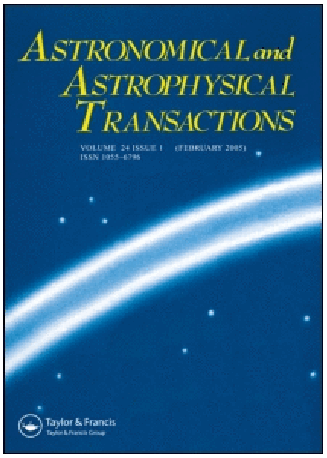

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PERTURBATED ROTATIONAL MOTION OF WEAKLY DEFORMABLE CELESTIAL BODIES

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The rotation equations of weakly deformable celestial bodies (in canonical and non-canonical Andoyer variables) are developed in detail. A theory of the perturbed rotational motion of an isolated weakly deformable body has been developed. Applications to Earth's rotation theory are given.

KEY WORDS Rotation of deformable body, Chandler pole motion, perturbation theory

INTRODUCTION

Liouville's equations have wide applications in the theory of Earth rotation (in the first instance, of the polar motion) (Munk and Mac-Donald, 1960; Lambeck, 1980; Moritz and Mueller, 1992). Usually a certain reduced linear form of these equations is used. However, in connection with the increase of accuracy of observations, the problem of taking into account new terms, additional to linear, and of more accurate analytical description of the corresponding effects in the Earth's rotation become actual (Podobed and Nesterov, 1975). This necessity has a place, for example, for an explanation of the observed discrepancies in the values of amplitudes of some Earth axis nutations. An important problem is a full account of unperturbed Chandler–Euler motion properties in Earth rotation theory of its perturbed motion (Barkin, 1996a; Barkin, Ferrandiz, and Getino, 1995b). These points, of course, refer to all bodies of the solar system. In particular, they are more relevant for construction of the theories of rotation of Venus, Mars, the asteroids and others.

For a solution of these problems, it is very important to have and to use different forms of rotation equations, having a clear geometrical and dynamic sense. For a model of rigid celestial bodies, for example, the canonical equations in Andoyer variables, (Andoyer, 1923) have very important applications in the theory rotation of the Earth (Kinoshita, 1977; Kinoshita and Souchay, 1991) of the Moon (Beletskij, 1971; 1972; 1975; Henrard and Moons, 1978; Lidov and Neishtadt, 1975; Barkin, 1978, 1987), of Mercury and Venus (Beletskij, 1975; Beletskij, Levin, and Pogorelov, 1979; Barkin, 1988) and of other bodies of the solar system (Barkin, 1984b). Equations in Andoyer variables have been used in artificial satellite dynamics (Beletskij, 1965; 1975; Chernousko, 1963; Beletskij and Khentov, 1985 and others; see also review in, for example, Barkin *et al.*, 1982a; Barkin and Demin, 1982b).

Equations in the angle-action variables of the Euler–Poinsot problem have created efficient applications in Earth rotation theory (Kinoshita, 1977), for a study of the polar motion of the Earth (Barkin, 1996a; 1998b), in the theory of asteroid rotation (Barkin, 1984b; Kinoshita, 1992) and in many other works. (See, for example, reviews in the monographs of Beletskij, 1975; Beletskij and Hentov, 1985 and others).

Wide and interesting studies have used equations in Andoyer and angle-action variables applied to the quantitative and analytical dynamics of a rigid body (Arkhangelskij, 1977; Kozlov, 1980; Barkin and Borisov, 1989 and others).

Differential equations in Andoyer variables (Getino and Ferrandiz, 1990; 1991) and in angle-action variables, for Euler–Chandler unperturbed rotational motion of deformable celestial bodies (Barkin, Ferrandiz and Getino, 1995b) have obtained some important applications for the study of some particularities of the Earth's polar motion and of the perturbations in the Earth's rotation (Barkin, 1996b).

In this paper, analogous equations in Andoyer variables are developed for an other model of a celestial body (Liouville's model). It is assumed that the rigid or elastic external envelope of the Earth is covered with a deformable layer. Relative displacements of the particles of this layer are specified by functions of time.

This model is the basis for wide studies of the influence of different geophysical and tectonic processes on the Earth's rotation, and for a study of rotation of other celestial bodies.

Equations of rotation were obtained for the first time by the author in a more general statement of the problem of translatory–rotary motion of a planetary system of mutually gravitating deformable bodies in 1979. First integrals of this problem were efficiently used for a generalization of the classical Laplace and Laplace–Lagrange theorems about the stability of a planetary system, proved for a planet system of material points and for a planet system of rigid celestial bodies (Barkin, 1977) on the above-mentioned planetary system of weakly deformable bodies. These results, including the main form of the rotation equations in Andoyer variables and in the angle-action variables, were presented for the first time at Douboshine seminar (May 1979). However, in spite of the great importance of these results, they were not published, except for some short annotations (Barkin and Demin, 1979a; 1984, and others).

In this paper, the rotation equations of weakly-deformable celestial bodies (in canonical and non-canonical Andoyer variables) are developed in details. A theory of the perturbed rotational motion of an isolated weakly deformable body has been developed. The components of the inertia tensor of this body are considered as definite conditionally periodic functions of time.

Unperturbed rotational motion is an Euler–Chandler motion of an axisymmetric elastic body. Perturbations in the rotation were obtained for arbitrary values of the unperturbed motion parameters (among them, for example, the angle θ between the vector of the angular momentum and the polar axis of the body). The results obtained are very important for the studies of rotation of different bodies in the solar system, for example for Venus, for which angle θ is not small (about 13°, Williams, *et al.*, 1983).

The solution obtained of the problem of perturbed rotational motion of an isolated celestial body also presents important interest for the development of Earth rotation theory as well as for an explanation of the main mechanical phenomena in the Earth's polar motion and for a refinement of the amplitudes of the perturbations adding some additional terms, for example, proportional to the small angle θ, and to study new fine effects due to different geophysical processes.

In this paper, we discuss some of these effects. In our paper, the Chandler motion of the pole of the Earth and its properties have been explained by means of a new approach to the problem on the basis of the non-canonical equations in Andoyer variables. It corresponds to classical explanation of the Chandler pole motion (Munk and MacDonald, 1960) and its modern modifications (Kubo, 1991).

Periodic variations of the components of the angular velocity of the Earth, due to tidal lunar–solar variations of the components of the inertia tensor, have been determined. Corresponding effects in the polar motion are fine and the main perturbations are characterized by amplitudes of the order of 0.0001 arc seconds.

The results obtained give new opportunities for a study of rotational motion of solar system bodies and in the first case of Earth rotation. In a future paper, we will investigate in detail secular effects in the rotation of a weakly deformable body caused by slow redistributions of its masses: secular motion of the axis rotation pole, secular motion of the pole of the angular momenum vector, secular motion of the poles of the principal axes of inertia, acceleration of the axial rotation, secular variations of the Euler–Chandler motion, etc. These effects are estimated for definite mechanisms of subduction and mass accumulation of the oceanic plates (Barkin, 1995a; 1996b; 1999). As a result of these fulfilled preliminary studies, in the last papers the paleomigration of the Earth's pole in the present geological epoch has been revealed.

1 EQUATIONS OF WEAKLY DEFORMABLE BODY ROTATION

1.1 Main Kinematical and Dynamical Characteristics

Consider a weakly deformable body, assuming that its particles in the process of the body motion are weakly displaced from their initial positions, or are moved in a given manner in the time but with small velocity. The body has an inner rigid envelope, with which we connect the Cartesian reference system $C\xi\eta\zeta$, and an external deformable envelope. The origin of these reference systems coincides with the centre of mass of the body.

Let $Cxyz$ be the main reference system with the same origin and with axes having their permanent orientation in space. We define the orientation of the axes

$C\xi\eta\zeta$ with respect to the reference system $Cxyz$ by the Eulerian angles Ψ, Θ and Φ (precession angle, nutation angle, and angle of own rotation) (Douboshine, 1975). Let $\bar{\omega}$ be the angular velocity vector of rotation of the body reference system $C\xi\eta\zeta$ in the main reference system $Cxyz$. Its projections on the axes $C\xi$, $C\eta$, and $C\zeta$ are defined by the Eulerian kinematical equations:

$$\begin{aligned} p &= \sin\Theta\sin\Phi\dot{\Psi} + \cos\Phi\dot{\Theta}\,, \\ q &= \sin\Theta\cos\Phi\dot{\Psi} - \sin\Phi\dot{\Theta}\,, \\ r &= \cos\Theta\dot{\Psi} + \dot{\Phi}\,, \end{aligned} \tag{1.1}$$

where $\dot{\Psi}$, $\dot{\Theta}$, and $\dot{\Phi}$ are the corresponding generalized velocities.

We define the position of an arbitrary point of the body in its undeformed state (or in some initial state) and in its deformed state by radius vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ and introduce the bias vector $\boldsymbol{u} = \boldsymbol{r}' - \boldsymbol{r}$.

We denote the components of this vector in the $C\xi$, $C\eta$ and $C\zeta$, axes as u, v and w. Let (ξ, η, ζ) and (ξ', η', ζ') be Cartesian coordinates of an arbitrary point of the body in the $C\xi\eta\zeta$ reference system for the two abovementioned states of the body. If $\boldsymbol{i}_b$, $\boldsymbol{j}_b$, $\boldsymbol{k}_b$, are unit vectors of the coordinate axes $C\xi$, $C\eta$ and $C\zeta$, we have the following representations for the vectors $\boldsymbol{u}$, $\boldsymbol{r}'$, $\boldsymbol{r}$:

$$\begin{aligned} \boldsymbol{r} &= x\boldsymbol{i}_b + y\boldsymbol{j}_b + z\boldsymbol{k}_b\,, \\ \boldsymbol{r}' &= x'\boldsymbol{i}_b + y'\boldsymbol{j}_b + z'\boldsymbol{k}_b\,, \\ \boldsymbol{u} &= u\boldsymbol{i}_b + v\boldsymbol{j}_b + w\boldsymbol{k}_b\,. \end{aligned} \tag{1.2}$$

We point out that the vector $\boldsymbol{u}$ is a given function of time. From the mechanical point of view, the processes of the mass redistribution of the body are considered as given and independent from its rotation.

In our paper, the general forms of equations of rotational motion under the action of an arbitrary moment of forces $\boldsymbol{L}$ are obtained. The general case of non-potential forces and the case of potential forces are considered. Canonical and non-canonical equations of motion in Eulerian, Andoyer and Poinsot variables are obtained. In the general case, the body axes $C\xi\eta\zeta$ are not principal.

The angular moment of the body rotation in the reference system $Cxyz$ is defined by the following integral

$$\boldsymbol{G} = \int\limits_{\tau'} \rho(\boldsymbol{r}')\boldsymbol{r}' \times \boldsymbol{v}'\,\mathrm{d}\tau'\,, \tag{1.3}$$

where $\boldsymbol{v}'$ is the velocity of an arbitrary particle of the body, $\mathrm{d}\boldsymbol{r}' = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$ is an elementary volume, and integration in (1.3) is spread over the full volume of the body in its deformed state,

$$\boldsymbol{v}' = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{r}'\,. \tag{1.4}$$

The first term in (1.4) is the relative velocity of the particle in the body reference system.

Neglecting terms of the second order with respect to $\boldsymbol{u}$ and $\partial \boldsymbol{u}/\partial t$, we present the expression of the angular momentum (1.3) as the sum of integrals:

$$\begin{aligned} \boldsymbol{G} &= \int_{\tau'} \rho(\boldsymbol{r}+\boldsymbol{u})\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})\, \mathrm{d}\tau' \\ &+ \int_{\tau'} \rho(\boldsymbol{r})\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{u})\, \mathrm{d}\tau' \\ &+ \int_{\tau'} \rho(\boldsymbol{r})\boldsymbol{u} \times (\boldsymbol{\omega} \times \boldsymbol{r})\, \mathrm{d}\tau' . \end{aligned} \tag{1.3'}$$

In the general case, the density variation in an arbitrary body point has first order and we can use for the density the following approximation:

$$\rho(\boldsymbol{r}') = \rho(\boldsymbol{r}) + \frac{\partial \rho}{\partial x} u + \frac{\partial \rho}{\partial y} v + \frac{\partial \rho}{\partial z} w . \tag{1.5}$$

In the integrals (1.3), (1.3') the variables x', y', z' are replaced by the variables x, y, z and integration over the volume of the deformed body is replaced by integration over the undeformed body. The transformation of the volumes is given by the formula:

$$\mathrm{d}\tau' = \left(1 + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \mathrm{d}\tau . \tag{1.6}$$

In a particular case, for example, for the motion of separate blocks of the Earth (or plates), the density in an arbitrary point practically does't change and formulae (1.5), (1.6) are simplified:

$$\rho(\boldsymbol{r}') = \rho(\boldsymbol{r}), \quad \mathrm{d}\tau' = \mathrm{d}\tau . \tag{1.7}$$

Retaining in (1.3') the main order terms, after some algebra with the help of formulae (1.5)–(1.7), we obtain the following expression for the angular momentum:

$$\boldsymbol{G} = G_\xi \boldsymbol{i}_b + G_\eta \boldsymbol{j}_b + G_\zeta \boldsymbol{k}_b . \tag{1.8}$$

Projections of the vector $\boldsymbol{G}$ on the body coordinate axes are defined by the formulae:

$$\begin{aligned} G_\xi &= Ap - Fp - Er + P , \\ G_\eta &= -Fp + Bq - Dr + Q , \\ G_\zeta &= -Ep - Dq + Cr + R . \end{aligned} \tag{1.9}$$

Here A, B, C and F, E, D are axial and centrifugal moments of inertia of the body in the body axes, but calculated for the changed body density

$$\rho(\boldsymbol{r}) = \rho_0(\boldsymbol{r}) + \rho_1(\boldsymbol{r}, \boldsymbol{u}) , \quad \rho_1(\boldsymbol{r}, \boldsymbol{u}) = \frac{\partial \rho}{\partial \xi} u + \frac{\partial \rho}{\partial \eta} v + \frac{\partial \rho}{\partial \zeta} w \tag{1.10}$$

(according to our assumption, it is a known function of time). P, Q and R are projections of the angular momentum vector of the relative motion of the body particles (in the $C\xi\eta\zeta$ reference system) on axes $C\xi$, $C\eta$ and $C\zeta$.

Thus, the components of the body inertia tensor are represented as a sum of two terms:

$$A = A_0 + A_1\,, \quad B = B_0 + B_1\,, \quad C = C_0 + C_1\,,$$
$$F = F_0 + F_1\,, \quad E = E_0 + E_1\,, \quad D = D_0 + D_1\,, \tag{1.11}$$

where A_0, B_0, C_0, F_0, E_0, D_0 are the components of the tensor of inertia in the undeformed state of the body, and A_1, B_1, C_1, F_1, E_1, D_1 are some additional terms due to variations of the density redistribution. These characteristics are defined by an integration over the full volume of the body in its undeformed state:

$$\begin{aligned}
A_0 &= \int_\tau \rho_0(\boldsymbol{r})(\eta^2+\zeta^2)\,\mathrm{d}\tau\,,\\
B_0 &= \int_\tau \rho_0(\boldsymbol{r})(\zeta^2+\xi^2)\,\mathrm{d}\tau\,,\\
C_0 &= \int_\tau \rho_0(\boldsymbol{r})(\xi^2+\eta^2)\,\mathrm{d}\tau\,,\\
F_0 &= \int_\tau \rho_0(\boldsymbol{r})\eta\xi\,\mathrm{d}\tau\,,\\
E_0 &= \int_\tau \rho_0(\boldsymbol{r})\xi\zeta\,\mathrm{d}\tau\,,\\
D_0 &= \int_\tau \rho_0(\boldsymbol{r})\zeta\eta\,\mathrm{d}\tau\,.
\end{aligned} \tag{1.12}$$

$$\begin{aligned}
A_1 &= \int_\tau \{\rho_1(\boldsymbol{r},\boldsymbol{u})(\eta^2+\zeta^2) + 2\rho_0(\boldsymbol{r})(\eta v + \zeta w)\}\,\mathrm{d}\tau\,,\\
B_1 &= \int_\tau \{\rho_1(\boldsymbol{r},\boldsymbol{u})(\zeta^2+\xi^2) + 2\rho_0(\boldsymbol{r})(\zeta w + \xi u)\}\,\mathrm{d}\tau\,,\\
C_1 &= \int_\tau \{\rho_1(\boldsymbol{r},\boldsymbol{u})(\xi^2+\eta^2) + 2\rho_0(\boldsymbol{r})(\xi u + \eta v)\}\,\mathrm{d}\tau\,,\\
F_1 &= \int_\tau \{\rho_1(\boldsymbol{r},\boldsymbol{u})\eta\xi + \rho_0(\boldsymbol{r})(\eta u + \xi v)\}\,\mathrm{d}\tau\,,\\
E_1 &= \int_\tau \{\rho_1(\boldsymbol{r},\boldsymbol{u})\xi\zeta + \rho_0(\boldsymbol{r})(\zeta u + \xi w)\}\,\mathrm{d}\tau\,,
\end{aligned}$$

$$D_1 = \int_\tau \{\rho_1(\boldsymbol{r}, \boldsymbol{u})\eta\zeta + \rho_0(\boldsymbol{r})(\zeta v + \eta w)\}\, \mathrm{d}\tau. \tag{1.13}$$

In (1.9), the components of the relative angular momentum of the body particles in the $C\xi\eta\zeta$ reference system are defined by the following volume integrals:

$$\begin{aligned} P &= \int_\tau \rho_0(\boldsymbol{r}) \left(\eta\frac{\partial w}{\partial t} - \zeta\frac{\partial v}{\partial t}\right) \mathrm{d}\tau, \\ Q &= \int_\tau \rho_0(\boldsymbol{r}) \left(\zeta\frac{\partial u}{\partial t} - \xi\frac{\partial w}{\partial t}\right) \mathrm{d}\tau, \\ R &= \int_\tau \rho_0(\boldsymbol{r}) \left(\xi\frac{\partial v}{\partial t} - \eta\frac{\partial u}{\partial t}\right) \mathrm{d}\tau. \end{aligned} \tag{1.14}$$

Formulae (1.12)–(1.14) enable us to present (1.11) and (1.9) as explicit functions of time. The kinetic energy of the rotational motion of the weakly deformable body with respect to the reference system $Cxyz$ is defined by the following volume integral:

$$T = \frac{1}{2}\int_{\tau'} \rho(\boldsymbol{r}')\boldsymbol{v}' \cdot \boldsymbol{v}'\, \mathrm{d}\tau', \tag{1.15}$$

were $\mathrm{d}\tau' = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$ and the integration is spread over the full volume of the deformed body.

