-2}}[-4mR\ddot{R} + (1+\dot{R}^2)(6m - R)] + E}{32m\pi R\sqrt{1+\dot{R}^2}\sqrt{\dot{v}^2 - \psi^{-2}}}, \\ N_2 &= \frac{R - 2m}{8m}, \\ E &= \sqrt{1+\dot{R}^2}[\dot{R}\dot{v}(2m + R) - 4mR\ddot{v}]. \end{aligned}$$

In the case of dust $P = 0$.

The equation of motion for shell in the Kruskal coordinates (33a) is very difficult to solve, therefore we will use the constraint equations (28), (32b) to get the independent equations of motion of the shell.

Similarly the equations of motion for the shell between two Schwarzschild spacetime with different masses m can be constructed.

4.2 The Motion Of Two-Shells

In order to follow the intersections of two shells in the similar way as in the Section 3.2 it is more appropriate to use the same independent variable for both shells. Expressing the proper times of observers on both shells by means of the Kruskal coordinate v in the space-time between both shells and using the constraint conditions (28) and (32b) we get for shells A and B the equations of motion

$$\begin{aligned} R'_{A-} &= \frac{8m_1^2 q_A^2 \psi_{1A}^2}{64m_1^4 u_{1A}^2 + R_A^2 q_A^2 \psi_{1A}^2 \exp(R_A/m_1)} \left(-R_A v_1 \exp\left(\frac{R_A}{2m_1}\right) + H\right), \\ u'_{A-} &= \frac{1}{u_{1A}} \left(v_1 + \frac{R_A R'_{A-}}{8m_1^2} \exp\left(\frac{R_A}{2m_1}\right)\right), \end{aligned} \tag{36}$$

$$\begin{aligned} R'_{B+} &= \frac{8m_1^2 q_B^2 \psi_{1B}^2}{64m_1^4 u_{1B}^2 + R_B^2 q_B^2 \psi_{1B}^2 \exp(R_B/m_1)} \left(-R_B v_1 \exp\left(\frac{R_B}{2m_1}\right) + G\right), \\ u'_{B+} &= \frac{1}{u_{1B}} \left(v_1 + \frac{R_B R'_{B+}}{8m_1^2} \exp\left(\frac{R_B}{2m_1}\right)\right), \end{aligned} \tag{37}$$

where $R' = \mathrm{d}R/\,\mathrm{d}v$,

$$\begin{aligned} q_A &= -\sqrt{\left(\frac{m_1}{M_A} + \frac{M_A}{2R_A}\right)^2 - 1}, \\ q_B &= -\sqrt{\left(\frac{m - m_1}{M_B} + \frac{M_B}{2R_B}\right)^2 + \frac{2m_1}{R_B} - 1}, \\ H &= u_{1A} q_A^{-1} \sqrt{q_A^2 R_A^2 \exp\left(\frac{R_A}{m_1}\right) - 2m_1 R_A \phi_{1A} \exp\left(\frac{R_A}{2m_1}\right)}, \\ G &= u_{1B} q_B^{-1} \sqrt{q_B^2 R_B^2 \exp\left(\frac{R_B}{m_1}\right) - 2m_1 R_B \phi_{1B} \exp\left(\frac{R_B}{2m_1}\right)}, \end{aligned}$$

and

$$\begin{aligned} \dot{v}_1 &= \frac{1}{\sqrt{\psi_1^2 (1 - u_1'^2)}}, \\ \psi_1^2 &= \frac{32m_1^3}{R} \exp\left(-\frac{R}{2m_1}\right), \\ \phi_1 &= \left(1 - \frac{R}{2m_1}\right) \exp\left(\frac{R}{2m_1}\right). \end{aligned}$$

The motion of shells after intersection is calculated using a method similar to that in the previous section. We calculate m_2 from the condition (25), (36) and (37),

$$(u'_{A+} u'_{B-} - 1)^2 - Z^2 (1 - u'^2_{A+})(1 - u'^2_{B-}) = 0, \tag{38}$$

where

$$Z = \frac{(u'_{A-}u'_{B+} - 1)}{\sqrt{(1 - u'^2_{A-})(1 - u'^2_{B+})}}.$$

The velocities of shells after the intersection will be

$$\begin{aligned}
R'_{A+} &= \frac{8m_2^2 Z_A^2 \psi_{2A}^2}{64m_2^4 u_{2A}^2 + R_A^2 Z_A^2 \psi_{2A}^2 \exp(R_A/m_2)} \left(-R_A v_2 \exp\left(\frac{R_A}{2m_2}\right) + H1 \right), \\
u'_{A+} &= \frac{1}{u_{2A}} \left[v_2 + \frac{R_A R'_{A+}}{8m_2^2} \exp\left(\frac{R_A}{2m_2}\right) \right],
\end{aligned} \quad (39)$$

