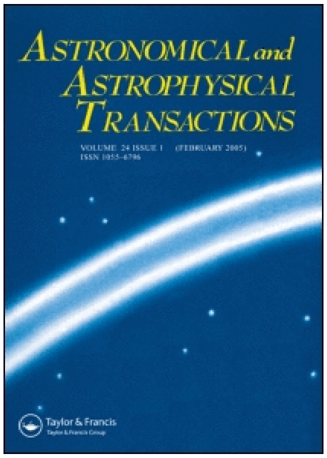

Astronomical & Astrophysical Transactions
The Journal of the Eurasian Astronomical Society
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713453505

On the some properties of gravitational lens equation near cusps

A. F. Zakharov [a]
[a] Institute of Theoretical and Experimental Physics, Moscow, Russia

Online Publication Date: 01 August 1999
To cite this Article: Zakharov, A. F. (1999) 'On the some properties of gravitational lens equation near cusps', Astronomical & Astrophysical Transactions, 18:1, 17 - 25
To link to this article: DOI: 10.1080/10556799908203029
URL: http://dx.doi.org/10.1080/10556799908203029

Astronomical and Astrophysical Transactions, 1999,
Vol. 18, pp. 17–25
Reprints available directly from the publisher
Photocopying permitted by license only

ON THE SOME PROPERTIES OF GRAVITATIONAL LENS EQUATION NEAR CUSPS

A. F. ZAKHAROV

Institute of Theoretical and Experimental Physics, B. Cheremushkinskaya, 25, 117259, Moscow, Russia

(Received December 25, 1996)

The common solution of the gravitational lens equation near cusps is presented. We use the gravitational lens equation by Schneider and Weiss. Using the symmetrical polynomials oh the roots of a polynomial of third degree we obtained a generalization of Schneider and Weiss statement on the magnification near different solutions of the gravitational lens equation. Analytical expressions for magnifications of different images near cusps are presented.

KEY WORDS Gravitational lens

1 INTRODUCTION

When we consider mappings of a two-dimensional surface into the plane it is well known that there are only two types of stable singularities: folds and cusps (pleats). There are also similar singularties of caustics in gravitational lens optics. Schneider and Weiss (1986, 1992) studied gravitational lens mapping near cusps. Some properties of the mappings are very important for solving different problems of gravitational lensing, for example, in consideration of the mutual coherence of images near the cusps (Mandzhos, 1995). The analytical expressions are very useful for these purposes. Therefore, we will obtain the analytical expressions for the solution of the gravitational lens equation and magnifications of different images near the cusp.

2 BASIC EQUATIONS

We recall the basic equations from Schneider and Weiss (1992) before our consideration of their gravitational lens equation near a cusp. As shown by Schneider, Ehlers and Falco (1992), hereafter SEF, the gravitational lens equation may be written in the following form when the distance between a source and an observer is D_s, the distance between a gravitational lens and an observer is D_d, and D_{ds} is the distance between the gravitational lens and an observer. Since we have pseudoeuclidean lengths, it is necessary to define more correctly the measure of length. Therefore more correctly to consider angular distances against linear distances. If we suppose a small angle of deflection then we have the following simple expression for the lens equation

$$\boldsymbol{\eta} = D_s\boldsymbol{\xi}/D_d + D_{ds}\boldsymbol{\alpha}(\boldsymbol{\xi}),$$

where the vectors $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ define the coordinates in the lens plane and in the source plane respectively,

$$\boldsymbol{\alpha}(\boldsymbol{\xi}) = 4G/c^2 \int \rho(\boldsymbol{R})(\boldsymbol{r} - \boldsymbol{R})/|\boldsymbol{r} - \boldsymbol{R}|^2 \, \mathrm{d}X \, \mathrm{d}Y,$$

where $\boldsymbol{R} = \{X, Y\}$ is a point vector in lens plane and $\rho(\boldsymbol{R})$ is the surface mass density of the gravitational lens. We introduce the following variables

$$\boldsymbol{x} = \boldsymbol{\xi}/R_0, \quad \boldsymbol{y} = D_s\boldsymbol{\eta}/(R_0 D_d),$$