The absolute velocity of the body particle is defined by the following formula:

$$\boldsymbol{v}' = \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{\omega} \times \boldsymbol{r} + \boldsymbol{\omega} \times \boldsymbol{u} \tag{1.16}$$

(the derivative of the vector $\boldsymbol{u}$ is calculated with respect to the body reference system).

Substituting (1.16) into expression (1.15) and neglecting the second order terms with respect to the components of $\boldsymbol{u}$ and $\mathrm{d}\boldsymbol{u}/\mathrm{d}t$, we describe the kinetic energy of the body as:

$$\begin{aligned} T &= \frac{1}{2}\int_{\tau'} \rho(\boldsymbol{r}')(\boldsymbol{\omega} \times \boldsymbol{r})^2\, \mathrm{d}\tau' \\ &+ \int_{\tau'} \rho(\boldsymbol{r}')(\boldsymbol{\omega} \times \boldsymbol{r})(\boldsymbol{\omega} \times \boldsymbol{u})\, \mathrm{d}\tau' \\ &+ \int_{\tau'} \rho(\boldsymbol{r}')\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}(\boldsymbol{\omega} \times \boldsymbol{r})\, \mathrm{d}\tau'. \end{aligned} \tag{1.17}$$

Going to the integration over the undeformed state of the body and taking into account formulae (1.6), (1.7), after some algebra we obtain the following simplified expression of the kinetic energy:

$$T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2 - 2Fpq - 2Epr - 2Drq) + pP + qQ + rR\,, \tag{1.18}$$

where p, q, r are the components of the angular velocity vector (1.1), and the components of the inertia tensor of the body and the components of the relative angular moment are defined by formulae (1.12)–(1.14).

Formula (1.18) presents the kinetic energy of the body as a function of the components of the angular velocity (1.1) and time, or as a function of the generalized coordinates and velocities:

$$\Psi,\ \Theta,\ \Phi,\ \dot{\Psi},\ \dot{\Theta},\ \dot{\Phi}. \tag{1.19}$$

The dynamic characteristics of the weakly deformable body (1.8), (1.9) and (1.18) permit us to obtain different forms of the differential equations of the rotational motion.

1.2 *Liouville's Equations*

We suppose that the body is subject to the action of definite forces and the principal moment of these forces with respect to the centre of mass C is $\boldsymbol{L}$. In accordance with the theorem about the angular momentum for a mechanical system, we obtain the following vector equation of the body rotation:

$$\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}t} + \boldsymbol{\omega} \times \boldsymbol{G} = \boldsymbol{L} \tag{1.20}$$

or in projections on the axes of the body $\xi\eta\zeta$:

$$\begin{aligned}
\frac{\mathrm{d}G_\xi}{\mathrm{d}t} + qG_\zeta - rG_\eta &= L_\xi\,,\\
\frac{\mathrm{d}G_\eta}{\mathrm{d}t} + rG_\xi - pG_\zeta &= L_\eta\,,\\
\frac{\mathrm{d}G_\zeta}{\mathrm{d}t} + pG_\eta - qG_\xi &= L_\zeta\,.
\end{aligned} \tag{1.21}$$

The derivative of the vector $\boldsymbol{G}$ in (1.20) is taken with respect to the moving reference system $C\xi\eta\zeta$.

Substituting values of the projections (1.9) into (1.21), we obtain the Liouville equations (Liouville, 1858; Tisserand, 1891):

$$\begin{aligned}
L_\xi &= \frac{\mathrm{d}}{\mathrm{d}t}(Ap - Fq - Er + P)\\
&+ D(r^2 - q^2) + (C - B)qr + (Fr - Eq) + qR - rQ\,,\\
L_\eta &= \frac{\mathrm{d}}{\mathrm{d}t}(-Fp + Bq - Dr + Q)\\
&+ E(p^2 - r^2) + (A - C)rp + (Dp - Fr)q + rP - pR\,,\\
L_\zeta &= \frac{\mathrm{d}}{\mathrm{d}t}(-Ep - Dq + Cr + R)\\
&+ F(q^2 - p^2) + (B - A)pq + (Eq - Dp)r + pQ - qP\,.
\end{aligned} \tag{1.22}$$

These equations are combined with Euler's kinematic equations (1.1) and are integrated for concrete values of the components of the tensor of inertia and of the components of the relative angular momentum as definite functions of time:

$$A(t),\ B(t),\ C(t);\quad F(t),\ E(t),\ D(t);\quad P(t),\ Q(t),\ R(t). \tag{1.23}$$

Projections of the principal moment of the forces acting on the body are considered as known functions of the generalized coordinates and velocities (1.19) and time.

In the absence of this moment (when $\boldsymbol{L} = 0$), the integrable cases of equations (1.22) and their integrability were studied by Barkin and Demin (1979; 1984), Barkin (1998), Borisov (1991).

1.3 Canonical Equations of the Rotational Motion in Euler Variables

Taking the Eulerian variables as generalized coordinates, we define the conjugate canonical momenta by formulae (Douboshine, 1975)

$$p_\Psi = \frac{\partial T}{\partial \dot\Psi}, \quad p_\Phi = \frac{\partial T}{\partial \dot\Phi}, \quad p_\Theta = \frac{\partial T}{\partial \dot\Theta}. \tag{1.24}$$

Substituting the expression for T (1.17) into (1.24), we obtain:

$$\begin{aligned}
p_S &= Ap\frac{\partial p}{\partial \dot S} + Bq\frac{\partial q}{\partial \dot S} + Cr\frac{\partial r}{\partial \dot S} - F\left(q\frac{\partial p}{\partial \dot S} + p\frac{\partial q}{\partial \dot S}\right) - E\left(r\frac{\partial p}{\partial \dot S} + p\frac{\partial r}{\partial \dot S}\right)\\
&- D\left(q\frac{\partial r}{\partial \dot S} + r\frac{\partial q}{\partial \dot S}\right) + P\frac{\partial p}{\partial \dot S} + Q\frac{\partial q}{\partial \dot S} + R\frac{\partial r}{\partial \dot S}\\
&= (Ap - Fq - Er + P)\frac{\partial p}{\partial \dot S} + (-Fp + Bq - Dr + Q)\frac{\partial q}{\partial \dot S}\\
&+ (-Ep - Dq + Cr + R)\frac{\partial r}{\partial \dot S},
\end{aligned} \tag{1.25}$$

where $S = (\Psi, \Theta, \Phi)$, $\dot S = (\dot\Psi, \dot\Theta, \dot\Phi)$.

Taking into account formulae (1.9) and (1.1), we find from (1.25) the following relationships:

$$p_S = G_\xi \frac{\partial p}{\partial \dot S} + G_\eta \frac{\partial q}{\partial \dot S} + G_\zeta \frac{\partial r}{\partial \dot S},$$

where

$$\begin{aligned}
\frac{\partial p}{\partial \dot\Psi} &= \sin\Theta\sin\Phi, & \frac{\partial q}{\partial \dot\Psi} &= \sin\Theta\cos\Phi, & \frac{\partial r}{\partial \dot\Psi} &= \cos\Theta,\\
\frac{\partial q}{\partial \dot\Phi} &= 0, & \frac{\partial q}{\partial \dot\Phi} &= 0, & \frac{\partial r}{\partial \dot\Phi} &= 1,\\
\frac{\partial p}{\partial \dot\Theta} &= \cos\Phi, & \frac{\partial q}{\partial \dot\Theta} &= -\sin\Phi, & \frac{\partial r}{\partial \dot\Theta} &= 0.
\end{aligned}$$

Thus

$$
\begin{aligned}
p_\Psi &= G_\xi \sin\Theta \sin\Phi + G_\eta \sin\Theta \cos\Phi + G_\zeta \cos\Theta \,, \\
p_\Phi &= G_\zeta \,, \\
p_\Theta &= G_\xi \cos\Phi - G_\eta \sin\Phi \,.
\end{aligned} \tag{1.26}
$$

Solving (1.26) for the components of the angular momentum, we obtain the following expressions:

$$
\begin{aligned}
G_\xi &= p_\Theta \cos\Phi + \frac{(p_\Psi - p_\Phi \cos\Theta)}{\sin\Theta} \sin\Phi \,, \\
G_\eta &= -p_\Theta \sin\Phi + \frac{(p_\Psi - p_\Phi \cos\Theta)}{\sin\Theta} \cos\Phi \,, \\
G_\zeta &= p_\Phi \,,
\end{aligned} \tag{1.27}
$$

and for the modulus of the vector $\boldsymbol{G}$

$$
G = \sqrt{G_\xi^2 + G_\eta^2 + G_\zeta^2} = \sqrt{p_\Theta^2 + p_\Phi^2 + \frac{(p_\Psi - p_\Phi \cos\Theta)}{\sin^2\Theta}} \,. \tag{1.28}
$$

Let us assume that the body motion is executed under the action of potential forces and the problem admits a definite force function $U(\Psi, \Theta, \Phi, t)$. The Lagrangian of this problem will be $L = T + U$, and the Hamiltonian (generalized energy) is defined by (Olkhovskij, 1975):

$$
K = p_\Psi \dot{\Psi} + p_\Phi \dot{\Phi} + p_\Theta \dot{\Theta} - T - U \,. \tag{1.29}
$$

This function must be presented as an explicit function of the canonical variables of the problem using formulae (1.26), (1.18), (1.1):

$$
\Psi,\ \Theta,\ \Phi,\ p_\Psi,\ p_\Theta,\ p_\Phi \,. \tag{1.30}
$$

From the general theory of the canonical system, for the Hamiltonian (1.29) we have the following representation:

$$
K = T^{(2)} - U \,,
$$

where $T^{(2)}$ is the quadratic part of the kinetic energy with respect to the generalized velocities. It means that in the function T, it is sufficient to retain only the quadratic terms with respect to p, q, r, expressed in terms of the variables (1.30).

Let us solve equations (1.9) with respect to components of the angular velocity p, q and r. We will have:

$$
\begin{aligned}
p &= a(G_\xi - P) - f(G_\eta - Q) - e(G_\zeta - R) \,, \\
q &= -f(G_\xi - P) + b(G_\eta - Q) - d(G_\zeta - R) \,, \\
r &= -e(G_\xi - P) - d(G_\eta - Q) + c(G_\zeta - R) \,,
\end{aligned} \tag{1.31}
$$

where we used the new notation:

$$a = \frac{BC - D^2}{\Delta}, \quad b = \frac{AC - E^2}{\Delta}, \quad c = \frac{AB - F^2}{\Delta},$$
$$f = -\frac{ED + FC}{\Delta}, \quad e = -\frac{FD + BE}{\Delta}, \quad d = -\frac{FE + AD}{\Delta},$$
$$\Delta = ABC - AD^2 - BE^2 - CF^2 - 2FED. \tag{1.32}$$

Let us substitute these formulae (1.31) into the expression of the quadratic part of the kinetic energy in (1.18). Neglecting terms depending only on the time, we obtain the following expression for the Hamiltonian of the problem:

$$\begin{aligned} K &= \frac{1}{2}\{aG_\xi^2 + bG_\eta^2 + cG_\zeta^2 - 2fG_\xi G_\eta - 2eG_\zeta G_\xi - 2dG_\zeta G_\eta\} \\ &- \Omega_\xi G_\xi - \Omega_\eta G_\eta - \Omega_\zeta G_\zeta - U(\Psi, \Theta, \Phi, t), \end{aligned} \tag{1.33}$$

where Ω_ξ, Ω_η, Ω_ζ are the components of some angular velocity:

$$\begin{aligned} \Omega_\xi &= aP - fQ - eR, \\ \Omega_\eta &= -fP + bQ - dR, \\ \Omega_\zeta &= -eP - dQ + cR \end{aligned} \tag{1.34}$$

and are known functions of time, and the components of the angular momentum are defined by formulae (1.27), (1.28).

Thus, the canonical equations of the rotational motion of a deformable body in variables (1.30) have the following form:

$$\frac{dS}{dt} = \frac{\partial K}{\partial p_S}, \quad \frac{dp_S}{dt} = -\frac{\partial K}{\partial S} \ (S = \Psi, \Theta, \Phi), \tag{1.35}$$

$$\begin{aligned} K &= \frac{1}{2}(H_{11}p_\Psi^2 + H_{22}p_\Theta^2 + H_{33}p_\Phi^2 + 2H_{12}p_\Psi p_\Theta + 2H_{13}p_\Psi p_\Phi + 2H_{23}p_\Theta p_\Phi) \\ &+ h_1 p_\Psi + h_2 p_\Theta + h_3 p_\Phi - U(\Psi, \Theta, \Phi, t). \end{aligned}$$

The coefficients H_{ij}, h_i in (1.35) are defined by the formulae:

$$\begin{aligned} H_{11} &= \sec^2\Theta(a\sin^2\Phi + b\cos^2\Phi - f\sin 2\Phi), \\ H_{22} &= a\cos^2\Phi + b\sin^2\Phi + f\sin 2\Phi, \\ H_{33} &= \cot^2\Theta(a\sin^2\Phi + b\cos^2\Phi - f\sin 2\Phi) + c + 2c\tan\Theta(e\sin\Phi + d\cos\Phi), \\ H_{12} &= \sec\Theta[\sin 2\Phi(a - b) - 2f\cos 2\Phi], \\ H_{13} &= -\cos\Theta\sec^2\Theta(a\sin^2\Phi + b\cos^2\Phi - f\sin 2\Phi) - \sec\Theta(e\sin\Phi + d\cos\Phi), \\ H_{23} &= -\frac{1}{2}\cos\Theta\sec\Theta[\sin 2\Phi(a - b) - 2f\cos 2\Phi] - d\sin\Phi - e\cos\Phi \\ h_1 &= \sec\Theta(\Omega_\xi\sin\Phi + \Omega_\eta\cos\Phi), \\ h_2 &= \Omega_\xi\cos\Phi - \Omega_\eta\sin\Phi, \\ h_3 &= -\cot\Theta(\Omega_\xi\sin\Phi + \Omega_\eta\cos\Phi) + \Omega_\zeta. \end{aligned} \tag{1.36}$$

The force function U in (1.35) is a function ot the direction cosines a_{ij} of the body axes $C\xi\eta\zeta$ with respect to the main reference system $Cxyz$ and can be presented as a function of the Euler angles with the help of the well-known formulae (Douboshine, 1975)

$$\begin{aligned}
a_{11} &= \cos\Psi\cos\Phi - \sin\Psi\cos\Theta\sin\Phi\,,\\
a_{21} &= \sin\Psi\cos\Phi + \cos\Psi\cos\Theta\sin\Phi\,,\\
a_{31} &= \sin\Theta\sin\Phi\,,\\
a_{12} &= -\cos\Psi\sin\Phi - \sin\Psi\cos\Theta\cos\Phi\,,\\
a_{22} &= -\sin\Psi\sin\Phi - \cos\Psi\cos\Theta\cos\Phi\,,\\
a_{32} &= \sin\Theta\cos\Phi\,,\\
a_{13} &= \sin\Psi\sin\Theta\,,\\
a_{23} &= -\cos\Psi\sin\Theta\\
a_{33} &= \cos\Theta\,.
\end{aligned} \tag{1.37}$$

1.4 Canonical Equations of Rotational Motion of a Deformable Body in Andoyer Variables

Let us introduce into consideration the Andoyer variables

$$G,\ \theta,\ \rho,\ l,\ g,\ h \tag{1.38}$$

which are connected with the angular momentum vector $\boldsymbol{G}$ (1.8), (1.9) (see for example Barkin, 1977; Kinoshita, 1977).

Let $CG_1G_2G_3$ be an intermediate reference system, connected with vector $\boldsymbol{G}$. The axis CG_3 is directed along the vector $\boldsymbol{G}$, and the axis CG_1 is situated in the plane Cxy of the main reference system $Cxyz$ and is directed along the line of intersection of the planes CG_1G_2 and Cxy to the ascending node of the plane CG_1G_2. Let $G = |\boldsymbol{G}|$ be the modulus of the angular momenum and ρ and h are the angles determining the orientation of the reference system $CG_1G_2G_3$ with respect to the reference system $Cxyz$: ρ is the angle between Cz axis and the angular momentum vector $\boldsymbol{G}$, and h is the angle between the positive directions of the coordinate axes Cx and CG_1 (h is the longitude of the ascending node of the intermediate plane CG_1G_2).

We define the orientation of the body axes $C\xi\eta\zeta$ with respect to intermediate reference system $CG_1G_2G_3$ which we define by Eulerian angles l, g, θ. The nutation angle θ is the angle between the positive directions of the axes CG_3 and $C\zeta$. The precession angle g is the angle between the CG_1 axis and the line of intersection of the coordinate planes CG_1G_2 and $C\xi\eta$ (or the angle between the positive direction of CG_1 axis and the direction toward the ascending node of the body plane $C\xi\eta$ on the intermediate plane CG_1G_2). The angle of own rotation l is the angle between the indicated direction to the ascending node of the plane $C\xi\eta$ and the axis $C\xi$.

Thus, the Eulerian angles $\Psi = h$, $\Theta = \rho$, $\Phi = 0$ give the orientation of the intermediate reference system $CG_1G_2G_3$ with respect to the reference system $Cxyz$, and the Eulerian angles $\Psi = g$, $\Theta = \theta$, $\Phi = l$ give the orientation of the body axes $C\xi\eta\zeta$ with respect to the intermediate reference system $CG_1G_2G_3$.

We will denote unit vectors of the Cartesian reference systems $Cxyz$, $CG_1G_2G_3$ and $C\xi\eta\zeta$ as

$$\boldsymbol{i}_s,\ \boldsymbol{j}_s,\ \boldsymbol{k}_s,\quad \boldsymbol{i}_G,\ \boldsymbol{j}_G,\ \boldsymbol{k}_G,\quad \boldsymbol{i}_b,\ \boldsymbol{j}_b,\ \boldsymbol{k}_b. \tag{1.39}$$

Also, we introduce into consideration other unit vectors, $\boldsymbol{e}_{bs}$, $\boldsymbol{e}_{Gs}$ and $\boldsymbol{e}_{bG}$, directed along the lines of intersection of the coordinate planes Cxy, $C\xi\eta$; Cxy, CG_1G_2 and CG_1G_2, $C\xi\eta$.

Let us define three Andoyer variables: L, G, H. L is the projection of the vector $\boldsymbol{G}$ on the polar axis of the body $C\zeta$, H is the projection of the vector $\boldsymbol{G}$ on the Cz axis, and G is the modulus of the vector $\boldsymbol{G}$. Obviously,

$$L = \boldsymbol{G}\boldsymbol{k}_b, \quad G = \boldsymbol{G}\boldsymbol{k}_G, \quad H = \boldsymbol{G}\boldsymbol{k}_s$$

or

$$L = G\cos\theta, \quad G = |G|, \quad H = G\cos\rho.$$

Now we will prove that the transformation of Euler's variables (1.30) to Andoyer's variables

$$L,\ G,\ H,\quad l,\ g,\ h \tag{1.40}$$

is canonical.