$$\begin{aligned}
R'_{B-} &= \frac{8m_2^2 Z_B^2 \psi_{2B}^2}{64m_2^4 u_{2B}^2 + R_B^2 Z_B^2 \psi_{2B}^2 \exp(R_B/m_2)} \left(-R_B v_2 \exp\left(\frac{R_B}{2m_2}\right) + G1 \right), \\
u'_{B-} &= \frac{1}{u_{2B}} \left[v_2 + \frac{R_B R'_{B-}}{8m_2^2} \exp\left(\frac{R_B}{2m_2}\right) \right],
\end{aligned} \quad (40)$$

where

$$\begin{aligned}
Z_A &= -\sqrt{\left(\frac{m - m_2}{M_A} + \frac{M_B}{2R_A}\right)^2 + \frac{2m_2}{R_A} - 1}, \\
Z_B &= -\sqrt{\left(\frac{m_2}{M_B} + \frac{M_B}{2R_B}\right)^2 - 1}, \\
H1 &= u_{2A} Z_A^{-1} \sqrt{Z_A^2 R_A^2 \exp\left(\frac{R_A}{m_2}\right) - 2m_2 R_A \phi_{2A} \exp\left(\frac{R_A}{2m_2}\right)}, \\
G1 &= u_{2B} Z_B^{-1} \sqrt{Z_B^2 R_B^2 \exp\left(\frac{R_B}{m_2}\right) - 2m_2 R_B \phi_{2B} \exp\left(\frac{R_B}{2m_2}\right)},
\end{aligned}$$

and

$$\begin{aligned}
\dot{v}_2 &= \frac{1}{\sqrt{\psi_2^2(1 - u_2'^2)}}, \\
\psi_2^2 &= \frac{32m_2^3}{R} \exp\left(-\frac{R}{2m_2}\right), \\
\phi_2 &= \left(1 - \frac{R}{2m_2}\right) \exp\left(\frac{R}{2m_2}\right).
\end{aligned}$$

After each intersection the mass between shells will be changed; it is determined by (25). The relations for m_2 are rather complicated and we solved them in the cases considered numerically.

5 INITIAL DATA FOR MOTION OF TWO SHELLS

The first order equations of motion (19) and (20) for two shells depend on parameters M_A and M_B, the rest masses of the shells, and the Schwarzschild masses m_1

and m. In order to determine their motion the initial value of R_A and R_B must be given. Suppose that the initial data have been chosen so that the shells intersect, i.e. there is a time T_0 such that $R_A = R_B = R_0$. At this point the solution ceases to be uniquely determined by the initial data. At the point of intersection we know the following information:

1. the location of shells $R_A = R_B = R_0$,
2. the two velocities at the point of intersection,
3. the total mass of the system, i.e. the gravitational mass outside m, which remains constant.

After intersection the shells interchange their position, the inner shell becomes the outer shell and vice versa. To follow the motion of the shells, one determines the Schwarzschild mass between the shells after the intersection from condition (26). Since the number of the shells and the energy-momentum are conserved, then the problem is uniquely determined. The rest masses are constant during the evolution.

We study all these three cases to determine the value of m_1 and m with different initial velocities of shells. We choose the time between shells as the independent variable.

1. Let us suppose that the velocities of both shells would tend to zero for their radii going to infinity. It corresponds to the choice $m_1 = M_A$ and $m = m_1 + M_B$. The initial velocities of shells are given by

$$R'_{A0} = -\frac{1 - (2M_A/R_{A0})}{(1 - (M_A/(2R_{A0}))}\sqrt{\frac{M_A}{R_{A0}} + \frac{M_A^2}{4R_{A0}^2}},$$
$$R'_{B0} = -\frac{1 - (2M_A/R_{B0})}{(1 + (M_B/(2R_{B0}))}\sqrt{\frac{2M_A}{R_{B0}} + \frac{M_B}{R_{B0}} + \frac{M_B^2}{4R_{B0}^2}},$$

and the Schwarzschild masses are given by

$$m_1 = M_A\sqrt{1 + \frac{M_A}{R_{A0}} + \frac{M_A^2}{4R_{A0}^2}} - \frac{M_A^2}{2R_{A0}}$$

and

$$m = m_1 + M_B\sqrt{1 + \frac{M_B}{R_{B0}} + \frac{M_B^2}{4R_{B0}^2}} - \frac{M_B^2}{2R_{B0}}$$

2. Let $R'_{A0} = R'_{B0} = 0$ (the initial velocities of shells equal zero). The Schwarzschild masses are given by

$$m_{1\pm} = \frac{M_A}{2R_{A0}}(\pm 2R_{A0} - M_A),$$

$$m_+ = \frac{1}{2R_{B0}}(2m_1 R_{B0} - M_B^2 \pm 2M_B\sqrt{R_{B0}^2 - 2m_1 R_{B0}}.$$

Because m_1 and m are supposed to be positive the upper sign has to be taken.