where $R_0 = \sqrt{2r_g D_d D_{ds}/D_s}$, the Einstein–Chwolson radius (see Historical Remarks in SEF). The distance between the different images is approximately equal to $2R_0$. Then the angle for the Einstein radius is approximately equal to

$$\theta_0 = \xi_0/D_d = \sqrt{2r_g D_{ds}/(D_d D_s)},$$

and we have an angle of $2''$ for galactic distances and galactic masses with a value $10^{12} M_\odot$. If we consider gravitational lensing of stars with solar masses then it is more convenient to use the following expression

$$\theta_0 = 10^{-6}\sqrt{M/M_\odot},$$

i.e. the value is approximately equal to a few microseconds (therefore we call stellar lensing in galaxies microlensing) for typical values of distances and galactic masses with value $10^{12} M_\odot$ (Wambsganss, 1990). We also introduce the following notation for the scaled angle

$$\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha} D_{ds} D_d/(D_s R_0).$$

In modelling gravitational lenses, the surface mass density is normalized with the critical surface mass density (Wambsganss, 1990)

$$\rho_{\mathrm{cr}} = \frac{c^2 D_s}{4\pi G D_d D_{ds}}.$$

For typical lensing situations the critical surface mass density is of the order $\rho_{\mathrm{cr}} = 10^4 M_\odot \ \mathrm{pc}^{-2}$ (Wambsganss, 1990). Therefore, if we define the scaled surface mass density by following expression

$$\sigma = \rho/\rho_{\mathrm{cr}},$$

then we have the following expression for the $\hat{\boldsymbol{\alpha}}$:

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{x}) = \int \sigma(\boldsymbol{x}')(\boldsymbol{x} - \boldsymbol{x}')/|\boldsymbol{x} - \boldsymbol{x}'|^2 \, \mathrm{d}^2 x.$$

As Schneider (1985) showed, we may introduce the scalar potential ψ, such that

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{x}) = \nabla\psi(\boldsymbol{x}),$$

where

$$\psi(\boldsymbol{\xi}) = \int (\boldsymbol{x}') \ln |\boldsymbol{x} - \boldsymbol{x}'| \, \mathrm{d}^2 x.$$

Then

$$\boldsymbol{y} = \boldsymbol{x} - \hat{\boldsymbol{\alpha}}(\boldsymbol{x}). \tag{1}$$

If we introduce

$$\phi(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x} - \boldsymbol{y})^2/2 - \psi(\boldsymbol{x}),$$

then we can write the lens equation as (Schneider, 1985)

$$\nabla(\boldsymbol{x}, \boldsymbol{y}) = 0.$$

It is easy to see that the mapping $x \to y$ is Lagrange's mapping (Arnold, 1979), since that is a gradient mapping (Arnold, 1983). In fact if we consider the function

$$S = \boldsymbol{x} \cdot \boldsymbol{x}/2 - \psi(\boldsymbol{x}),$$

then

$$\boldsymbol{y} = \nabla S.$$

Thus the mapping is the gradient mapping, so the mapping is Lagrange's mapping. Singularities of Lagrange's mappings are described in Arnold's paper (1972), and in Arnold's review (1983), and their reconstuctions are described in Arnold's paper (1976). According to the paper if we consider Lagrange's mapping of a plane into a plane then we have singularities of only the two following types: folds and pleats (cusps). According to the paper if we consider Lagrange's mapping in three-dimensional space then we can have moreover the following singularities: the swallowtail, the purse and the pyramyd. If we consider the moving caustic in three-dimensional space then singularities of the butterfly type and the parabolic umbilic type can appear. Arnold (1983) showed that the ray family is Langange's family in a more common situation than the ray family in gravitational lens optics.

Equation (1) defined the mapping of points on the lens plane into points on the source plane. Using the Jacobian matrix we define the local mapping:

$$\boldsymbol{y} = A\delta\boldsymbol{x},$$

$$A = \begin{pmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 \end{pmatrix} = \begin{pmatrix} 1 + \psi_{11} & \psi_{12} \\ \psi_{21} & 1 + \psi_{22} \end{pmatrix} \tag{2}$$

where

$$\psi_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j} \quad (i, j = 1, 2). \tag{3}$$

Let us consider lenses which are systems of point masses. Outside the points where there are masses, we have the following equality (see SEF)

$$\psi_{22} = -\psi_{11} \tag{4}$$

(since Laplace's equation is valid for the potential).