This fact follows from the differential form establishing the canonicity of the transformation. This form, described in Euler's (1.30) and in Andoyer's (1.40) variables, is equal to the scalar product of the angular momentum vector $\boldsymbol{G}$ and the elementary angle of rotation $\mathrm{d}\boldsymbol{\Omega}$ of the body axes $C\xi\eta\zeta$,

$$p_\Psi\,\mathrm{d}\Psi + p_\Theta\,\mathrm{d}\Theta + p_\Phi\,\mathrm{d}\Phi = L\,\mathrm{d}l + G\,\mathrm{d}g + H\,\mathrm{d}h = \boldsymbol{G}\,\mathrm{d}\boldsymbol{\Omega}\,. \tag{1.41}$$

To prove this equality we twice calculate the scalar product $\boldsymbol{G}\,\mathrm{d}\boldsymbol{\Omega}$, using the Eulerian variables and Andoyer's variables, and show that these products are equal:

$$(\boldsymbol{G}\,\mathrm{d}\boldsymbol{\Omega}) = (\boldsymbol{G}\,\mathrm{d}\boldsymbol{\Omega})_E = (\boldsymbol{G}\,\mathrm{d}\boldsymbol{\Omega})_A\,.$$

We will use some of the simplest properties of the direction cosines of the axes and of the corresponding unit vectors. First we point out that in the abovementioned variables the elementary angle of rotation (it is collinear with the angular velocity vector) is defined by the formulae (Archangelskij, 1977):

$$(\mathrm{d}\boldsymbol{\Omega})_E = \boldsymbol{k}_s\,\mathrm{d}\Psi + \boldsymbol{k}_b\,\mathrm{d}\Phi + \boldsymbol{e}_{bs}\,\mathrm{d}\Theta\,, \tag{1.42}$$

$$(\mathrm{d}\boldsymbol{\Omega})_A = \boldsymbol{k}_s\,\mathrm{d}h + \boldsymbol{i}_G\,\mathrm{d}\rho + \boldsymbol{k}_G\,\mathrm{d}g + \boldsymbol{k}_b\,\mathrm{d}l + \boldsymbol{e}_{bG}\,\mathrm{d}\theta\,. \tag{1.43}$$

For the angular momentum vector, we have similar representations:

$$(\boldsymbol{G})_E = G_\xi\boldsymbol{i}_b + G_\eta\boldsymbol{j}_b + G_\zeta\boldsymbol{k}_b\,, \tag{1.44}$$

$$(\boldsymbol{G})_A = G\boldsymbol{k}_G\,, \tag{1.45}$$

where the components G_ξ, G_η, G_ζ and G are defined by formulae (1.27), (1.28).

Multiplying the expressions (1.42), (1.44), we obtain:

$$\begin{aligned}(\boldsymbol{G}\,\mathrm{d}\boldsymbol{\Omega})_E &= (G_\xi \boldsymbol{i}_b\boldsymbol{k}_s + G_\eta \boldsymbol{j}_b\boldsymbol{k}_s + G_\zeta \boldsymbol{k}_b\boldsymbol{k}_s)\,\mathrm{d}\Psi \\ &+ (G_\xi \boldsymbol{i}_b\boldsymbol{k}_b + G_\eta \boldsymbol{j}_b\boldsymbol{k}_b + G_\zeta \boldsymbol{k}_b\boldsymbol{k}_b)\,\mathrm{d}\Phi \\ &+ (G_\xi \boldsymbol{i}_b\boldsymbol{e}_{bs} + G_\eta \boldsymbol{j}_b\boldsymbol{e}_{bs} + G_\zeta \boldsymbol{k}_b\boldsymbol{e}_{bs})\,\mathrm{d}\Theta\,. \end{aligned} \tag{1.46}$$

Using now the simple geometric relationship

$$\boldsymbol{e}_{bs} = \boldsymbol{i}_b \cos\Phi - \boldsymbol{j}_b \sin\Phi$$

for the scalar products of the unit vectors in (1.46), we obtain the following table of their values:

$$\begin{array}{lllllllll} (\boldsymbol{i}_b\boldsymbol{k}_s) &=& \sin\Phi\sin\Theta\,, & (\boldsymbol{i}_b\boldsymbol{k}_b) &=& 0\,, & (\boldsymbol{i}_b\boldsymbol{e}_{bs}) &=& \cos\Phi\,, \\ (\boldsymbol{j}_b\boldsymbol{k}_s) &=& \cos\Phi\sin\Theta\,, & (\boldsymbol{j}_b\boldsymbol{k}_b) &=& 0\,, & (\boldsymbol{j}_b\boldsymbol{e}_{bs}) &=& -\sin\Phi\,, \\ (\boldsymbol{k}_b\boldsymbol{k}_s) &=& \cos\Theta\,, & (\boldsymbol{k}_b\boldsymbol{k}_b) &=& 1\,, & (\boldsymbol{k}_b\boldsymbol{e}_{bs}) &=& 0\,. \end{array} \tag{1.47}$$

Taking into account formulae (1.47) , we can present relationship (1.46) in a more compact form:

$$\begin{aligned}(\boldsymbol{G}\,\mathrm{d}\boldsymbol{\Omega})_E &= (G_\xi \sin\Phi\sin\Theta + G_\eta\cos\Phi\sin\Theta + G_\zeta\cos\Theta)\,\mathrm{d}\Psi \\ &+ G_\zeta\,\mathrm{d}\Phi + (G_\xi\cos\Phi - G_\eta\sin\Phi)\,\mathrm{d}\Theta\,, \end{aligned} \tag{1.48}$$

or, taking into account (1.26),

$$(\boldsymbol{G}\,\mathrm{d}\boldsymbol{\Omega})_E = p_\Psi\,\mathrm{d}\Psi + p_\Theta\,\mathrm{d}\Theta + p_\Phi\,\mathrm{d}\Phi\,. \tag{1.49}$$

Let us prove now that the similar representation

$$(\boldsymbol{G}\,\mathrm{d}\boldsymbol{\Omega})_A = L\,\mathrm{d}l + G\,\mathrm{d}g + H\,\mathrm{d}h \tag{1.50}$$

holds.

In fact, multiplying vectors (1.43) and (1.45), we find:

$$(\boldsymbol{G}\,\mathrm{d}\boldsymbol{\Omega})_A = G[(\boldsymbol{k}_G\boldsymbol{k}_s)\,\mathrm{d}h + (\boldsymbol{k}_G\boldsymbol{i}_G)\,\mathrm{d}\rho + (\boldsymbol{k}_G\boldsymbol{k}_G)\,\mathrm{d}g + (\boldsymbol{k}_G\boldsymbol{k}_b)\,\mathrm{d}l + (\boldsymbol{k}_G\boldsymbol{e}_{bG})\,\mathrm{d}\theta]\,. \tag{1.51}$$

We obtain the following values of the products of the unit vectors (1.51), using the definition of the Andoyer variables:

$$\begin{array}{lllllllll} (\boldsymbol{k}_G\boldsymbol{k}_s) &=& \cos\rho\,, & (\boldsymbol{k}_G\boldsymbol{i}_G) &=& 0\,, & (\boldsymbol{k}_G\boldsymbol{k}_G) &=& 1\,, \\ (\boldsymbol{k}_G\boldsymbol{k}_b) &=& \cos\theta\,, & (\boldsymbol{k}_G\boldsymbol{e}_{bG}) &=& 0 & & & \end{array} \tag{1.52}$$

and, consequently,

$$(\boldsymbol{G}\,\mathrm{d}\boldsymbol{\Omega})_A = G(\cos\rho\,\mathrm{d}h + \,\mathrm{d}g + \cos\theta\,\mathrm{d}l) = L\,\mathrm{d}l + G\,\mathrm{d}g + H\,\mathrm{d}h\,. \tag{1.53}$$

Obviously, the scalar products (1.48) and (1.53) are equal and, consequently, the transformation from variables (1.30) to variables (1.40) is canonical.

For the presentation of the Hamiltoman of the problem in the new variables (1.40), it is sufficient to use in (1.33) the following expressions of the projections of the angular momentum vector with respect to the body axes:

$$\begin{aligned} G_\xi &= G\boldsymbol{k}_b = G\boldsymbol{k}_G\boldsymbol{i}_b = G\sin\theta\sin l = \sqrt{G^2-L^2}\sin l\,, \\ G_\eta &= G\boldsymbol{j}_b = G\boldsymbol{k}_G\boldsymbol{j}_b = G\sin\theta\cos l = \sqrt{G^2-L^2}\cos l\,, \\ G_\zeta &= G\boldsymbol{k}_b = G\boldsymbol{k}_G\boldsymbol{k}_b = G\cos\theta = L\,. \end{aligned} \tag{1.54}$$

Now we obtain the canonical equations of the rotational motion of the weakly deformable

$$\begin{aligned} \frac{\mathrm{d}l}{\mathrm{d}t} &= \frac{\partial K}{\partial L}\,, \quad \frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{\partial K}{\partial l}\,, \\ \frac{\mathrm{d}g}{\mathrm{d}t} &= \frac{\partial K}{\partial G}\,, \quad \frac{\mathrm{d}G}{\mathrm{d}t} = -\frac{\partial K}{\partial g}\,, \\ \frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{\partial K}{\partial H}\,, \quad \frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial K}{\partial h}\,. \end{aligned} \tag{1.55}$$

$$\begin{aligned} K &= \frac{1}{2}G^2\{(a\sin^2 l + b\cos^2 l - f\sin 2l)\sin^2\theta + c\cos^2\theta - \sin 2\theta(e\sin l + d\cos l)\} \\ &- G[(\Omega_\xi\sin l + \Omega_\eta\cos l)\sin\Theta + \Omega_\zeta\cos\theta] - U(L,G,H,l,g,h,t)\,. \end{aligned} \tag{1.56}$$

The force function U in (1.56) must be presented as a function of the canonical variables (1.40) and of time.

The last problem is solved with the help of the known representations of direction cosines a_{ij} (1.32) of the body axes $C\xi\eta\zeta$ with respect to the main reference system $Cxyz$ (Barkin, 1992):

$$\begin{aligned} a_{11} &= \frac{1}{4}(1+\cos\theta)(1+\cos\rho)\cos(l+g+h) \\ &+ \frac{1}{4}(1+\cos\theta)(1-\cos\rho)\cos(l+g-h) \\ &+ \frac{1}{4}(1-\cos\theta)(1-\cos\rho)\cos(l-g+h) \\ &+ \frac{1}{4}(1-\cos\theta)(1+\cos\rho)\cos(l-g-h) \\ &+ \frac{1}{2}\sin\theta\sin\rho\cos(l-h) - \frac{1}{2}\sin\theta\sin\rho\cos(l+h)\,, \\ a_{21} &= \frac{1}{4}(1+\cos\theta)(1+\cos\rho)\sin(l+g+h) \\ &- \frac{1}{4}(1+\cos\theta)(1-\cos\rho)\sin(l+g-h) \\ &+ \frac{1}{4}(1-\cos\theta)(1-\cos\rho)\sin(l-g+h) \\ &- \frac{1}{4}(1-\cos\theta)(1+\cos\rho)\sin(l-g-h) \end{aligned}$$

$$
\begin{aligned}
&- && \frac{1}{2}\sin\theta\sin\rho\sin(l+h) - \frac{1}{2}\sin\theta\sin\rho\sin(l-h)\,,\\
a_{31} &= && \frac{1}{2}\sin\rho(1+\cos\theta)\sin(l+g) - \frac{1}{2}\sin\rho(1-\cos\theta)\sin(l-g)\\
&+ && \cos\rho\sin\theta\sin l\,,\\
a_{12} &= && -\frac{1}{4}(1+\cos\theta)(1+\cos\rho)\sin(l+g+h)\\
&+ && \frac{1}{4}(1+\cos\theta)(1-\cos\rho)\sin(l+g-h)\\
&- && \frac{1}{4}(1-\cos\theta)(1-\cos\rho)\sin(l-g+h)\\
&+ && \frac{1}{4}(1-\cos\theta)(1+\cos\rho)\sin(l-g-h)\\
&+ && \frac{1}{2}\sin\theta\sin\rho\sin(l+h) - \frac{1}{2}\sin\theta\sin\rho\sin(l-h)\,,\\
a_{22} &= && \frac{1}{4}(1+\cos\theta)(1+\cos\rho)\cos(l+g+h)\\
&- && \frac{1}{4}(1+\cos\theta)(1-\cos\rho)\cos(l+g-h)\\
&+ && \frac{1}{4}(1-\cos\theta)(1-\cos\rho)\cos(l-g+h)\\
&- && \frac{1}{4}(1-\cos\theta)(1+\cos\rho)\cos(l-g-h)\\
&- && \frac{1}{2}\sin\theta\sin\rho\cos(l+h) - \frac{1}{2}\sin\theta\sin\rho\cos(l-h)\,,\\
a_{32} &= && \frac{1}{2}\sin\rho(1+\cos\theta)\cos(l+g) - \frac{1}{2}\sin\rho(1-\cos\theta)\cos(l-g)\\
&+ && \sin\theta\cos\rho\cos l\,,\\
a_{13} &= && \frac{1}{2}\sin\theta(1+\cos\rho)\sin(g+h) + \frac{1}{2}\sin\theta(1-\cos\rho)\sin(g-h)\\
&+ && \sin\rho\cos\theta\sin h\,,\\
a_{23} &= && -\frac{1}{2}\sin\theta(1+\cos\rho)\cos(g+h) + \frac{1}{2}\sin\theta(1-\cos\rho)\cos(g-h)\\
&- && \sin\rho\cos\theta\cos h\,,\\
a_{33} &= && -\sin\rho\sin\theta\cos g + \cos\rho\cos\theta\,.
\end{aligned}
\tag{1.57}
$$

In formulae (1.56), (1.57):

$$
\sin\theta = \frac{\sqrt{G^2-L^2}}{G}\,,\quad \cos\theta = \frac{L}{G}\,,\quad \sin\rho = \frac{\sqrt{G^2-H^2}}{G}\,,\quad \cos\rho = \frac{H}{G}\,.
$$

1.5 Canonical Equations in the Angle-action Variables

For a wide class of weakly deformable celestial bodies executing rotation close to the unperturbed rotational motion of the rigid body in accordance with the Euler–

Poinsot solution, the corresponding equations of rotation in the angle-action variables can be adopted (Barkin, 1992; 1996a).

Let the values considered (1.11)-(1.14) admit the following representations for the body:

$$A = A_0 + \mu A_1 + \mu^2 A_2 + \dots, \quad B = B_0 + \mu B_1 + \mu^2 B_2 + \dots,$$
$$C = C_0 + \mu C_1 + \mu^2 C_2 + \dots,$$
$$F = \mu F_1 + \mu^2 F_2 + \dots, \quad E = \mu E_1 + \mu^2 E_2 + \dots, \quad D = \mu D_1 + \mu^2 D_2 + \dots,$$
$$P = \mu P_1 + \mu^2 P_2 + \dots, \quad Q = \mu Q_1 + \mu^2 Q_2 + \dots, \quad R = \mu R_1 + \mu^2 R_2 + \dots, \tag{1.58}$$

where μ is some small parameter; A_0, B_0, C_0 are constant unperturbed values of the axial moments of inertia, and the terms of the second and higher orders with respect to μ include the effects of the change of the dynamic structure of the body and are known functions of time.

Similar to (1.58), representations are obtained for values (1.32) and (1.34):

$$a = a_0 + \mu a_1 + \mu^2 a_2 + \dots, \quad b = b_0 + \mu b_1 + \mu^2 b_2 + \dots,$$
$$c = c_0 + \mu c_1 + \mu^2 c_2 + \dots,$$
$$f = \mu f_1 + \mu^2 f_2 + \dots, \quad e = \mu e_1 + \mu^2 e_2 + \dots, \quad d = \mu d_1 + \mu^2 d_2 + \dots,$$
$$\Omega_\xi = \mu \Omega_\xi^{(1)} + \mu^2 \Omega_\xi^{(2)} + \dots, \quad \Omega_\eta = \mu \Omega_\eta^{(1)} + \mu^2 \Omega_\eta^{(2)} + \dots, \quad \Omega_\zeta = \mu \Omega_\zeta^{(1)} + \mu^2 \Omega_\zeta^{(2)} + \dots, \tag{1.59}$$

where the coefficients for different powers of the parameter μ are defined by the formulae:

$$\begin{aligned}
a_0 &= \frac{1}{A_0}, & b_0 &= \frac{1}{B_0}, & c_0 &= \frac{1}{C_0}, \\
a_1 &= -\frac{A_1}{A_0^2}, & b_1 &= -\frac{B_1}{B_0^2}, & c_1 &= -\frac{C_1}{C_0^2}, \\
f_1 &= \frac{-F_1}{A_0 B_0}, & e_1 &= \frac{-E_1}{C_0 A_0}, & d_1 &= \frac{-D_1}{B_0 C_0}, \\
\Omega_\xi^{(1)} &= \frac{P_1}{A_0}, & \Omega_\eta^{(1)} &= \frac{Q_1}{B_0}, & \Omega_\zeta^{(1)} &= \frac{R_1}{C_0},
\end{aligned}$$

$$\begin{aligned}
a_2 &= \frac{-A_2}{A_0^2} + \frac{1}{A_0^2}\left(\frac{A_1^2}{A_0} + \frac{E_1^2}{C_0} + \frac{F_1^2}{B_0}\right), \\
b_2 &= \frac{-B_2}{B_0^2} + \frac{1}{B_0^2}\left(\frac{B_1^2}{B_0} + \frac{D_1^2}{C_0} + \frac{F_1^2}{A_0}\right), \\
c_2 &= \frac{-C_2}{C_0^2} + \frac{1}{C_0^2}\left(\frac{C_1^2}{C_0} + \frac{D_1^2}{B_0} + \frac{E_1^2}{A_0}\right), \\
f_2 &= \frac{F_2}{A_0 B_0} - \frac{F_1}{A_0 B_0}\left(\frac{A_1}{A_0} + \frac{B_1}{B_0}\right) + \frac{D_1 E_1}{A_0 B_0 C_0},
\end{aligned}$$

$$
\begin{aligned}
e_2 &= \frac{E_2}{C_0A_0} - \frac{E_1}{C_0A_0}\left(\frac{C_1}{C_0} + \frac{A_1}{A_0}\right) + \frac{F_1D_1}{A_0B_0C_0}, \\
d_2 &= \frac{D_2}{B_0C_0} - \frac{D_1}{B_0C_0}\left(\frac{B_1}{B_0} + \frac{C_1}{C_0}\right) + \frac{E_1F_1}{A_0B_0C_0}, \\
\Omega_\xi^{(2)} &= \frac{P_2}{A_0} - \frac{1}{A_0}\left(\frac{E_1R_1}{C_0} + \frac{F_1Q_1}{B_0} + \frac{A_1P_1}{A_0}\right), \\
\Omega_\eta^{(2)} &= \frac{Q_2}{B_0} - \frac{1}{B_0}\left(\frac{E_1P_1}{A_0} + \frac{B_1Q_1}{B_0} + \frac{D_1R_1}{C_0}\right), \\
\Omega_\zeta^{(2)} &= \frac{R_2}{C_0} - \frac{1}{C_0}\left(\frac{E_1P_1}{A_0} + \frac{D_1Q_1}{B_0} + \frac{C_1R_1}{C_0}\right). \qquad (1.60)
\end{aligned}
$$