3. Let $m_1 = M_A$ as in the case 1. The corresponding initial velocity for the shell A is then

$$R'_{A0} = -\frac{1-(2M_A/R_{A0})}{1-(M_A/(2R_{A0}))}\sqrt{\frac{M_A}{R_{A0}} + \frac{M_A^2}{4R_{A0}^2}}.$$

Let the initial velocity of the shell B is

$$R'_{B0} = R'_{A0} + \Delta R',$$

where $\Delta R'$ is a very small quantity. Then

$$m = m_1 - \frac{M_B^2}{2R_{B0}} + M_B\sqrt{f_{1B} + \frac{f_{1B}R'^2_{BO}}{(f_{1B}^2 - R'^2_{B0})}}.$$

6 NUMERICAL SOLUTION AND DISCUSSION

Here we study the last case 3 in more detail, while the other cases will be discussed in the dissertation. Let the initial value of masses and radii be

$$M_A = M_B = M = 100M_0, \quad R_{A0} = 10M', \quad R_{B0} = 1.001R_{A0},$$

$$v_1 = 0, \quad \tau_A = \tau_B = 0, \quad \Delta R' = -0.001, \quad t_M = 0.$$

From (28) we get the value of u_1. Since the velocity of the exterior shell is greater than the velocity of the inner shell, the exterior shell crosses the inner one and its velocity will decrease while the velocity of the other shell will increase. Then they will intersect again and so on; i.e. one of shells oscillates around the other. We found many intersections for two collapsing shells before the singularity.

During the collapse the rate of change of the proper time between the points of intersection is diminishing; the Schwarzschild mass between shells after each intersection is increasing as shown in Table 1. The difference of radii of shells increases after each intersection.

Dependence of intersection on the initial data. We study this case with different initial data. The number of points of intersection depends on the initial distance between shells. It is shown in Table 2. If the difference of radii between shells was increased then the number of intersections decreased.

Table 1. The change of the Schwarzschild mass m_i between shells during the intersections, where i is the number of intersection. The change of the proper time: $\Delta\tau_{Ai} = \rho_{Ai} - \tau_{Ai-1}$, $\Delta\tau_{Bi} = \tau_{Bi} - \tau_{Bi-1}$. The Schwarzschild mass in the external space is $m = 190.05906$.

ith of inters.	m_i	$\Delta\tau_{Ai}$	$\Delta\tau_{Bi}$
1	100.00000	271.44516	271.44514
2	100.53523	234.71529	234.71530
3	101.16942	201.12761	201.12760
4	101.92327	170.77800	170.77802
5	102.82291	143.66511	143.66510
6	103.90164	119.70755	119.70757
...	...	...	...
12	117.78661	31.75030	31.75032
13	122.55509	24.22199	24.22197
...	...	...	...
23	308.59983	1.16527	1.16526
24	376.13217	0.80758	0.80758
25	483.38334	0.52464	0.52462

7 CONCLUSION

The main motivation for our work has been to show how far one thin shell can approximate the evolution of a thick layer of matter. Of course, two shells as model of the thick layer is still very rough but it gives some hints with respect of the general situation.

In the case that no intersection of shells occurs we can choose a thin shell with the proper mass equal to the sum of proper masses of both shells which during all its collapse remains in the range of radial variable limited by radii of both shells. At the beginning the radius of this shell is nearby the outer shell and during its

Table 2. The relation between the number of the points of intersection and the difference in radius of two shells for different initial radii of the inner shell, where $\Delta R_{B0} = (1 + \Delta R_0)R_{A0}$.

No. of inters.	$\Delta R_0 = R_{B0} - R_{A0}$			
	$R_{A0}=10^2M$	$R_{A0}=10M$	$R_{A0}=5M$	$R_{A0}=2.5M$
Multi-Points	$\Delta R_0 < 0.001$	$\Delta R_0 < 0.001$	$\Delta R_0 < 0.001$	$\Delta R_0 < 0.001$
Many-Points	$\Delta R_0 \leq 0.22$	$\Delta R_0 < 0.06$	$\Delta R_0 < 0.06$	$\Delta R_0 < 0.01$
Two-Points	$\Delta R_0 \leq 0.2272$	$\Delta R_0 \leq 0.169$	$\Delta R_0 \leq 0.113$	$\Delta R_0 \leq 0.0215$
One-Point	$\Delta R_0 \leq 0.2273$	$\Delta R_0 \leq 0.17$	$\Delta R_0 \leq 0.114$	$\Delta R_0 \leq 0.02158$
No-Point	$\Delta R_0 \geq 0.22735$	$\Delta R_0 \geq 0.171$	$\Delta R_0 \geq 0.115$	$\Delta R_0 \geq 0.0216$

collapse is approaching the radius of the inner shell. However, in order to ensure these conditions the original distance of two shells in this model must be relatively large. This seems to show that a singular shall can be a good model for the thick layer which is not too dense far away from the centre.

If the thickness original two-shells layer is small, the intersections of shells occurs. We took the points of intersection of shells in the case of 25-intersections and approximated their positions by a smooth curve. Taking this curve as world line of a thin shell we calculated its equation of state i.e. the relation between the surface density and the surface pressure using (34) and (35). This relation cannot be approximated by a simple formula. For this reason using a one shell model to approximate the two-shell model does not seem reasonable. It points to the more general conclusion that it is not possible to approximate the motion of the thick layer of matter by one thin shell with some reasonable equation of state in the case that the layer is really dense and consequently a mixing of particles in the radial direction occurs.

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