The magnification of an image at $\boldsymbol{x}$ is

$$\mu(\boldsymbol{x}) = 1/\det A(\boldsymbol{x}),$$

since the square for the transformation is determined by the Jaconian matrix. The singular points of the mapping are characterized by

$$\det A(\boldsymbol{x}) = 0$$

(where the mapping is not one-to-one). The set is formed by so-called critical curves (SEF). If $\gamma^{(c)}(\lambda)$ is a critical curve, we can map the curve into the source plane to obtain the corresponding caustic curve (SEF)

$$\mathbf{\Gamma}(\lambda) = \gamma^{(c)}(\lambda) - \boldsymbol{\alpha}(\gamma^{(c)}(\lambda)).$$

Accordingto SEF, we give the geometrical definition for cusps: cusps have the property that the tangent vector of the critical curve is an eigenvector of the corresponding vanishing eigenvalue (we suppose that A is not a zero matrix). It is clear that the tangent vector for cusps has the expression

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbf{\Gamma}^{(c)} = A(\gamma^{(c)}(\lambda))\frac{\mathrm{d}}{\mathrm{d}\lambda}\gamma^{(c)}(\lambda).$$

Since $\det A(\gamma^{(c)}) = 0$, it is therefore possible to find coordinates in which $\phi_{11} \neq 0$, $\phi_{12} = \phi_{22} = 0$. As shown by Schneider and Weiss (1992), the lens equation near a cusp can be represented as

$$\begin{aligned} y_1 &= cx_1 + \frac{b}{2}x_2^2, \\ y_2 &= bx_1x_2 + ax_2^3, \end{aligned} \tag{5}$$

where $a = \frac{1}{6}\phi_{2222}$, $b = \phi_{12}$, $c = \phi_{11}$ and $c \neq 0$, $b \neq 0$, $2ac - b^2 \neq 0$. It was shown in SEF that additional terms do not affect the local properties of the mapping. This can be seen by direct comparison of terms (for example the term with ϕ_{112} is less than the term with ϕ_{11}) or by using Newton's polygon or Bruno's truncating rules (see for example Bruno, 1989). Similarly (Schneider and Weiss, 1992), we introduce the notation

$$\hat{x}_1 = \frac{c}{b}x_1, \quad \hat{x}_2 = x_2, \quad \hat{y}_1 = y_1/b, \quad \hat{y}_2 = \frac{c}{b^2}y_2; \tag{6}$$

then we have

$$\begin{aligned}\hat{y}_1 &= \hat{x}_1 + \frac{1}{2}\hat{x}_2^2,\\ \hat{y}_2 &= \hat{x}_1\hat{x}_2 + s\hat{x}_2^3,\end{aligned}$$

where $s = ac/b^2$. We introduce also $S = 1 - 2s$, $\tilde{y}_1 = 2\hat{y}_1/3S$, $\tilde{y}_2 = \hat{y}_2/S$. Then we have (Schneider and Weiss, 1992)

$$\hat{x}_1 = \frac{3S}{2}\tilde{y}_1 - \frac{1}{2}\hat{x}_2^2, \tag{7}$$

$$\hat{x}_2^3 - 3\tilde{y}_1\hat{x}_2 + 2\tilde{y}_2 = 0. \tag{8}$$

Now we have the most convenient form for solving the gravitational lens equation.

3 SOLUTION OF GRAVITATIONAL LENS EQUATION NEAR THE CUSP

It is easy to see that the caustic near the cusp is described by the semicubical parabola

$$\tilde{y}_1^3 = \tilde{y}_2^2. \tag{9}$$

Let us consider the solution of equation (8). If $\tilde{y}_1^3 > \tilde{y}_2^2$ then the point is inside the cusp region and we have three solutions of (8) (since we have three images for the point). In fact, the discriminant for the pure cubic polynomial is

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = -\tilde{y}_1^3 + \tilde{y}_2^2 < 0. \tag{10}$$