The Hamiltonian of the problem of the rotation of a weakly deformable body can be presented in the standard form :

$$K = K_0 + K_1\,, \qquad (1.61)$$

where

$$K_0 = \frac{1}{2}\left(\frac{\sin^2 l}{A_0} + \frac{\cos^2 l}{B_0}\right)(G^2 - L^2) + \frac{L^2}{2C_0} \qquad (1.62)$$

is a Hamiltonian of the unperturbed Eulerian rotation of the undeformable body with constant principal moments of inertia A_0, B_0 and C_0, and K_1 is a perturbing function, including the small terms of the quadratic part of the kinetic energy of the rotational motion of the body and a force function of the problem, which is comparatively small with respect to the main components of the Hamiltonian K_0:

$$
\begin{aligned}
K_1 &= \sum_{\sigma=1} \mu^\sigma \Big\{ \frac{1}{2}G^2\big[\sin^2\theta(a_\sigma \sin^2 l + b_\sigma \cos^2 l - f_\sigma \sin 2l) \\
&+ c_\sigma \cos^2\theta - \sin 2\theta(d_\sigma \cos l + e_\sigma \sin l)\big] \\
&- G\big[(\Omega_\xi^\sigma \sin l + \Omega_\eta^\sigma \cos l)\sin\theta + \Omega_\zeta^\sigma \cos\theta\big]\Big\} - U(\theta, \rho, l, g, h, t)\,. \qquad (1.63)
\end{aligned}
$$

Angle-action variables of the Euler–Poinsot problem were studied by many authors (Sadov, 1970; Kinoshita, 1972; Barkin, 1992; 1998b, and others). Let us denote these variables as I_i, φ_i ($i = 1, 2, 3$) and introduce them using the formulae of the canonical transformation from Andoyer variables (Barkin, 1992; 1998b):

$$
\begin{aligned}
L &= G\frac{\pi k}{K\sqrt{k^2+\lambda^2}} \sum_{m=0}^{\infty} \frac{\cos 2m\varphi}{\cosh 2md}(1+\delta_{m0})^{-1}, \\
G &= I_2\,, \\
H &= I_3\,, \\
l &= \varphi_1 + \sum_{m=1}^{\infty} \frac{\cosh m\sigma}{m\cosh(2md)} \sin 2m\varphi_1\,, \\
g &= \varphi_2 + \sum_{m=1}^{\infty} \frac{\sinh m\sigma}{m\sinh(2md)} \sin 2m\varphi_1\,,
\end{aligned}
$$

$$h = \varphi_3 , \tag{1.64}$$

where $\delta_{00} = 1$, $\delta_{m=0} = 0$ $(m \geq 1)$,

$$\begin{aligned} d &= \frac{\pi K(\lambda')}{2K(\lambda)} , \\ \sigma &= \frac{\pi}{2K(\lambda)} F(\arctan \frac{k}{\lambda}, \lambda') , \\ \lambda' &= \sqrt{1 - \lambda^2} . \end{aligned}$$

Here $K(\lambda)$ and F are complete and incomplete elliptic integrals of the first kind. The modulus of these integrals λ and the parameter k are defined by the initial conditions of the problem:

$$\lambda^2 = k^2 \frac{A_0^2}{C_0^2} \frac{p_0^2}{r_0^2} , \quad k^2 = \frac{C_0(A_0 - B_0)}{A_0(B_0 - C_0)} , \tag{1.65}$$

where p_0, r_0 $(q_0 = 0)$ are initial values of the components of the angular velocity.

The equations of the rotational motion in the angle-action variables can be described in the following way:

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = \frac{\partial H}{\partial I_i} , \quad \frac{\mathrm{d}I_i}{\mathrm{d}t} = -\frac{\partial H}{\partial \varphi_i} \quad (i = 1, 2, 3) , \tag{1.66}$$

$$H = H_0(I_1, I_2) + H_1(I_1, I_2, I_3, \varphi_1, \varphi_2, \varphi_3, t) . \tag{1.67}$$

Here H_0 is the Hamiltonian of the unperturbed motion (Sadov, 1970; Barkin, 1992)

$$H_0 = \frac{I_2^2}{2A} \left[1 + \frac{A - C}{C} \frac{k^2}{(\lambda^2 + k^2)} \right] \tag{1.68}$$

and H_1 is the perturbing function (1.63), which must be presented as a function of the angle-action variables. This is done with the help of formulae (1.64) and a wide set of formulae of the unperturbed Eulerian motion obtained in the course of the Saragossa lectures (Barkin, 1992; 1998b). The Fourier series for the direction cosines a_{ij}, for their mutual products and squares, etc. (Barkin, 1992) have a very important significance for applications (Barkin, 1996a; 1998b).

Equations (1.61), (1.62) and (1.66)–(1.68) admit efficient application of different methods of perturbation theory to the study of rotational motion of deformable celestial bodies.

1.6 Equations of Motion in the Andoyer Variables (Case of Non-potential Forces)

The canonical equations of rotational motion, obtained in Sections 1.4, 1.5, hold only in the case of the existence of the force function of the problem. The generalization of these canonical equations to the case of generalized potential forces can be given in accordance with similar results, obtained for rigid body rotation (Barkin, 1999b).

In this connection, we obtain here other forms of the equations, which can be used for studies of rotational motion of celestial bodies in arbitrary force fields.

We will use the theorem of angular momentum conservation of a mechanical system in its motion about the center of mass:

$$\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}t} = \boldsymbol{L}. \tag{1.69}$$

The angular momentum $\boldsymbol{G}$ is defined by formulae (1.8), (1.9). $\boldsymbol{L}$ is the principal moment of all the forces acting on the body, with respect to its centre of mass.

In the general case, some volume force $\boldsymbol{f}(\boldsymbol{r}', t)$ acts on an elementary volume of the body $\mathrm{d}\tau'$ and some surface force $\boldsymbol{P}_n(\boldsymbol{r}', t)$ acts on an elementary element of the surface $\mathrm{d}\sigma'$. The principal moment of these forces $\boldsymbol{L}$ is defined by formula

$$\boldsymbol{L} = \int\limits_{\tau'} \boldsymbol{r}' \times \boldsymbol{f}(\boldsymbol{r}', t)\,\mathrm{d}\tau' + \oint\!\!\!\oint\limits_{\sigma'} \boldsymbol{r}' \times \boldsymbol{P}_n(\boldsymbol{r}', t)\,\mathrm{d}\sigma', \tag{1.70}$$

where the integrals are spread over the full volume of the body and over its surface in a deformable state.

Here we don't consider the procedure of calculation and description of the components of moment (1.70), but we point out that this vector is determined with respect to the centre of mass of the body and in the general case is presented as a function of the Euler angles, of the components of angular velocity (1.1), and of time

$$\boldsymbol{L} = \boldsymbol{L}(\Psi, \Theta, \Phi, p, q, r, t). \tag{1.71}$$

To introduce the equations of motion in the Andoyer variables, we will use different representations of the vectors $\boldsymbol{G}$, $\boldsymbol{L}$ in the basis of the reference systems $Cxyz$, $C\xi\eta\zeta$, $CG_1G_2G_3$, and also we introduce into consideration a few new geometric and kinematic characteristics of the relative motion of the above-mentioned reference systems.

The vector of the angular velocity $\boldsymbol{\omega}$ of rotation of the reference system $C\xi\eta\zeta$ with respect to the reference system $Cxyz$ admits the following representations:

$$\boldsymbol{\omega} = p\boldsymbol{i}_b + q\boldsymbol{j}_b + r\boldsymbol{k}_b, \tag{1.72}$$

$$\boldsymbol{\omega} = p_s\boldsymbol{i}_s + q_s\boldsymbol{j}_s + r_s\boldsymbol{k}_s, \tag{1.73}$$

where $\boldsymbol{i}_b$, $\boldsymbol{j}_b$, $\boldsymbol{k}_b$ and $\boldsymbol{i}_s$, $\boldsymbol{j}_s$, $\boldsymbol{k}_s$ are systems of unit vectors of the corresponding reference systems; and p, q, r and p_s, q_s, r_s are projections of the vector $\boldsymbol{\omega}$ on the axes of these reference systems. In Euler's angles, these projections are defined by the formulae (Suslov, 1946):

$$\begin{aligned} p &= \sin\Phi\sin\Theta\dot{\Psi} + \cos\Phi\dot{\Theta}, \\ q &= \cos\Phi\sin\Theta\dot{\Psi} - \sin\Phi\dot{\Theta}, \\ r &= \cos\Theta\dot{\Psi} + \dot{\Phi} \end{aligned} \tag{1.74}$$

and

$$\begin{aligned} p_s &= \dot{\Theta}\cos\Psi + \dot{\Phi}\sin\Theta\sin\Psi\,, \\ q_s &= \dot{\Theta}\sin\Psi - \dot{\Phi}\sin\Theta\cos\Psi\,, \\ r_s &= \dot{\Psi} + \dot{\Phi}\cos\Theta\,. \end{aligned} \tag{1.75}$$

Using formulae (1.72)–(1.75), we obtain similar kinematic formulae for the relative motion of the reference systems $CG_1G_2G_3$ and $Cxyz$, and also of the systems $C\xi\eta\zeta$ and $CG_1G_2G_3$.

Let ω_{Gs} be the angular velocity of rotation of the intermediate reference system $CG_1G_2G_3$ with respect to the main reference system. This vector $Cxyz$ can be presented in the two bases:

$$\omega_{Gs} = p_G\boldsymbol{i}_G + q_G\boldsymbol{j}_G + r_G\boldsymbol{k}_G\,, \tag{1.76}$$

$$\omega_{Gs} = p_{Gs}\boldsymbol{i}_s + q_{Gs}\boldsymbol{j}_s + r_{Gs}\boldsymbol{k}_s\,. \tag{1.77}$$

It is easy to obtain the values of the projections of this vector in (1.76), (1.77) on the basis of formulae (1.72)–(1.75), substituting $\Psi = h$, $\Theta = \rho$, $\Phi = 0$:

$$p_G = \dot{\rho}\,, \quad q_G = \sin\rho\dot{h}\,, \quad r_G = \cos\rho\dot{h} \tag{1.78}$$

and

$$p_{Gs} = \dot{\rho}\cos h\,, \quad q_{Gs} = \dot{\rho}\sin h\,, \quad r_{Gs} = \dot{h}\,. \tag{1.79}$$

Here p_G, q_G , r_G are projections of the vector ω_{Gs} on the axes of the intermediate reference system $CG_1G_2G_3$,and p_{Gs}, q_{Gs}, r_{Gs} are projections of the same vector on the axes of the reference system Cx, Cy and Cz.

In a similar way, we obtain on the basis (1.37) the values of the direction cosines of the intermediate axes $CG_1G_2G_3$ in the main reference system:

$$\begin{aligned} g_{11} &= \cos h, & g_{21} &= \sin h, & g_{31} &= 0, \\ g_{12} &= -\sin h\cos\rho, & g_{22} &= \cos h\cos\rho, & g_{32} &= \sin\rho, \\ g_{13} &= \sin h\sin\rho, & g_{23} &= -\cos h\sin\rho, & g_{33} &= \cos\rho. \end{aligned} \tag{1.80}$$

Substituting now $\Psi = g$, $\Theta = 0$, $\Phi = l$, on the basis of formulae (1.72)–(1.75), we obtain corresponding representations for the vector ω_{bG} of the angular velocity of the body reference system $C\xi\eta\zeta$ with respect to intermediate reference system $CG_1G_2G_3$:

$$\begin{aligned} \omega_{bG} &= p_b\boldsymbol{i}_b + q_b\boldsymbol{j}_b + r_b\boldsymbol{k}_b\,, \\ \omega_{bG} &= p_{bG}\boldsymbol{i}_G + q_{bG}\boldsymbol{j}_G + r_{bG}\boldsymbol{k}_G\,. \end{aligned} \tag{1.81}$$

Here p_b, q_b, r_b are projections of the vector ω_{bG} on the body axes, and p_{bG}, q_{bG}, r_{bG} are projections of the vector ω_{bG} on the axes of the intermediate reference system:

$$\begin{aligned} p_b &= \sin l\sin\theta\dot{g} + \cos l\dot{\theta}\,, \\ q_b &= \cos l\sin\theta\dot{g} - \sin l\dot{\theta}\,, \\ r_b &= \cos\theta\dot{g} + \dot{l}\,, \end{aligned} \tag{1.82}$$

$$
\begin{aligned}
p_{bG} &= \dot{\theta}\cos g + \dot{l}\sin\theta\sin g\,, \\
q_{bG} &= \dot{\theta}\sin g - \dot{l}\sin\theta\cos g\,, \\
r_{bG} &= \dot{g} + \dot{l}\cos\theta\,.
\end{aligned} \tag{1.83}
$$

Substituting now $\Psi = g$, $\Theta = 0$, $\Phi = l$ in (1.37) we obtain representations for the direction cosines b_{ij} of the body reference system $C\xi\eta\zeta$ with respect to the intermediate reference system $CG_1G_2G_3$:

$$
\begin{aligned}
b_{11} &= \cos g\cos l - \sin g\cos\theta\sin l\,, \\
b_{21} &= \sin g\cos l + \cos g\cos\theta\sin l\,, \\
b_{31} &= \sin\theta\sin l\,, \\
b_{12} &= -\cos g\sin l - \sin g\cos\theta\cos l\,, \\
b_{22} &= -\sin g\sin l - \cos g\cos\theta\cos l\,, \\
b_{32} &= \sin\theta\cos l\,, \\
b_{13} &= \sin g\sin\theta\,, \\
b_{23} &= -\cos g\sin\theta\,, \\
b_{33} &= \cos\theta\,.
\end{aligned} \tag{1.84}
$$

Using the vector equations (1.69)–(1.71) and the kinematic formulae (1.78)–(1.84), we obtain the differential equations of the rotational motion of the body in the Andoyer variable (Andoyer, 1923):

$$
G,\ \theta,\ \rho,\ l,\ g,\ h. \tag{1.85}
$$

To obtain the first three equations (for the variables G, ρ, h) we use a representation of the absolute derivative $\mathrm{d}\boldsymbol{G}/\,\mathrm{d}t$ in terms of the local derivative, which is calculated with respect to the intermediate reference system.

The following vector equation

$$
\frac{\mathrm{d}^G\boldsymbol{G}}{\mathrm{d}t} + \boldsymbol{\omega}_{Gs}\times\boldsymbol{G} = \boldsymbol{L} \tag{1.86}
$$

and equations in projections on the coordinate axes CG_1, CG_2 and CG_3:

$$
\begin{aligned}
\frac{\mathrm{d}G_1}{\mathrm{d}t} + q_G G_3 - r_G G_2 &= L_{G_1}\,, \\
\frac{\mathrm{d}G_2}{\mathrm{d}t} + r_G G_1 - p_G G_3 &= L_{G_2}\,, \\
\frac{\mathrm{d}G_3}{\mathrm{d}t} + p_G G_2 - r_G G_1 &= L_{G_3}
\end{aligned} \tag{1.87}
$$

follow from (1.69).

Here G_i are projections of the vector $\boldsymbol{G}$ on the axes of the intermediate reference system. Obviously, $G_1 = G_2 = 0$, $G_3 = G$. Taking into account the values p_G, q_G,

r_G (1.78), the equations (1.87) are represented in the following form:

$$\begin{aligned} G\dot{h}\sin\rho &= L_{G_1}\,, \\ -\dot{\rho}G &= L_{G_2}\,, \\ \dot{G} &= L_{G_3}\,. \end{aligned} \tag{1.88}$$

Projections of the principal-moment vector $\boldsymbol{L}$ in (1.88) can be expressed in terms of projections of this vector on the body axes $C\xi\eta\zeta$ or on the axes of the main reference system $Cxyz$:

$$\begin{aligned} L_{G_i} &= L_\xi b_{i1} + L_\eta b_{i2} + L_\zeta b_{i3}\,, \\ L_{G_i} &= L_x g_{i1} + L_y g_{i2} + L_z g_{i3}\,, \quad (i = 1,2,3) \end{aligned} \tag{1.89}$$

or, in more detail,

$$\begin{aligned} L_{G_1} &= L_x \cos h + L_y \sin h\,, \\ L_{G_2} &= (-L_x \sin h + L_y \cos h)\cos\rho + L_z \sin\rho\,, \\ L_{G_3} &= (L_x \sin h - L_y \cos h)\sin\rho + L_z \cos\rho\,. \end{aligned} \tag{1.90}$$

In their form, equations (1.88)–(1.90) coincide with analogous equations of rotational motion of the rigid body (Chernousko, 1963; Beletskij, 1965).

To obtain the next three equations (for variables θ and l, g), we use the vector equation

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{Gs} + \boldsymbol{\omega}_{bG} \tag{1.91}$$

which expresses the well-known theorem about composite rotational motion of the rigid body (Suslov, 1946).

Equation (1.91) admits the presentation

$$\boldsymbol{\omega}_{bG} = \boldsymbol{\omega} - \boldsymbol{\omega}_{Gs}$$

and can be written in the projections on the axes of the intermediate reference system, CG_1, CG_2 and CG_3. Taking into account formulae (1.81)–(1.84) for components of vectors ω_{bG} and ω_{Gs}, we will have:

$$\begin{aligned} \dot{\theta}\cos g + \dot{l}\sin\theta\sin g &= \omega_{G_1} - \dot{\rho}\,, \\ \dot{\theta}\sin g - \dot{l}\sin\theta\cos g &= \omega_{G_2} - \sin\rho\dot{h}\,, \\ \dot{g} + \dot{l}\cos\theta &= \omega_{G_3} - \cos\rho\dot{h}\,. \end{aligned} \tag{1.92}$$

Solving equations (1.92) with respect to $\dot{\theta}$, $\dot{l}$, $\dot{h}$ we obtain the following set of equations:

$$\begin{aligned} \dot{\theta} &= \omega_{G_1}\cos g + \omega_{G_2}\sin g - [\dot{\rho}\cos g + \sin\rho\dot{h}\sin g]\,, \\ \dot{l} &= \sec\theta(\omega_{G_1}\sin g - \omega_{G_2}\cos g) - \sec\theta[\dot{\rho}\sin g - \sin\rho\dot{h}\cos g]\,, \\ \dot{g} &= \omega_{G_3} - \cot\theta(\omega_{G_1}\sin g - \omega_{G_2}\cos g) - \cos\rho\dot{h} \\ &+ \cot\theta[\dot{\rho}\sin g - \sin\rho\dot{h}\cos g]\,. \end{aligned} \tag{1.93}$$

In (1.92), (1.93), ω_{G_i} are projections of the angular velocity vector in the intermediate reference system. They can be expressed in terms of projections of the angular velocity p, q, r on the body axes:

$$\omega_{G_i} = pb_{i1} + qb_{i2} + rb_{i3}, \quad (i = 1, 2, 3), \tag{1.94}$$

where the direction cosines b_{ij} are defined by formulae (1.84), and p, q, r are defined by formulae (1.31), (1.32), (1.54).

The free terms in the right-hand sides of equations (1.93) can be transformed with the help of the formulae (1.94), (1.84). The derivatives $\dot{\rho}$, $\dot{h}$ can be expressed in terms of the projections of moment $\boldsymbol{L}$ in the intermediate axes with the help of the formulae (1.89). After some algebra, we obtain an intermediate form of the differential equations of rotational motion of a weakly deformable body in the Andoyer variables (1.85):

$$\begin{aligned}
\dot{G} &= L_{G_3}, \\
\dot{\rho} &= -\frac{1}{G} L_{G_2}, \\
\dot{h} &= \frac{1}{G} \sec\rho L_{G_1}, \\
\dot{\theta} &= p\cos l - q\sin l + \frac{1}{G}(L_{G_2}\cos g - L_{G_1}\sin g), \\
\dot{l} &= r - \cot\theta(p\sin l + q\cos l) + \frac{1}{G}(L_{G_1}\cos g + L_{G_2}\sin g)\sec\theta, \\
\dot{g} &= \sec\theta(p\sin l + q\cos l) - \frac{1}{G}\cot\rho L_{G_1} \\
&- \frac{1}{G}\cot\theta(L_{G_1}\cos g + L_{G_2}\sin g),
\end{aligned} \tag{1.95}$$

where the components of the angular velocity p, q, r are defined as functions of the variables θ, l, G and of time by formulae (1.31), (1.32), and in a general case projections of the principal moment of the forces L_{G_1}, L_{G_2}, L_{G_3} are presented as functions of the variables (1.85) and of time.

The right-hand sides of equations (1.95) can be expressed in terms of the projections of the vector $\boldsymbol{L}$ in the body axes $C\xi\eta\zeta$ (1.89). As a result, we come to the following set of equations:

$$\begin{aligned}
\dot{G} &= L_\xi b_{31} + L_\eta b_{32} + L_\zeta b_{33}, \\
\dot{\rho} &= -\frac{1}{G}(L_\xi b_{21} + L_\eta b_{22} + L_\zeta b_{33}), \\
\dot{h} &= \frac{1}{G}\sec\rho(L_\xi b_{11} + L_\eta b_{12} + L_\zeta b_{13}), \\
\dot{\theta} &= p\cos l - q\sin l + \frac{1}{G}[\cos\theta(L_\xi\sin l + L_\eta\cos l) - \sin\theta L_\zeta], \\
\dot{l} &= r - \cot\theta(p\sin l + q\cos l) + \frac{1}{G}\sec\theta(L_\xi\cos l - L_\eta\sin l),
\end{aligned}$$

$$\begin{aligned}\dot{g} &= \sec\theta(p\sin l + q\cos l) - \frac{1}{G}\cot\rho(L_\xi b_{11} + L_\eta b_{12} + L_\zeta b_{13}) \\ &- \frac{1}{G}\cot\theta(L_\xi\cos l - L_\eta\sin l) \end{aligned} \tag{1.96}$$

where

$$L_{\xi,\eta,\zeta} = L_{\xi,\eta,\zeta}(G,\theta,\rho,l,g,h,t).$$

Finally, expressing in (1.95) the projections of the momentum $\boldsymbol{L}$ in terms of its projections in the main reference system (1.90), after some additional algebra we obtain the following equations of rotational motion of the body:

$$\begin{aligned}
\frac{dG}{dt} &= (L_x\sin h + L_y\cos h)\sin\rho + L_z\cos\rho\,, \\
\frac{d\rho}{dt} &= -\frac{1}{G}[-(L_x\sin h - L_y\cos h)]\cos\rho + L_z\sin\rho]\,, \\
\frac{dh}{dt} &= \frac{1}{G}\sec\rho(L_x\cos h + L_y\sin h)\,, \\
\frac{d\theta}{dt} &= p\cos l - q\sin l + \frac{1}{G}[L_x(-\sin h\cos\rho\cos g - \cos h\sin g) \\
&+ L_y(\cos h\cos\rho\cos g - \sin h\sin g) + L_z\sin\rho\cos g]\,, \\
\frac{dl}{dt} &= r - \cot\theta(p\sin l + q\cos l) + \frac{1}{G}\sec\theta[L_x(\cos h\cos g - \sin h\cos\rho\sin g) \\
&+ L_y(\sin h\cos g + \cos h\cos\rho\sin g) + L_z\sin\rho\sin g]\,, \\
\frac{dg}{dt} &= \sec\theta(p\sin l + q\cos l) - \frac{1}{G}\cot\rho(L_x\cos h + L_y\sin h) \\
&- \frac{1}{G}\cot\theta[L_x(\cos h\cos g - \sin h\cos\rho\sin g) \\
&+ L_y(\sin h\cos g + \cos h\cos\rho\sin g) + L_z\sin\rho\sin g]\,,
\end{aligned} \tag{1.97}$$

where

$$L_{x,y,z} = L_{x,y,z}(G,\theta,\rho,l,g,h,t).$$

Now we use formulae (1.31), (1.32) for the components of the angular velocity p, q, r, taking (1.54):

$$G_\xi = G\sin\theta\sin l, \quad G_\eta = G\sin\theta\cos l, \quad G_\zeta = G\cos\theta.$$

Substituting these formulae into the right-hand sides of equations (1.95), after some simple transformations, we obtain another form of the differential equations:

$$\begin{aligned}
\frac{dG}{dt} &= L_{G_3}\,, \\
\frac{d\rho}{dt} &= -\frac{1}{G}L_{G_2}\,, \\
\frac{dh}{dt} &= \frac{1}{G}\sec\rho L_{G_1}\,,
\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= G\sin\theta\left[\frac{1}{2}(a-b)\sin 2l - f\cos 2l\right] + G\cos\theta(d\sin l - e\cos l)\\
&+ \Omega_\eta \sin l - \Omega_\xi \cos l + \frac{1}{G}(L_{G_2}\cos g - L_{G_1}\sin g)\,,\\
\frac{\mathrm{d}l}{\mathrm{d}t} &= G\cos\theta(c - a\sin^2 l - b\cos^2 l + f\sin 2l) + G\sec\theta(e\sin l + d\cos l)\cos 2\theta\\
&- \Omega_\zeta + \cot\theta(\Omega_\xi \sin l + \Omega_\eta \cos l) + \frac{1}{G}\sec\theta(L_{G_1}\cos g + L_{G_2}\sin g)\,,\\
\frac{\mathrm{d}g}{\mathrm{d}t} &= G(a\sin^2 l + b\cos^2 l - f\sin 2l) - G\cot\theta(e\sin l + d\cos l)\\
&- \sec\theta(\Omega_\xi \sin l + \Omega_\eta \cos l) - \frac{1}{G}\cot\rho L_{G_1}\\
&- \frac{1}{G}\cot\theta(L_{G_1}\cos g + L_{G_2}\sin g)\,,
\end{aligned}
\tag{1.98}
$$

where $a, b, c, \ldots, \Omega_\zeta$ are known functions of time (1.32), (1.23) and projections of the the moment L_{G_1} are known functions of the Andoyer variables and of time.

Expressing L_{G_i} in terms of the projections of the vector $\boldsymbol{L}$ in the reference systems $Cxyz$ and $C\xi\eta\zeta$, we can also obtain two other forms of equations, analogous to the sets of equations (1.96) and (1.97).

In the particular case where the coordinate axes $C\xi\eta\zeta$ are principal and the central axes of inertia of the deformable body, equations (1.98) are reduced to:

$$
\begin{aligned}
\frac{\mathrm{d}G}{\mathrm{d}t} &= L_{G_3}\,,\\
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\frac{1}{G}L_{G_2}\,,\\
\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{1}{G}\sec\rho L_{G_1}\,,\\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= G\sin\theta\left(\frac{1}{A} - \frac{1}{B}\right)\sin l\cos l + \left(\frac{Q}{B}\sin l - \frac{P}{A}\cos l\right)\\
&+ \frac{1}{G}(L_{G_2}\cos g - L_{G_1}\sin g)\,,\\
\frac{\mathrm{d}l}{\mathrm{d}t} &= G\cos\theta\left(\frac{1}{C} - \frac{\sin^2 l}{A} - \frac{\cos^2 l}{B}\right) - \frac{R}{C} + \cot\theta\left(\frac{P}{A}\sin l + \frac{Q}{B}\cos l\right)\\
&+ \frac{1}{G}\sec\theta(L_{G_1}\cos g + L_{G_2}\sin g)\,,\\
\frac{\mathrm{d}g}{\mathrm{d}t} &= G\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right) - \sec\theta\left(\frac{P}{A}\sin l + \frac{Q}{B}\cos l\right)\\
&- \frac{1}{G}\cot\rho L_{G_1} - \frac{1}{G}\cot\theta(L_{G_1}\cos g + L_{G_2}\sin g)\,.
\end{aligned}
\tag{1.99}
$$

If $P = Q = R = 0$ and the moments of inertia A, B, C retain their constant values, the equations coincide with similar equations of rotational motion of a rigid body (Beletskij, 1975; Chernousko, 1963; Barkin, 1975).

- The canonical equations (1.55)–(1.57) were obtained under the condition that the dynamic characteristics of the deformable body (1.23) are definite functions of time. But equations (1.98), (1.99) are more universal and are valid for a wider class of problems. They conserve their form for some deformable body, the dynamic characteristics of which

$$Z = (A, B, C, F, E, D, P, Q, R)$$

are definite functions of the time, of the Euler angles and of the components of the angular velocity,

$$Z = Z(t, \Psi, \Theta, \Phi, p, q, r) \tag{1.100}$$

or, in terms of the Andoyer variables,

$$Z = Z(t, l, g, h, \theta, \rho, G). \tag{1.101}$$

In fact, the procedure of introducing the equations of the rotational motion (1.98) does not depend on the structure of the functions (1.23) and, consequently, these equations remain valid for a wider class of deformable bodies.

The dependences (1.100), (1.101) hold for celestial bodies deformed by their own rotation (Munk and MacDonald, 1960) and can be realized in different kinds of artificial satellite systems. Therefore, for generality we will assume that in the aboveobtained differential equations of the rotational motion (1.97), (1.98) the components of the tensor of inertia and of the relative moment of the body particles in the general case depend on time, on the Euler angles and on the projections of the angular vecity of the body (or on Andoyer variables and time (1.101)).

- For $L_{G_i} = 0$ $(i = 1, 2, 3)$, equations (1.98), (1.99) describe the rotation of the isolated deformable body. The integrable cases of the problem, which we called Liouville's problem, were obtained and studied in the abovementioned author's report of 1979. These results are described in recent papers (Borisov, 1991; Barkin, 1998a).

Let us use the formulae (1.58)–(1.60) and describe the main terms in the right-hand sides of equations (1.98):

$$\begin{aligned}
\frac{\mathrm{d}G}{\mathrm{d}t} &= L_{G_3}\,, \\
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\frac{1}{G}L_{G_2}\,, \\
\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{1}{G}\sec\rho L_{G_1}\,, \\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= G\sin\theta\left\{\frac{1}{2}\left[\left(\frac{1}{A_0}-\frac{1}{B_0}\right)-\frac{1}{2}\left(\frac{A_1}{A_0^2}-\frac{B_1}{B_0^2}\right)\right]\sin 2l+\frac{F_1}{A_0B_0}\cos 2l\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad G\cos\theta\left(\frac{D_1}{B_0C_0}\sin l - \frac{E_1}{C_0A_0}\cos l\right) + \frac{Q_1}{B_0}\sin l - \frac{P_1}{A_0}\cos l \\
&+ \quad \frac{1}{G}(L_{G_2}\cos g - L_{G_1}\sin g) + N_\theta\,, \\
\frac{\mathrm{d}l}{\mathrm{d}t} &= \quad G\cos\theta\left\{\frac{1}{C_0} - \frac{C_1}{C_0} - \left(\frac{1}{A_0} - \frac{A_1}{A_0^2}\right)\sin^2 l\right. \\
&- \quad \left.\left(\frac{1}{B_0} - \frac{B_1}{B_0^2}\right)\cos^2 l - \frac{F_1}{A_0B_0}\sin 2l\right\} \\
&- \quad G\sec\theta\cos 2\theta\left(\frac{E_1}{C_0A_0}\sin l + \frac{D_1}{B_0C_0}\cos l\right) \\
&- \quad \frac{R_1}{C_0} + \cot\theta\left(\frac{P_1}{A_0}\sin l + \frac{Q_1}{B_0}\cos l\right) \\
&+ \quad \frac{1}{G}\sec\theta(L_{G_1}\cos g + L_{G_2}\sin g) + N_l\,, \\
\frac{\mathrm{d}g}{\mathrm{d}t} &= \quad G\left\{\left(\frac{1}{A_0} - \frac{A_1}{A_0^2}\right)\sin^2 l + \left(\frac{1}{B_0} - \frac{B_1}{B_0^2}\right)\cos^2 l - \frac{F_1}{A_0B_0}\sin 2l\right\} \\
&+ \quad G\cot\theta\left(\frac{E_1}{C_0A_0}\sin l + \frac{D_1}{B_0C_0}\cos l\right) - \sec\theta\left(\frac{P_1}{A_0}\sin l + \frac{Q_1}{B_0}\cos l\right) \\
&- \quad \frac{1}{G}\cot\rho L_{G_1} - \frac{1}{G}\cot\theta(L_{G_1}\cos g + L_{G_2}\sin g) + N_g\,. \qquad (1.102)
\end{aligned}
$$

Here N_θ, N_l and N_g are small terms of the second and higher orders with respect to the small parameter. Obviously, functions N_θ, N_l and N_g are defined as differences of the free terms (by $L_{G_i} = 0$) of the corresponding right-hand sides of equations (1.98) and (1.102). The described terms in the right-hand sides of the equations (1.102) play the principal role for analysis of the effects of elasticity of a celestial body on its rotation.

2 DYNAMICS OF A CELESTIAL BODY, DEFORMED BY ITS OWN ROTATION

Here we will concentrate our attention on the study of some dynamic effects in rotation of an elastic celestial body caused by its own rotation. As an illustration and application of these results, we will study the corresponding effects in the Earth's rotation.

2.1 Equations of Motion

Let the body be isolated and no forces act on it. In this case, $L = 0$ and the equations of its rotational motion (1.98) can be written in the following way:

$$
\frac{\mathrm{d}\theta}{\mathrm{d}t} = G\sin\theta\left[\frac{1}{2}(a-b)\sin 2l - f\cos 2l\right] + G\cos\theta(d\sin l - e\cos l)
$$

$$
\begin{aligned}
& + \quad \beta \sin l - \alpha \cos l \,, \\
\frac{dl}{dt} &= G \cos\theta (c - a \sin^2 l - b \cos^2 l + f \sin 2l) \\
& + \quad G \sec\theta \cos 2\theta (e \sin l + d \cos l) - \gamma + \cot\theta (\alpha \sin l + \beta \cos l) \,, \\
\frac{dg}{dt} &= G (a \sin^2 l + b \cos^2 l - f \sin 2l) \\
& + \quad G \cot\theta (e \sin l + d \cos l) - \sec\theta (\alpha \sin l + \beta \cos l)
\end{aligned} \tag{2.1}
$$

(here and below we use the notation $\alpha = \Omega_\xi$, $\beta = \Omega_\eta$, $\gamma = \Omega_\zeta$).

Hereafter we will often use various simplifications and reductions of the exact equations (2.1) on the basis of some additional assumptions.

To concentrate our attention on the effects pointed out in this section we will suppose that the angular moment of the relative motion of the body particles is equal to zero ($P = Q = R = 0$). Assuming now that the body is weakly deformable and using representations of the main characteristics of the body (1.58)–(1.60), instead of equations (2.1), we will consider their simplified version (see (1.102)):

$$
\begin{aligned}
\frac{d\theta}{dt} &= G \sin\theta \left\{ \frac{1}{2} \left[\frac{1}{A_0} - \frac{1}{B_0} + \frac{B_1}{B_0^2} - \frac{A_1}{A_0^2} \right] \sin 2l + \frac{F_1}{A_0 B_0} \cos 2l \right\} \\
& + \quad \frac{G}{C_0} \cos\theta \left(\frac{E_1}{A_0} \cos l - \frac{D_1}{B_0} \sin l \right) \,, \\
\frac{dl}{dt} &= G \cos\theta \left\{ \frac{1}{C_0} - \frac{C_1}{C_0} - \left(\frac{1}{A_0} - \frac{A_1}{A_0^2} \right) \sin^2 l - \left(\frac{1}{B_0} - \frac{B_1}{B_0^2} \right) \cos^2 l \right. \\
& - \quad \left. \frac{F_1}{A_0 B_0} \sin 2l \right\} - \frac{G}{C_0} \sec\theta \cos 2\theta \left(\frac{E_1}{A_0} \sin l + \frac{D_1}{B_0} \cos l \right) \,, \\
\frac{dg}{dt} &= G \left[\left(\frac{1}{A_0} - \frac{A_1}{A_0^2} \right) \sin^2 l + \left(\frac{1}{B_0} - \frac{B_1}{B_0^2} \right) \cos^2 l - \frac{F_1}{A_0 B_0} \sin 2l \right] \\
& + \quad \frac{G}{C_0} \cot\theta \left(\frac{E_1}{A_0} \sin l + \frac{D_1}{B_0} \cos l \right) \,.
\end{aligned} \tag{2.2}
$$

In the case considered, the first three equations of (2.2) are separated from the general set and give three first integrals:

$$
G = G_0, \quad \rho = \rho_0, \quad h = h_0 \tag{2.3}
$$

implying that the angular momentum vector of the rotational motion of deformable celestial bodies is a constant. G_0,ρ_0 and h_0 are initial values of the corresponding Andoyer variables.