Then we have three solutions of equation (8) (Bronstein and Semendjajew, 1980)

$$\hat{x}_2^{(1)} = -2\mathrm{sign}\,(\tilde{y}_2)\sqrt{\tilde{y}_1}\cos\left\{\frac{\cos^{-1}[\tilde{y}_2\mathrm{sign}\,(\tilde{y}_2)/\sqrt{\tilde{y}_1^3}]}{3}\right\}, \tag{11}$$

$$\hat{x}_2^{(2)} = -2\mathrm{sign}\,(\tilde{y}_2)\sqrt{\tilde{y}_1}\cos\left\{\frac{\cos^{-1}[\tilde{y}_2\mathrm{sign}\,(\tilde{y}_2)/\sqrt{\tilde{y}_1^3}]}{3} + \frac{2\pi}{3}\right\}, \tag{12}$$

$$\hat{x}_2^{(3)} = -2\mathrm{sign}\,(\tilde{y}_2)\sqrt{\tilde{y}_1}\cos\left\{\frac{\cos^{-1}[\tilde{y}_2\mathrm{sign}\,(\tilde{y}_2)/\sqrt{\tilde{y}_1^3}]}{3} + \frac{4\pi}{3}\right\}, \tag{13}$$

We suppose that sign $(0) = 1$. If we consider the case when the point is outside the cusp region, then we have $D > 0$ and we have only one (real) root of equation (8) and we may present the solution in Cardan's form (Bronstein and Semendjajew, 1980)

$$\hat{x}_2^{(1)} = \sqrt[3]{-\tilde{y}_2 + \sqrt{\tilde{y}_2^2 - \tilde{y}_1^3}} + \sqrt[3]{-\tilde{y}_2 - \sqrt{\tilde{y}_2^2 - \tilde{y}_1^3}}. \tag{14}$$

If $\tilde{y}_1^3 = \tilde{y}_2^2$ (the point lies on the caustic curve) then we have one single root and one double root of (8). We suppose that $\tilde{y}_2 > 0$,

$$\hat{x}_2^{(1)} = -2\sqrt{\tilde{y}_1}, \quad \hat{x}_2^{(2)} = \hat{x}_2^{(3)} = \sqrt{\tilde{y}_1}, \tag{15}$$

Similarly we consider the case $\tilde{y}_2 < 0$. If we consider (similar to Schneider and Weiss, 1992) the mapping of the $\tilde{y}_2 = 0$ axis then we have from equations (11–13) (for $D < 0$)

$$\hat{x}_2^{(1)} = -\sqrt{3\tilde{y}_1}, \quad \hat{x}_2^{(1)} = \sqrt{3\tilde{y}_1}, \quad \hat{x}_2^{(1)} = 0. \tag{16}$$

Therefore we have the following expression for the critical curve

$$\hat{x}_1 = (\hat{x}_2)^2/2$$

and we have an expression for the curve which is the other image of the caustic curve at the source plane:

$$\hat{x}_1 = -(\hat{x}_2)^2/4$$

and for $D > 0$ we have from (13)

$$x_2^{(1)} = 0.$$

The expressions for $x_1^{(i)}$ are obtained from equation (7).

4 STATEMENT OF THE MAGNIFICATIONS OF IMAGES NEAR CUSPS

We recall that (Schneider and Weiss, 1992)

$$\hat{\mu} = (\det \hat{A})^{-1}, \quad \hat{\mu} = b^2\mu, \tag{17}$$

$$\det A = b^2 \det \hat{A} = b^2[\hat{x}_1 + (3s-1)\hat{x}_2^2]. \tag{18}$$

Consider the magnifications for different images of one point inside the cusp. This is valid following the equality

$$\hat{\mu}^{(1)} + \hat{\mu}^{(2)} + \hat{\mu}^{(3)} = 0 \tag{19}$$

for all sources inside the cusp. It is clear that from equation (19) we have the statement of Schneider and Weiss (1992) that

$$|\hat{\mu}^{(1)}| = |\hat{\mu}^{(2)} + \hat{\mu}^{(3)}|. \tag{20}$$

We use the following expression for the magnification:

$$\mu^{(i)} = \frac{1}{\hat{x}_i^{(i)} + (3s-1)(\hat{x}_2^{(i)})^2} \tag{21}$$

or using equation (7)