We use equations (2.2) for an analysis of the most important effects of elasticity and inelasticity in the Earth's rotation. If the body in its undeformed state is axisymmetric (in this case $A_0 = B_0$), equations (2.2) are simplified:

$$
\frac{d\theta}{dt} = G \sin\theta \left\{ \frac{F_1}{A_0 B_0} \cos 2l \right\} + \frac{G}{C_0 A_0} \cos\theta (E_1 \cos l - D_1 \sin l) \,,
$$

$$\begin{aligned}\frac{\mathrm{d}l}{\mathrm{d}t} &= G\cos\theta\left\{\frac{1}{C_0}-\frac{C_1}{C_0}-\frac{1}{A_0}+\frac{A_1}{A_0^2}-\frac{F_1}{A_0^2}\sin 2l\right\}\\ &- \frac{G}{C_0A_0}\sec\theta\cos 2\theta(E_1\sin l+D_1\cos l)\,,\\ \frac{\mathrm{d}g}{\mathrm{d}t} &= G\left(\frac{1}{A_0}-\frac{A_1}{A_0^2}-\frac{F_1}{A_0^2}\sin 2l\right)+\frac{G}{C_0A_0}\cot\theta(E_1\sin l+D_1\cos l)\,. \quad (2.4)\end{aligned}$$

Other modifications of equations (2.2), (2.4) are possible.

2.2 Variations of the Components of the Earth's Tensor of Inertia due to its Rotation

In accordance with the well-known classical approach, the Earth's rotation generates some additional variations of the components of the tensor of inertia, depending on the orientation of the angular velocity vector (on its components p, q, r). More simply, these variations are defined in the reference system $C\omega_1\omega_2\omega_3$, connected with the angular velocity vector $\boldsymbol{\omega}$. The axis $C\omega_3$ is directed along $\boldsymbol{\omega}$, the axis $C\omega_1$ is directed along the line of intersection of the plane orthogonal to the vector $\boldsymbol{\omega}$ and of the main coordinate plane Cxy, and the axis $C\omega_2$ completes the reference system to the right.

In the reference system $C\omega_1\omega_2\omega_3$, the components of the tensor of inertia of the Earth's equatorial bulges (caused by its rotational deformation) are defined by the formulae (Munk and MacDonald, 1960):

$$A_\omega = B_\omega = -\frac{ka_e^5\omega_0^2}{9f}\,,\quad C_\omega = \frac{2ka_e^5\omega_0^2}{9f}\,,$$

$$F_\omega = 0\,,\quad E_\omega = 0\,,\quad D_\omega = 0\,, \qquad (2.5)$$

where a_e is the equatorial radius of the Earth, ω_0 is the angular velocity of the Earth's rotation, k is Love's number, and f is the gravitational constant.

Here we use the following values of the Earth's parameters (Getino and Ferrandiz, 1991)

$$ma_e^2 = C/0.3307,\quad \omega = 7.292\times10^{-5}\ 1/\mathrm{s},$$

$$fm = 3.986\times10^{-14}\ \mathrm{m}^3/\mathrm{s}^2,\quad a_e = 6.378\times10^6\ \mathrm{m},\quad k=0.29 \qquad (2.6)$$

and the corresponding representations:

$$A_\omega = B_\omega = -\mu C,\quad C_\omega = 2\mu C, \qquad (2.7)$$

where $\mu = 0.00116k = 0.3364\times10^{-3}$.

The components of the inertia tenser (axial and centrifugal moments of inertia) in the $C\xi\eta\zeta$ reference system are defined in terms of the moments (2.5):

$$\begin{aligned}A_r &= \omega_{11}^2A_\omega+\omega_{21}^2B_\omega+\omega_{31}^2C_\omega-2\omega_{11}\omega_{21}F_\omega-2\omega_{11}\omega_{31}E_\omega-2\omega_{21}\omega_{31}D_\omega\,,\\ B_r &= \omega_{12}^2A_\omega+\omega_{22}^2B_\omega+\omega_{32}^2C_\omega-2\omega_{12}\omega_{22}F_\omega-2\omega_{12}\omega_{32}E_\omega-2\omega_{22}\omega_{32}D_\omega\,,\end{aligned}$$

$$
\begin{aligned}
C_r &= \omega_{13}^2 A_\omega + \omega_{23}^2 B_\omega + \omega_{33}^2 C_\omega - 2\omega_{13}\omega_{23} F_\omega - 2\omega_{13}\omega_{33} E_\omega - 2\omega_{23}\omega_{33} D_\omega\,, \\
F_r &= \omega_{11}\omega_{12}(C_\omega - A_\omega) + \omega_{21}\omega_{22}(C_\omega - B_\omega) + (\omega_{11}\omega_{22} + \omega_{21}\omega_{12}) F_\omega \\
&+ (\omega_{11}\omega_{32} + \omega_{31}\omega_{12}) E_\omega + (\omega_{31}\omega_{22} + \omega_{21}\omega_{32}) D_\omega\,, \\
E_r &= \omega_{11}\omega_{13}(C_\omega - A_\omega) + \omega_{21}\omega_{23}(C_\omega - B_\omega) + (\omega_{11}\omega_{23} + \omega_{21}\omega_{13}) F_\omega \\
&+ (\omega_{11}\omega_{33} + \omega_{31}\omega_{13}) E_\omega + (\omega_{31}\omega_{23} + \omega_{21}\omega_{33}) D_\omega\,, \\
D_r &= \omega_{12}\omega_{13}(C_\omega - A_\omega) + \omega_{22}\omega_{23}(C_\omega - B_\omega) + (\omega_{12}\omega_{23} + \omega_{22}\omega_{13}) F_\omega \\
&+ (\omega_{12}\omega_{33} + \omega_{32}\omega_{13}) E_\omega + (\omega_{32}\omega_{23} + \omega_{22}\omega_{33}) D_\omega\,,
\end{aligned}
\tag{2.8}
$$

where ω_{ij}, are the direction cosines of the axes $C\omega_1\omega_2\omega_3$ with respect to the $C\xi\eta\zeta$ reference system.

If g_ω, l_ω and θ_ω, are Euler's angles (precession, own rotation and nutation), determining the orientation of the $C\omega_1\omega_2\omega_3$ reference system in the $C\xi\eta\zeta$ reference system, then:

$$
\begin{aligned}
\omega_{11} &= \cos g_\omega \cos l_\omega - \sin g_\omega \cos\theta_\omega \sin l_\omega\,, \\
\omega_{21} &= \sin g_\omega \cos l_\omega + \cos g_\omega \cos\theta_\omega \sin l_\omega\,, \\
\omega_{31} &= \sin\theta_\omega \sin l_\omega\,, \\
\omega_{12} &= -\cos g_\omega \sin l_\omega - \sin g_\omega \cos\theta_\omega \cos l_\omega\,, \\
\omega_{22} &= -\sin g_\omega \sin l_\omega + \cos g_\omega \cos\theta_\omega \cos l_\omega\,, \\
\omega_{32} &= \sin\theta_\omega \cos l_\omega\,, \\
\omega_{13} &= \sin\theta_\omega \sin g_\omega\,, \\
\omega_{23} &= -\sin\theta_\omega \cos g_\omega\,, \\
\omega_{33} &= \cos\theta_\omega\,.
\end{aligned}
\tag{2.9}
$$

With the help of (2.5), (2.7), formulae (2.8) are significantly reduced:

$$
\begin{aligned}
A_r &= \mu C(-1 + 3\omega_{31}^2)\,, \\
B_r &= \mu C(-1 + 3\omega_{32}^2)\,, \\
C_r &= \mu C(-1 + 3\omega_{32}^2)\,, \\
F_r &= -3\mu C\omega_{31}\omega_{32}\,, \\
E_r &= -3\mu C\omega_{31}\omega_{33}\,, \\
D_r &= -3\mu C\omega_{32}\omega_{33}\,.
\end{aligned}
\tag{2.10}
$$

Here the direction cosines are expressed in terms of the components of the angular velocity ω and are presented in the following form:

$$
\begin{aligned}
\omega_{31} &= \frac{p}{\omega} = \frac{a(G\sin\theta\sin l - P)}{\omega} - \frac{f(G\sin\theta\cos l - Q)}{\omega} - \frac{e(G\cos\theta - R)}{\omega}\,, \\
\omega_{32} &= \frac{q}{\omega} = \frac{-f(G\sin\theta\sin l - P)}{\omega} + \frac{b(G\sin\theta\cos l - Q)}{\omega} - \frac{d(G\cos\theta - R)}{\omega}\,, \\
\omega_{33} &= \frac{r}{\omega} = \frac{-e(G\sin\theta\sin l - P)}{\omega} - \frac{d(G\sin\theta\cos l - Q)}{\omega} + \frac{c(G\cos\theta - R)}{\omega}\,, \\
\omega &= \sqrt{p^2 + q^2 + r^2}\,.
\end{aligned}
\tag{2.11}
$$

In the case of small values of the angle θ, moments F, E, D and relative angular momentum components P, Q, R, we obtain reduced expressions of the moments (2.10), (2.11). In this variant, having an important meaning in the Earth's theory of rotation,

$$\theta_\omega \cong \theta, \quad l_\omega \cong l \tag{2.12}$$

and the formulae (2.10) can be written in the following way:

$$\begin{aligned}
A_r &= \mu C(-1 + 3\sin^2\theta \sin^2 l)\,, \\
B_r &= \mu C(-1 + 3\sin^2\theta \cos^2 l)\,, \\
C_r &= \mu C(-1 + 3\cos^2\theta)\,, \\
F_r &= -3\mu C \sin^2\theta \sin l \cos l\,, \\
E_r &= -3\mu C \sin\theta \cos\theta \sin l \\
D_r &= -3\mu C \sin\theta \cos\theta \cos l\,.
\end{aligned} \tag{2.13}$$

In this paper, we will consider a simplified variant of the problem. In expressions (2.13), we neglect small terms of order $\mu \sin^2\theta$. In this simplified variant of the problem, we have:

$$\begin{gathered}
A_r = -\mu C, \quad B_r = -\mu C, \quad C_r = 2\mu C, \\
F_r = 0, \quad E_r = -3\mu C \sin\theta \cos\theta \sin l, \\
D_r = -3\mu C \sin\theta \cos\theta \cos l.
\end{gathered} \tag{2.14}$$

2.3 Chandler's Unperturbed Motion and its Properties

Substituting formulae for variations of the components of the inertia tensor due to rotational deformation (2.14) into equations (2.2), we retain me main terms in the right-hand sides of these equations. Let us conserve the terms of the first order with respect to μ, but neglect the terms of the third order with respect to small parameters:

$$\mu, \quad \frac{A_0 - B_0}{A_0}, \quad \theta$$

(in reality, we add these small terms to the perturbing terms in the right-hand sides of the equations (1.102)).

As a result of simple transformations, the equations of the rotational motion of the isolated celestial body can be presented in the following form:

$$\begin{aligned}
\frac{d\theta}{dt} &= G \sin\theta \sin l \cos l \left(\frac{1}{A_0} - \frac{1}{B_0}\right)(1 - 2\mu)\,, \\
\frac{dl}{dt} &= G \cos\theta \left[\frac{1}{C_0} - \left(\frac{\sin^2 l}{A_0} + \frac{\cos^2 l}{B_0}\right)\right](1 - 2\mu)\,, \\
\frac{dg}{dt} &= G\left(\frac{\sin^2 l}{A_0} + \frac{\cos^2 l}{B_0}\right)(1 - 2\mu)\,.
\end{aligned} \tag{2.15}$$

Note that here A_0, B_0 and C_0 are principal central moments of inertia of the body in the undeformed state.

Equations (2.15) fully coincide with the equations of the Euler–Poinsot problem described in the Andoyer variables (Beletskij, 1965; ¡Barkin, 1975) for an absolutely rigid body with principal moment of inertia:

$$\tilde{A} = A_0(1 + 2\mu), \quad \tilde{B} = B_0(1 + 2\mu), \quad \tilde{C} = C_0(1 + 2\mu). \tag{2.16}$$

We remark that in the deformed state (in the observed rotational motion of the Earth) the axial moments of inertia are

$$\bar{A} = A_0(1 - \mu), \quad \bar{B} = B_0(1 - \mu), \quad \bar{C} = C_0(1 + 2\mu). \tag{2.17}$$

This means that the moments of inertia (2.16) are different from the real values (2.17) and are connected with them by simple relationships:

$$\tilde{A} = \bar{A}_0(1 + 3\mu), \quad \tilde{B} = \bar{B}_0(1 + 3\mu), \quad \tilde{C} = \bar{C}_0. \tag{2.18}$$

As a result, we come to the following important theorem (Barkin, 1998b).

Theorem. The rotational motion of an elastic body deformed by its own rotation is executed according to the Euler–Poinsot laws for an equivalent rigid body with changed principal central moments of inertia:

$$\tilde{A} = \bar{A}(1 + 3\mu), \quad \tilde{B} = \bar{B}(1 + 3\mu), \quad \tilde{C} = \bar{C}, \tag{2.18'}$$

where μ is the coefficient of elasticity, and $\bar{A}$, $\bar{B}$ and $\bar{C}$ are average values of the principal moments of inertia of the rotating body.

This means that the elastic body rotates as an absolutely rigid body with equatorial moments of inertia increased by $3\mu C_0$. The polar moment of inertia of the model body is equal to the mean moment of inertia of the deformable body.

This theorem was proved for the first time on the basis of the Hamiltonian formalism by means of a study, of the unperturbed Chandler–Euler rotational motion (Barkin, Getino, and Ferrandiz, 1995b; Barkin. 1996a; 1998b). In these papers, the rotation equations in 'the elastic Andoyer variables' (in the Ferrandiz, Getino terminology) have efficient applications in Earth rotation studies.

Equations (2.15) let us make another interpretation of the deformable body rotation.

The motion of a body deformed by its own rotation, which is described by differential equations (2.15), is executed according to the Euler–Poinsot laws for some fictitious rigid body with undeformed principal moments of inertia A_0, B_0 and C_0 in the 'slowed' time $\tau = (1 - 2\mu)t$.

This interpretation is very important and lets us write the expressions of the two frequencies of the Euler–Chandler motion:

$$\omega_{CH} = \omega_E(1-2\mu), \quad \Omega_{CH} = \Omega_E(1-2\mu), \tag{2.19}$$

where ω_E, Ω_E are frequencies of the Euler–Poinsot motion of a rigid body with the principal moments of inertia A_0, B_0 and C_0 (Sadov, 1970; Barkin, 1992):

$$\begin{aligned}
\omega_0 &= \frac{G}{C}\left(1 - \frac{A_0 - C_0}{A_0}\,\frac{\Pi(\pi/2, k_0^2, \lambda_0)}{K(\lambda_0)}\right), \\
\Omega_0 &= \frac{G(A_0 - C_0)}{2A_0C_0}\,\frac{\pi K}{\sqrt{(1+k_0^2)(k_0^2+\lambda_0^2)}K(\lambda_0)}\,.
\end{aligned} \tag{2.20}$$

Here $K(\lambda_0)$ and $\Pi(\pi/2, k_0^2, \lambda_0)$ are complete elliptical integrals of the first and third kinds. The module λ of these integrals and the parameter k_0 are defined by the initial conditions of the problem:

$$\lambda_0^2 = k_0^2\frac{A_0^2p_0^2}{C_0^2r_0^2}\,, k_0^2 = \frac{C_0(A_0 - B_0)}{A_0(B_0 - C_0)}\,, \tag{2.21}$$

where p_0, r_0 ($q_0 = 0$) are initial values of the components of the angular velocity p, r and (q), respectively.

For the adopted values of the parameters of the problem (Barkin, 1996a):

$$\begin{gathered}
A_0 = 8.086206 \times 10^{44}\ \mathrm{g\ cm}^2,\ B_0 = 8.086380 \times 10^{44}\ \mathrm{g\ cm}^2, \\
C_0 = 8.104309 \times 10^{44}\ \mathrm{g\ cm}^2, \\
k_0 = 0.981975 \times 10^{-1}, \quad \lambda_0 = 0.120103 \times 10^{-6}
\end{gathered} \tag{2.22}$$

the values of the frequencies (2.19), (2.20) are

$$\omega_{CH} = 0.997774\omega_0, \quad \Omega_{CH} = -2.2263 \times 10^{-3}\omega_0,$$

where ω_0 is the mean diurnal velocity of the Earth's rotation.

If the body in its undeformed state is axisymmetric (in this case, $A_0 = B_0$), equations (2.15) are simplified:

$$\begin{aligned}
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= 0\,, \\
\frac{\mathrm{d}l}{\mathrm{d}t} &= \Omega = G\cos\theta\left(\frac{1}{C_0} - \frac{1}{A_0}\right)(1-2\mu)\,, \\
\frac{\mathrm{d}g}{\mathrm{d}t} &= \omega = \frac{G}{A_0}(1-2\mu)\,.
\end{aligned} \tag{2.23}$$

If the angle θ is neglected the velocity of rotation of the deformable body is

$$\omega_0 = \Omega + \omega = \frac{G}{C_0}(1-2\mu) = \frac{G}{\bar{C}}\,. \tag{2.24}$$

Thus, in this case the elastic properties of the body do not influence the value of its angular velocity, but they influence the Chandler–Euler frequencies of the motion. The modulus of both frequencies are reduced, but their sum (2.24) remains the same.

Let us point out some of the main properties of the Chandler–Euler motion of the Earth (Barkin, 1998b).

1. The projection of the trajectory of the end of the angular velocity vector ω on the equatorial plane of the body Cxy is an ellipse with excentricity $e = 0.095835$ and with the minor semi-axis directed parallel to the principal central axes of inertia of the Earth, corresponding to the moment of inertia A (this axis is located 14.5° west from the Greenwich meridian).

2. The mean Chandler frequency of the motion of the ellipse $\Omega = -2.308643 \times 10^{-3}\omega_0$ (ω_0 is the mean value of the angular velocity of the Earth) defines the straight polar motion (in the counter clockwise direction, if viewed from the end of the Earth's polar axis (Cz)) with a period of 433.154 days.

3. The polar motion along the ellipse is executed with a variable velocity. The maximal velocity is achieved at the moment of crossing of the smallest of the equatorial axes of the Earth's ellipsoid of inertia (the corresponding Chandler's period is 433.079 days), and the minimal velocity is achieved at the moment of crossing the major of the equatorial axes (the corresponding value of Chandler's period is 437.112 days). This means that the corresponding variation of the Chandler period is 4.033 days.

These numerical values of the Earth's rotation parameters have been obtained for the model values of the main parameters (2.22).

Due to the results of this section by the construction of the perturbation theory of rotation for a weakly deformable body we can use Euler–Chandler motion of the axisymmetric body with the frequencies from (2.23). This means that in unperturbed motion we take into account the more important elastic properties of the body and their influence on the body rotation. In this sense we talk about Euler–Chandler motion.

3 VARIATIONS OF THE DEFORMABLE BODY ROTATION DUE TO CYCLIC PROCESSES OF THE MASS REDISTRIBUTION

3.1 Perturbations of the Andoyer Variables

We will assume that in the external envelope of the body cyclic displacements of masses (for example, analogous to the seasonal mass redistribution of the hydrosphere, atmosphere, ice envelope of the Earth, etc.) are executed. In the general case this mass redistribution has a conditionally periodic character and is characterized by definite frequencies: $\Omega_1, \Omega_2, \ldots, \Omega_N$.

Let us suppose that, as a result of a study of the corresponding characteristics of these processes, the components of the tensor of inertia and the components of the angular moment of the relative motion of the redistributed masses were defined in the main body reference system and were presented by definite Fourier series by the arguments $U_\sigma = \Omega_\sigma t + U_\sigma^{(0)}$ $(\sigma = 1, 2, \dots N)$.

Let these series be constructed for the parameters

$$Z = (a, b, c, f, e, d, \alpha, \beta, \gamma) \tag{3.1}$$

and have the following form:

$$\begin{aligned} Z &= Z_0 + \sum_{\|\nu\| \geq 1} Z_\nu \cos \Theta_\nu + Z_\nu^* \sin \Theta_\nu \, , \\ \Theta_\nu &= \nu_1 U_1 + \nu_2 U_2 + \dots + \nu_N U_N \, . \end{aligned} \tag{3.2}$$

The summation in (3.2) is produced by corresponding numerical values of the indices $\nu_1, \nu_2, \dots, \nu_N$. Z_0 are constant components of these parameters. In accordance with the representation (1.58), (1.59) we have:

$$f_0 = 0, \ e_0 = 0, \ d_0 = 0, \ \alpha_0 = 0, \ \beta_0 = 0, \ \gamma_0 = 0 \tag{3.3}$$

and we refer periodic components (3.2) to perturbations of the first order with respect to the small parameter μ.

Taking into account these assumptions, the equations of rotational motion in the variables θ, l and g can be written in the standard form of a two-frequency oscillating system with a small parameter:

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= N_z^{(\theta)} + A_z(\theta, t) \cos 2l + B_z(\theta, t) \sin 2l + C_z(\theta, t) \cos l \\ &+ D_z(\theta, t) \sin l + E_z(\theta, t) \, , \end{aligned} \tag{3.4}$$

where $z = (\theta, l, g)$, and the coefficients $A_z, \dots, E_z$ and frequencies N_z are known functions of the variable θ and of time:

$$\begin{aligned} A_\theta &= -fG \sin \theta \, , \\ B_\theta &= \frac{1}{2}(a - b) G \sin \theta \, , \\ C_\theta &= -eF \cos \theta - \alpha \, , \\ D_\theta &= dG \cos \theta - \alpha \, , \\ N_\theta &= 0 \, , \\ E_\theta &= 0 \, , \\ A_l &= \frac{1}{2}(a - b) G \cos \theta \, , \\ B_l &= fG \sin \theta \, , \\ C_l &= dG \sec \theta \cos 2\theta + \beta \cot \theta \, , \end{aligned}$$

$$
\begin{aligned}
D_l &= eG\sec\theta\cos 2\theta + \alpha\cot\theta\,, \\
E_l &= -\gamma\,, \\
N_l &= G\cos\theta\left[c - \frac{1}{2}(a+b)\right]\,, \\
A_g &= \frac{1}{2}(b-a)G\,, \\
B_g &= -fG\,, \\
C_g &= -dG\cot\theta - \beta\sec\theta\,, \\
D_g &= -eG\cot\theta - \alpha\sec\theta\,, \\
E_g &= 0\,, \\
N_g &= G\frac{1}{2}(a+b)\,. \qquad (3.5)
\end{aligned}
$$

All coefficients (3.5) (except for the frequencies N_l, N_g) are small in first order with respect to a small parameter which we don't introduce here for simplicity.

Perturbations of the first order of the problem are defined by simple quadratures:

$$
\begin{aligned}
\delta\theta &= \int [A_\theta(\theta,t)\cos 2l + B_\theta(\theta,t)\sin 2l + C_\theta(\theta,t)\cos l + D_\theta(\theta,t)\sin l]\,\mathrm{d}t\,, \\
\delta l &= \int \left(\frac{\partial N}{\partial\theta}\delta\theta\right)\,\mathrm{d}t \\
&+ \int [A_l(\theta,t)\cos 2l + B_l(\theta,t)\sin 2l + C_l(\theta,t)\cos l + D_l(\theta,t)\sin l + E_l(\theta,t)]\,\mathrm{d}t\,, \\
\delta g &= \int [A_g(\theta,t)\cos 2l + B_g(\theta,t)\sin 2l + C_g(\theta,t)\cos l + D_g(\theta,t)\sin l]\,\mathrm{d}t\,. \qquad (3.6)
\end{aligned}
$$

In (3.6), the variables θ and l in the integrands take the following unperturbed values:

$$
\theta = \theta_0 = \text{const}, \quad l = N_l t + l_0, \qquad (3.7)
$$

where θ_0 and l_0 are initial values of the variables θ and l.

For the problem considered, the integrands in (3.6) are presented by conditionally periodic functions of time in the form (3.1), (3.2). Therefore, the integrals are readily calculated. As a result, we find the perturbations of the first order:

$$
\delta Z = \sum_\nu \sum_{|\sigma|=1,2} Z_{\nu,\sigma}\cos(\Theta_\nu + \sigma l) + Z^*_{\nu,\sigma}\sin(\Theta_\nu + \sigma l), \qquad (3.8)
$$

where $Z = (\theta, l, g)$, and the coefficients are defined by the formulae:

$$
\begin{aligned}
\Theta_{\nu,\varepsilon} &= \frac{(l^*_\nu - \varepsilon d_\nu)G\cos\theta + \alpha^*_\nu - \varepsilon\beta_\nu}{2(\Omega_\nu + \varepsilon\Omega)}\,, \\
\Theta^*_{\nu,\varepsilon} &= -\frac{(l_\nu + \varepsilon d^*_\nu)G\cos\theta + \varepsilon\beta^*_\nu + \alpha_\nu}{2(\Omega_\nu + \varepsilon\Omega)}\,,
\end{aligned}
$$

$$
\begin{aligned}
\Theta_{\nu,2\varepsilon} &= \frac{\varepsilon(b_\nu - a_\nu) + 2f_\nu^*}{4(\Omega_\nu + 2\varepsilon\Omega)} G\sin\theta\,, \\
\Theta^*_{\nu,2\varepsilon} &= \frac{\varepsilon(b_\nu - a_\nu) - 2f_\nu}{4(\Omega_\nu + 2\varepsilon\Omega)} G\sin\theta\,, \\
L_{\nu,2\varepsilon} &= \frac{b_\nu^* - a_\nu^* - 2\varepsilon f_\nu}{4(\Omega_\nu + 2\varepsilon\Omega)} G\cos\theta\left(1 + \tan^2\theta\frac{\Omega\varepsilon}{\Omega_\nu + 2\varepsilon\Omega}\right), \\
L^*_{\nu,2\varepsilon} &= \frac{a_\nu - b_\nu - 2\varepsilon f_\nu^*}{4(\Omega_\nu + 2\varepsilon\Omega)} G\cos\theta\left(1 + \tan^2\theta\frac{\Omega\varepsilon}{\Omega_\nu + 2\varepsilon\Omega}\right), \\
L_{\nu,\varepsilon} &= \frac{-(\varepsilon l_\nu + d_\nu^*)}{2(\Omega_\nu + \varepsilon\Omega)} G\sec\theta\left(1 - \frac{2\Omega_\nu + \Omega\varepsilon}{\Omega_\nu + 2\varepsilon\Omega}\sin^2\theta\right) \\
&- \frac{(\beta_\nu^* + \varepsilon\alpha_\nu)\sec\theta}{2(\Omega_\nu + \varepsilon\Omega)\cos\theta}\left(1 - \frac{\Omega_\nu}{\Omega_\nu + \varepsilon\Omega}\sin^2\theta\right), \\
L^*_{\nu,\varepsilon} &= \frac{-(\varepsilon l_\nu^* + d_\nu)}{2(\Omega_\nu + \varepsilon\Omega)} G\sec\theta\left(1 - \frac{2\Omega_\nu + \Omega\varepsilon}{\Omega_\nu + 2\varepsilon\Omega}\sin^2\theta\right) \\
&- \frac{(\beta_\nu - \varepsilon\alpha_\nu^*)\sec\theta}{2(\Omega_\nu + \varepsilon\Omega)\cos\theta}\left(1 - \frac{\Omega_\nu}{\Omega_\nu + \varepsilon\Omega}\sin^2\theta\right), \\
G_{\nu,\varepsilon} &= \frac{(\varepsilon l_\nu + d_\nu^*)G\cot\theta + (\beta_\nu^* + \varepsilon\alpha_\nu)\sec\theta}{2(\Omega_\nu + \varepsilon\Omega)} = -\varepsilon\Theta^*_{\nu,\varepsilon}\,, \\
G^*_{\nu,\varepsilon} &= \frac{(\varepsilon l_\nu^* - d_\nu)G\cot\theta + (\varepsilon\alpha_\nu^* - \beta_\nu)\sec\theta}{2(\Omega_\nu + \varepsilon\Omega)} = \varepsilon\Theta_{\nu,\varepsilon}\,, \\
G_{\nu,2\varepsilon} &= \frac{2\varepsilon f_\nu + \alpha_\nu^* - \beta_\nu^*}{4(\Omega_\nu + 2\varepsilon\Omega)} G = -\varepsilon\Theta^*_{\nu,2\varepsilon}\sec\theta\,, \\
G^*_{\nu,2\varepsilon} &= \frac{(b_\nu - a_\nu) + 2\varepsilon f_\nu^*}{4(\Omega_\nu + 2\varepsilon\Omega)} G = \varepsilon\Theta^*_{\nu,2\varepsilon}\sec\theta\,.
\end{aligned} \tag{3.9}
$$

Formulae (3.7)–(3.9) present the approximate solution of the problem of rotation of an isolated deformable celestial body. The unperturbed motion is Euler rotation of an axysimmetric body with an arbitrary value of angle $\theta = \theta_0$ (between the angular momentum vector and the body axis of symmetry), although in accordance with the remark in Section 3.1, formulae (3.8), (3.9) retain their form for Chandler–Euler unperturbed motion of the axisymmetric body.

3.2 Periodic Perturbations of the Components of the Angular Velocity

Now we present the solution of the problem in variables p, q, r (1.1). For this purpose, we use the following expressions of the first-order perturbations of these variables, which are obtained from formulae (1.31), (1.32):

$$
\begin{aligned}
\delta p &= -\frac{G\delta A}{A_0^2}\sin\theta\sin l - \frac{\delta P}{A_0} + \frac{\delta F}{A_0 B_0} G\sin\theta\cos l + \frac{\delta E}{A_0 C_0} G\cos\theta \\
&+ \frac{G}{A_0}\cos\theta\sin l\delta\theta + \frac{G}{A_0}\sin\theta\cos l\delta l\,,
\end{aligned}
$$

$$
\begin{aligned}
\delta q &= -\frac{G\delta B}{B_0^2}\sin\theta\cos l - \frac{\delta Q}{B_0} + \frac{\delta F}{A_0 B_0} G\sin\theta\sin l + \frac{\delta D}{B_0 C_0} G\cos\theta \\
&+ \frac{G}{B_0}\cos\theta\cos l\delta\theta - \frac{G}{B_0}\sin\theta\sin l\delta l\,, \\
\delta r &= -\frac{G\delta C}{C_0^2}\cos\theta - \frac{\delta R}{C_0} + \frac{\delta E}{A_0 C_0} G\sin\theta\sin l \\
&+ \frac{\delta D}{B_0 C_0} G\sin\theta\cos l - \frac{G}{C_0}\sin\theta\delta\theta\,. \qquad (3.10)
\end{aligned}
$$

Substituting formulae (3.8), (3.9) into (3.10) after some algebra we obtain the following formulae for perturbations of the first order of the angular velocity components.

Perturbations of component p:

$$
\delta p = \sum_{\nu}\sum_{\sigma=-3}^{3}\{p_{\nu,\sigma}\cos(\Theta_\nu + \sigma l) + q^*_{\nu,\sigma}\sin(\Theta_\nu + \sigma l)\}\,, \qquad (3.11)
$$

where

$$
\begin{aligned}
p_{\nu,0} &= -\frac{P_\nu}{A_0} + \omega C_0 e_\nu\cos\theta \\
&- \frac{\omega G(e_\nu + d^*_\nu)}{2(\Omega_\nu + \Omega)}\left(1 - \frac{3\Omega_\nu + 2\Omega}{2(\Omega_\nu + \Omega)}\sin^2\theta\right) \\
&+ \frac{\omega G(e_\nu - d^*_\nu)}{2(\Omega_\nu - \Omega)}\left(1 - \frac{3\Omega_\nu - 2\Omega}{2(\Omega_\nu - \Omega)}\sin^2\theta\right) \\
&- \frac{\omega(\alpha_\nu + \beta^*_\nu)}{2\cos\theta(\Omega_\nu + \Omega)}\left(1 - \frac{2\Omega_\nu + \Omega}{2(\Omega_\nu + \Omega)}\sin^2\theta\right) \\
&+ \frac{\omega(\alpha_\nu - \beta^*_\nu)}{2\cos\theta(\Omega_\nu - \Omega)}\left(1 - \frac{2\Omega_\nu - \Omega}{2(\Omega_\nu - \Omega)}\sin^2\theta\right)\,, \\
p^*_{\nu,0} &= -\frac{P^*_\nu}{A_0} + \omega C_0 e^*_\nu\cos\theta \\
&- \frac{\omega G(e^*_\nu + d_\nu)}{2(\Omega_\nu - \Omega)}\left(1 - \frac{3\Omega_\nu - 2\Omega}{2(\Omega_\nu - \Omega)}\sin^2\theta\right) \\
&+ \frac{\omega G(e^*_\nu - d_\nu)}{2(\Omega_\nu + \Omega)}\left(1 - \frac{3\Omega_\nu + 2\Omega}{2(\Omega_\nu + \Omega)}\sin^2\theta\right) \\
&- \frac{\omega(\alpha^*_\nu + \beta_\nu)}{2\cos\theta(\Omega_\nu - \Omega)}\left(1 - \frac{2\Omega_\nu - \Omega}{2(\Omega_\nu - \Omega)}\sin^2\theta\right) \\
&+ \frac{\omega(\alpha^*_\nu - \beta_\nu)}{2\cos\theta(\Omega_\nu + \Omega)}\left(1 - \frac{2\Omega_\nu + \Omega}{2(\Omega_\nu + \Omega)}\sin^2\theta\right)\,, \\
p_{\nu,\varepsilon} &= \varepsilon\frac{\omega A^*_\nu\sin\theta}{2A_0} - \varepsilon\frac{\omega}{2}C_0 f_\nu\sin\theta \\
&+ \frac{\omega G\sin\theta\cos\theta(b^*_\nu - a^*_\nu - 2\varepsilon f_\nu)}{4(\Omega_\nu + 2\varepsilon\Omega)}\left(1 + \frac{\Omega\varepsilon}{2(\Omega_\nu + 2\varepsilon\Omega)}\tan^2\theta\right)\,,
\end{aligned}
$$

$$
\begin{aligned}
p^*_{\nu,\varepsilon} &= -\varepsilon\frac{\omega A_\nu \sin\theta}{2A_0} + \frac{\omega}{2}C_0 f^*_\nu \sin\theta \\
&+ \frac{\omega G \sin\theta\cos\theta(a_\nu - b_\nu - 2\varepsilon f_\nu)}{4(\Omega_\nu + 2\varepsilon\Omega)}\left(1 + \frac{\Omega\varepsilon}{2(\Omega_\nu + 2\varepsilon\Omega)}\tan^2\theta\right), \\
p_{\nu,2\varepsilon} &= \frac{\omega\sin^2\theta}{4(\Omega_\nu + \varepsilon\Omega)^2}\frac{[(\varepsilon e_\nu + d^*_\nu)G\Omega_\nu\cos\theta - \varepsilon\Omega(\beta^*_\nu + \varepsilon\alpha_\nu)]}{\cos\theta}, \\
p^*_{\nu,2\varepsilon} &= \frac{\omega\sin^2\theta}{4(\Omega_\nu + \varepsilon\Omega)^2}\frac{[(\varepsilon e_\nu - d_\nu)G\Omega_\nu\cos\theta + \Omega(-\alpha^*_\nu + \varepsilon\beta_\nu)]}{\cos\theta}, \\
p_{\nu,3\varepsilon} &= -\frac{G\omega\Omega\varepsilon}{8\cos\theta(\Omega_\nu + 2\varepsilon\Omega)^2}\sin^3\theta(a^*_\nu + b^*_\nu + 2\varepsilon f_\nu), \\
p^*_{\nu,3\varepsilon} &= \frac{G\omega\Omega\varepsilon}{8\cos\theta(\Omega_\nu + 2\varepsilon\Omega)^2}\sin^3\theta(b_\nu - a_\nu + 2\varepsilon f_\nu).
\end{aligned}
\tag{3.12}
$$

Perturbations of component q:

$$
\delta q = \sum_\nu \sum_{\sigma=-3}^{3} \{q_{\nu,\sigma}\cos(\Theta_\nu + \sigma l) + q^*_{\nu,\sigma}\sin(\Theta_\nu + \sigma l)\}, \tag{3.13}
$$

where

$$
\begin{aligned}
q_{\nu,0} &= -\frac{Q_\nu}{B_0} + \omega C_0 d_\nu \cos\theta \\
&+ \frac{\omega G(e^*_\nu + d_\nu)}{2(\Omega_\nu - \Omega)}\left(1 - \frac{3\Omega_\nu - 2\Omega}{2(\Omega_\nu - \Omega)}\sin^2\theta\right) \\
&+ \frac{\omega G(e^*_\nu - d_\nu)}{2(\Omega_\nu + \Omega)}\left(1 - \frac{3\Omega_\nu + 2\Omega}{2(\Omega_\nu + \Omega)}\sin^2\theta\right) \\
&+ \frac{\omega(\alpha^*_\nu + \beta_\nu)}{2\cos\theta(\Omega_\nu - \Omega)}\left(1 - \frac{2\Omega_\nu - \Omega}{2(\Omega_\nu + \Omega)}\sin^2\theta\right) \\
&+ \frac{\omega(\alpha^*_\nu - \beta_\nu)}{2\cos\theta(\Omega_\nu + \Omega)}\left(1 - \frac{2\Omega_\nu + \Omega}{2(\Omega_\nu + \Omega)}\sin^2\theta\right), \\
q^*_{\nu,0} &= -\frac{Q^*_\nu}{B_0} + \omega C_0 d^*_\nu \cos\theta \\
&- \frac{\omega G(e_\nu + d_\nu)}{2(\Omega_\nu - \Omega)}\left(1 - \frac{3\Omega_\nu - 2\Omega}{2(\Omega_\nu - \Omega)}\sin^2\theta\right) \\
&+ \frac{\omega G(e^*_\nu - d_\nu)}{2(\Omega_\nu + \Omega)}\left(1 - \frac{3\Omega_\nu + 2\Omega}{2(\Omega_\nu + \Omega)}\sin^2\theta\right) \\
&- \frac{\omega(\alpha^*_\nu + \beta_\nu)}{2\cos\theta(\Omega_\nu - \Omega)}\left(1 - \frac{2\Omega_\nu - \Omega}{2(\Omega_\nu - \Omega)}\sin^2\theta\right) \\
&+ \frac{\omega(\alpha^*_\nu - \beta_\nu)}{2\cos\theta(\Omega_\nu + \Omega)}\left(1 - \frac{2\Omega_\nu + \Omega}{2(\Omega_\nu + \Omega)}\sin^2\theta\right), \\
q_{\nu,\varepsilon} &= -\frac{\omega B_\nu \sin\theta}{2B_0} - \varepsilon\frac{\omega}{2}C_0 f^*_\nu \sin\theta
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{\varepsilon\omega G \sin\theta\cos\theta(b_\nu - a_\nu - 2\varepsilon f_\nu^*)}{4(\Omega_\nu + 2\varepsilon\Omega)} \left(1 + \frac{\Omega\varepsilon}{2(\Omega_\nu + 2\varepsilon\Omega)} \tan^2\theta\right), \\
q_{\nu,\varepsilon}^* &= -\frac{\omega B_\nu^* \sin\theta}{2B_0} + \varepsilon\frac{\omega}{2} C_0 f_\nu \sin\theta \\
& + \frac{\varepsilon\omega G \sin\theta\cos\theta(a_\nu^* - b_\nu^* - 2\varepsilon f_\nu)}{4(\Omega_\nu + 2\varepsilon\Omega)} \left(1 + \frac{\Omega\varepsilon}{2(\Omega_\nu + 2\varepsilon\Omega)} \tan^2\theta\right), \\
q_{\nu,2\varepsilon} &= \frac{\omega\sin^2\theta}{4(\Omega_\nu + \varepsilon\Omega)^2} \frac{[(e_\nu^* - \varepsilon d_\nu)G\Omega_\nu - \Omega(\beta_\nu - \varepsilon\alpha_\nu^*)]}{\cos\theta}, \\
q_{\nu,2\varepsilon}^* &= \frac{\omega^2\sin^2\theta}{4(\Omega_\nu + \varepsilon\Omega)^2} \frac{[-(\varepsilon e_\nu + d_\nu^*)G\Omega_\nu\varepsilon + \Omega(\beta_\nu^* + \varepsilon\alpha_\nu)]}{\cos\theta}, \\
q_{\nu,3\varepsilon} &= \frac{G\omega\Omega}{8\cos\theta(\Omega_\nu + 2\varepsilon\Omega)^2} \sin^3\theta(b_\nu - a_\nu + 2\varepsilon f_\nu^*), \\
q_{\nu,3\varepsilon}^* &= \frac{G\omega\Omega}{8\cos\theta(\Omega_\nu + 2\varepsilon\Omega)^2} \sin^3\theta(a_\nu^* - b_\nu^* + 2\varepsilon f_\nu). \qquad (3.14)
\end{aligned}
$$

Perturbation of component r:

$$
\delta r = \sum_\nu \sum_{\sigma=-3}^{3} \{r_{\nu,\sigma}\cos(\Theta_\nu + \sigma l) + r_{\nu,\sigma}^* \sin(\Theta_\nu + \sigma l)\}, \qquad (3.15)
$$

where

$$
\begin{aligned}
r_{\nu,0} &= -\omega\frac{C_\nu}{C_0}\cos\theta - \frac{R_\nu}{C_0}, \\
r_{\nu,0}^* &= -\omega\frac{C_\nu^*}{C_0}\cos\theta - \frac{R_\nu^*}{C_0}, \\
r_{\nu,\varepsilon} &= \frac{1}{2}\omega\sin\theta\left[C_0(d_\nu - \varepsilon e_\nu^*)\left(1 + \frac{\varepsilon\omega\cos\theta}{\Omega_\nu + \varepsilon\Omega}\tan^2\theta\right) + \frac{\varepsilon\beta_\nu - \alpha_\nu^*}{\Omega_\nu + \varepsilon\Omega}\right], \\
r_{\nu,\varepsilon}^* &= \frac{1}{2}\omega\sin\theta\left[C_0(d_\nu^* + \varepsilon e_\nu)\left(1 + \frac{\varepsilon\omega\cos\theta}{\Omega_\nu + \varepsilon\Omega}\tan^2\theta\right) + \frac{\beta_\nu^* + \varepsilon\alpha_\nu}{\Omega_\nu + \varepsilon\Omega}\right], \\
r_{\nu,2\varepsilon} &= -\frac{1}{4}\omega^2\sin^2\theta\frac{[\varepsilon(b_\nu - a_\nu) + 2f_\nu^*]}{\Omega_\nu + 2\varepsilon\Omega}, \\
r_{\nu,2\varepsilon}^* &= -\frac{1}{4}\omega^2\sin^2\theta\frac{[\varepsilon(b_\nu^* - a_\nu^*) - 2f_\nu]}{\Omega_\nu + 2\varepsilon\Omega}. \qquad (3.16)
\end{aligned}
$$

3.3 Variations of Earth Rotation due to Lunar–solar Tidal Variations of its Tensor of Inertia

For the unperturbed rotational motion of the Earth, the angle θ has a very small value of $\approx 10^{-6}$. Therefore, neglecting the second and higher order terms with respect to θ, we write formulae (3.11)–(3.16) in the following reduced form:

$$
\delta p = \sum_\nu p_\nu\cos\Theta_\nu + p_\nu^*\sin\Theta_\nu,
$$

$$\begin{aligned}\delta q &= \sum_\nu q_\nu \cos\Theta_\nu + q_\nu^* \sin\Theta_\nu\,,\\ \delta r &= \sum_\nu r_\nu \cos\Theta_\nu + r_\nu^* \sin\Theta_\nu\,.\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}
p_\nu &= p_{\nu,0}\\
&= -\frac{P_\nu}{A_0} + \omega C_0 e_\nu + \frac{\omega}{\Omega_\nu^2-\Omega^2}[(\Omega e_\nu - \Omega_\nu d_\nu^*)G + \Omega\alpha_\nu - \Omega_\nu\beta_\nu^*]\,,\\
p_\nu^* &= p_{\nu,0}^*\\
&= -\frac{P_\nu^*}{A_0} + \omega C_0 e_\nu^* + \frac{\omega}{\Omega_\nu^2-\Omega^2}[(\Omega e_\nu^* + \Omega_\nu d_\nu)G + \Omega\alpha_\nu^* + \Omega_\nu\beta_\nu]\,,\\
q_\nu &= q_{\nu,0}\\
&= -\frac{Q_\nu}{A_0} + \omega C_0 d_\nu + \frac{\omega}{\Omega_\nu^2-\Omega^2}[(\Omega e_\nu^* + \Omega d_\nu)G + \Omega\alpha_\nu^* + \Omega_\nu\beta_\nu]\,,\\
q_\nu^* &= q_{\nu,0}^*\\
&= -\frac{Q_\nu^*}{A_0} + \omega C_0 d_\nu^* + \frac{\omega}{\Omega_\nu^2-\Omega^2}[(-\Omega e_\nu + \Omega d_\nu^*)G - \Omega\alpha_\nu + \Omega_\nu\beta_\nu^*]\,,\\
r_\nu &= r_{\nu,0} = -\frac{R_\nu}{C_0} - \omega\frac{C_\nu}{C_0}\,,\\
r_\nu^* &= r_{\nu,0}^* = -\frac{R_\nu^*}{C_0} - \omega\frac{C_\nu^*}{C_0}\,.
\end{aligned} \tag{3.18}$$

In this paper, we will use formulae (3.17), (3.18) for the determination of the variations in the Earth's polar motion, caused by lunar–solar tidal deformations. For this purpose, we use tidal variations of the second-harmonic coefficients, presented in the form of the following series (Ferrandiz and Getino, 1993):

$$\begin{aligned}
\delta J_2 &= \sum_i K_2(i)\cos\Theta_i\,,\\
\delta C_{22} &= \sum_i K_{22a}(i)\cos(2l+2g-\Theta_i) + \sum_i K_{22b}(i)\cos(2l+2g+\Theta_i)\,,\\
\delta S_{22} &= -\sum_i K_{22a}(i)\sin(2l+2g-\Theta_i) - \sum_i K_{22b}(i)\sin(2l+2g+\Theta_i)\,,\\
\delta C_{21} &= \sum_i K_{21a}(i)\sin(l+g-\Theta_i) + \sum_i K_{21b}(i)\sin(l+g+\Theta_i)\,,\\
\delta S_{21} &= \sum_i K_{21a}(i)\cos(l+g-\Theta_i) + \sum_i K_{21b}(i)\cos(l+g+\Theta_i)\,,
\end{aligned} \tag{3.19}$$

where $K_{22a}(i), \ldots, K_{21b}(i)$ are numerical coefficients, the values of which are given in Table 1 of Ferrandiz and Getino (1993).

Argument Θ_i is the linear combination with numerical coefficients of the arguments of the Moon's orbital theory:

$$\begin{aligned}
\Theta_i &= m_1 l_M + m_2 l_S + m_3 F + m_4 D + m_5 \Omega\,, \\
F &= l_M + g_M\,, \\
D &= l_M + g_M + h_M - l_S - g_S - h_S\,, \\
i &= (m_1, m_2, m_3, m_4, m_5)\,.
\end{aligned}$$

Here l_M, g_M, $h_M$and l_S, s, h_S are the Delaunay variables for the Moon and the Sun. $l + g$ is the angle of the Earth's rotation.

Variations of the components of the Earth tensor of inertia are connected with variations (3.19) by simple relationships:

$$\begin{aligned}
\frac{\delta A}{C} &= -\frac{2}{J}\delta C_{22} - \frac{1}{3J}\delta J_2\,, \\
\frac{\delta B}{C} &= \frac{2}{J}\delta C_{22} - \frac{1}{3J}\delta J_2\,, \\
\frac{\delta C}{C} &= \frac{2}{3J}\delta J_2\,, \\
\frac{\delta F}{C} &= \frac{2}{J}\delta S_{22}\,, \\
\frac{\delta E}{C} &= \frac{1}{J}\delta C_{21}\,, \\
\frac{\delta D}{C} &= \frac{1}{J}\delta S_{21}\,,
\end{aligned} \qquad (3.20)$$

where $J = C/(mR^2)$ is a non-dimensional moment of inertia, and m, R are the mass and the radius of the Earth.

Substituting (3.18) into equations (3.20), we obtain analogous trigonometric series for the variations of the axial and the centrifugal moments of inertia:

$$\begin{aligned}
\frac{\delta A}{C} &= \sum_\sigma \{A_\sigma \cos\Theta_\sigma + A_{\sigma,-2}\cos(-2S + \Theta_\sigma) + A_{\sigma,2}\cos(\Theta_\sigma + 2S)\}\,, \\
\frac{\delta B}{C} &= \sum_\sigma \{B_\sigma \cos\Theta_\sigma + B_{\sigma,-2}\cos(-2S + \Theta_\sigma) + B_{\sigma,2}\cos(\Theta_\sigma + 2S)\}\,, \\
\frac{\delta C}{C} &= \sum_\sigma \{C_\sigma \cos\Theta_\sigma\}\,, \\
\frac{\delta F}{C} &= \sum_\sigma \{F^*_{\sigma,2}\sin(\Theta_\sigma + 2S) + F^*_{\sigma,-2}\sin(\Theta_\sigma - 2S)\}\,, \\
\frac{\delta E}{C} &= \sum_\sigma \{E^*_{\sigma,1}\sin(\Theta_\sigma + S) + E^*_{\sigma,-1}\sin(\Theta_\sigma - S)\}\,, \\
\frac{\delta D}{C} &= \sum_\sigma \{D_{\sigma,1}\cos(\Theta_\sigma + S) + D_{\sigma,-1}\cos(\Theta_\sigma - S)\}\,.
\end{aligned} \qquad (3.21)$$

Table 1. Coefficients of the main periodic tidal variations of the axial and centrifugal moments of inertia of the Earth (1 unit = 10^{-9})

N	l	l'	F	D	Ω	$A_{\nu,0} = B_{\nu,0}$	$A_{\nu,2} = F^*_{\nu,2} = B_{\nu,2}$	$A_{\nu,-2} = F^*_{\nu,-2} = B_{\nu,-2}$	$C_{\nu,0} = -2A_{\nu,0}$	$E^*_{\nu,1} = D_{\nu,1}$	$E^*_{\nu,-1} = D_{\nu,-1}$
1.	1	0	0	−2	0	−0.2049	−0.0321	−0.0321	0.4098	0.1470	0.1470
2.	1	0	0	0	0	−1.0716	−0.1669	−0.1669	2.1433	0.7690	0.7690
3.	0	0	0	2	0	−0.1778	−0.0278	−0.0278	0.3556	0.1276	0.1276
4.	1	0	2	0	1	−0.1610	−0.0073	0.1681	0.3220	0.0517	−0.3529
5.	0	0	2	0	1	−0.8409	−0.0381	0.8788	1.6818	0.2695	−1.8444
6.	0	0	0	0	1	0.8501	0.0381	−0.8885	−1.7001	−0.2725	1.8650
7.	−1	0	2	2	2	−0.0738	−0.0018	−0.8570	0.1476	0.0154	−0.3556
8.	−1	0	2	0	2	0.0574	0.0012	0.6665	−0.1147	−0.0118	0.2764
9.	1	0	2	0	2	−0.3885	−0.0085	−4.5131	0.7770	0.0804	−1.8725
10.	0	0	2	2	2	−0.0620	−0.0012	−0.7209	0.1240	0.0130	−0.2991
11.	0	0	2	0	2	−2.0286	−0.0436	−23.5732	4.0571	0.4207	−9.7796
12.	0	1	0	0	0	−0.1499	−0.0236	−0.0236	0.2998	0.1077	0.1077
13.	0	1	2	−2	2	−0.0552	−0.0012	−0.6411	0.1105	−0.0027	−0.2661
14.	0	0	2	−2	2	−0.9417	−0.0206	−10.9411	1.8834	−0.0484	−4.5394

The coefficients of the series (3.21) and (3.19) are connected by simple relationships:

$$\begin{aligned} A_\sigma &= B_\sigma = -\frac{1}{3J}K_2(\sigma)\,, \\ C_\sigma &= -2A_\sigma\,, \\ A_{\sigma,2} &= -B_{\sigma,2} = F^*_{\sigma,2} = \frac{2}{J}K_{22b}(\sigma)\,, \\ A_{\sigma,-2} &= -B_{\sigma,2} = F^*_{\sigma,-2} = \frac{2}{J}K_{22a}(\sigma)\,, \\ E^*_{\sigma,1} &= D_{\sigma,1} = -\frac{1}{J}K_{21b}(\sigma)\,, \\ E^*_{\sigma,-1} &= D_{\sigma,-1} = -\frac{1}{J}K_{21a}(\sigma)\,. \end{aligned} \tag{3.22}$$

Numerical values of coefficients (3.22) are presented in the Table 1.

The tidal variations of the moments of inertia of the Earth lead to the following perturbations in the poar motion:

$$\begin{aligned} \delta p &= \sum_\nu p^*_{\nu,1}\sin(\Theta_\nu + S) + p^*_{\nu,-1}\sin(\Theta_\nu - S)\,, \\ \delta q &= \sum_\nu q_{\nu,1}\cos(\Theta_\nu + S) + q_{\nu,-1}\cos(\Theta_\nu - S)\,, \end{aligned} \tag{3.23}$$

where $p^*_{\nu,\varepsilon} = q_{\nu,\varepsilon}$ and

$$\frac{p^*_{\nu,\varepsilon}}{\omega} = \frac{E^*_\nu}{C_0}\left[1 + \frac{\omega(\Omega + \Omega_\nu + \varepsilon\omega)}{(\Omega_\nu + \varepsilon\omega)^2 - \Omega^2}\right]. \tag{3.24}$$

Table 2. Coefficients of the periodic perturbations in the polar motion of Earth caused by its tidal deformations (1 unit = 10^{-3} arcsec).

N	l	l'	F	D	Ω	$p^*_{\nu,1}/\omega = q_{\nu,1}/\omega$	$p^*_{\nu,-1}/\omega = q_{\nu,-1}/\omega$
1.	1	0	0	−2	0	0.062	−0.001
2.	1	0	0	0	0	0.311	−0.006
3.	0	0	0	2	0	0.051	−0.028
4.	1	0	2	0	1	0.098	0.009
5.	0	0	2	0	1	0.107	0.031
6.	0	0	0	0	1	−0.056	−0.001
7.	−1	0	2	2	2	0.003	0.009
8.	−1	0	2	0	2	−0.005	−0.002
9.	1	0	2	0	2	0.015	0.049
10.	0	0	2	2	2	0.005	0.010
11.	0	0	2	0	2	0.081	0.165
12.	0	1	0	0	0	0.0443	0.000
13.	0	1	2	−2	2	−0.001	0.001
14.	0	0	2	−2	2	−0.020	0.007

Using formulae (3.24) we find the corresponding coefficients of the tidal lunar–solar variations in the polar motion of the Earth (Table 2).

Similar perturbations in the diurnal rotation of the Earth were described in detail earlier (Yoder *et al.*, 1981; Ferrandiz and Getino, 1993).

SUMMARY

In this paper new forms of the differential equations of the rotational motion of weakly deformable bodies were obtained (1.98), (1.99), (1.102), etc. These equations admit applications to various methods of celestial mechanics to study the rotational motion of planets, satellites (natural and artificial), asteroids and others bodies of the solar system. In next paper we will study secular effects in rotational motion of an isolated deformable body with application to the Earth's rotation and Venus's rotation.

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