$$\mu^{(i)} = \frac{2}{S[\tilde{y}_1 - (\hat{x}_2^{(i)})^2]}. \tag{22}$$

Therefore it is necessary to prove that

$$\frac{1}{\tilde{y}_1 - (\hat{x}_2^{(i)})^2} + \frac{1}{\tilde{y}_1 - (\hat{x}_2^{(i)})^2} + \frac{1}{\tilde{y}_1 - (\hat{x}_2^{(i)})^2} = 0 \tag{23}$$

or that

$$3\tilde{y}_1^2 - 2\tilde{y}_1[(\hat{x}_2^{(1)})^2 + (\hat{x}_2^{(2)})^2 + (\hat{x}_2^{(3)})^2] + [(\hat{x}_2^{(1)}\hat{x}_2^{(2)})^2 + (\hat{x}_2^{(1)}\hat{x}_2^{(3)})^2 + (\hat{x}_2^{(2)}\hat{x}_2^{(3)})^2 = 0. \tag{24}$$

Using Viet's theorem we have

$$\hat{x}_2^{(1)} + \hat{x}_2^{(2)} + \hat{x}_2^{(3)} = 0, \quad \hat{x}_2^{(1)}\hat{x}_2^{(2)} + \hat{x}_2^{(1)}\hat{x}_2^{(3)} + \hat{x}_2^{(2)}\hat{x}_2^{(3)} = -3\tilde{y}_1.$$

If we express the symmetric power polynomials in terms of symmetric elementary polynomials we have

$$(\hat{x}_2^{(1)})^2 + (\hat{x}_2^{(2)})^2 + (\hat{x}_2^{(3)})^2 = 6\tilde{y}_1, \tag{25}$$

$$(\hat{x}_2^{(1)}\hat{x}_2^{(2)})^2 + (\hat{x}_2^{(1)}\hat{x}_2^{(3)})^2 + (\hat{x}_2^{(2)}\hat{x}_2^{(3)})^2 = 9\tilde{y}_1^2, \tag{26}$$

Thus we obtain (24)

Similarly we have

$$q_1 = \hat{\mu}^{(1)}\hat{\mu}^{(2)}\hat{\mu}^{(3)} = \frac{2}{(3S)^3(\tilde{y}_1^3 - \tilde{y}_2^2)}, \tag{27}$$

$$p_1 = \hat{\mu}^{(1)}\hat{\mu}^{(2)} + \hat{\mu}^{(1)}\hat{\mu}^{(3)} + \hat{\mu}^{(2)}\hat{\mu}^{(3)} = \frac{-3\tilde{y}_1}{(3S)^2(\tilde{y}_1^3 - \tilde{y}_2^2)}. \tag{28}$$

Therefore we have an equation of third degree for magnifications near different images of a point inside the cusp:

$$\tilde{\mu}^3 + \tilde{p}_1\tilde{\mu} + \tilde{q}_1 = 0 \tag{29}$$

where $\tilde{\mu} = \hat{\mu}/S$, $\tilde{p}_1 = p_1(3S)^2$, $\tilde{q}_1 = q_1(3S)^3$. Similarly to the previous considerations we obtain expressions for the roots of the equation for a point inside the cusp:

$$\tilde{\mu}^{(1)} = -\frac{2}{3}\sqrt{\frac{\tilde{y}_1}{\tilde{y}_1^3 - \tilde{y}_2^2}} \cos\left\{\cos^{-1}\left[\sqrt{\frac{\tilde{y}_1^3 - \tilde{y}_2^2}{\tilde{y}_1^3}}\right]/3\right\}, \tag{30}$$

$$\tilde{\mu}^{(2)} = -\frac{2}{3}\sqrt{\frac{\tilde{y}_1}{\tilde{y}_1^3 - \tilde{y}_2^2}} \cos\left\{\cos^{-1}\left[\sqrt{\frac{\tilde{y}_1^3 - \tilde{y}_2^2}{\tilde{y}_1^3}}\right]/3 + 2\pi/3\right\}, \tag{31}$$

$$\tilde{\mu}^{(3)} = -\frac{2}{3}\sqrt{\frac{\tilde{y}_1}{\tilde{y}_1^3 - \tilde{y}_2^2}} \cos\left\{\cos^{-1}\left[\sqrt{\frac{\tilde{y}_1^3 - \tilde{y}_2^2}{\tilde{y}_1^3}}\right]/3 + 4\pi/3\right\}, \tag{32}$$

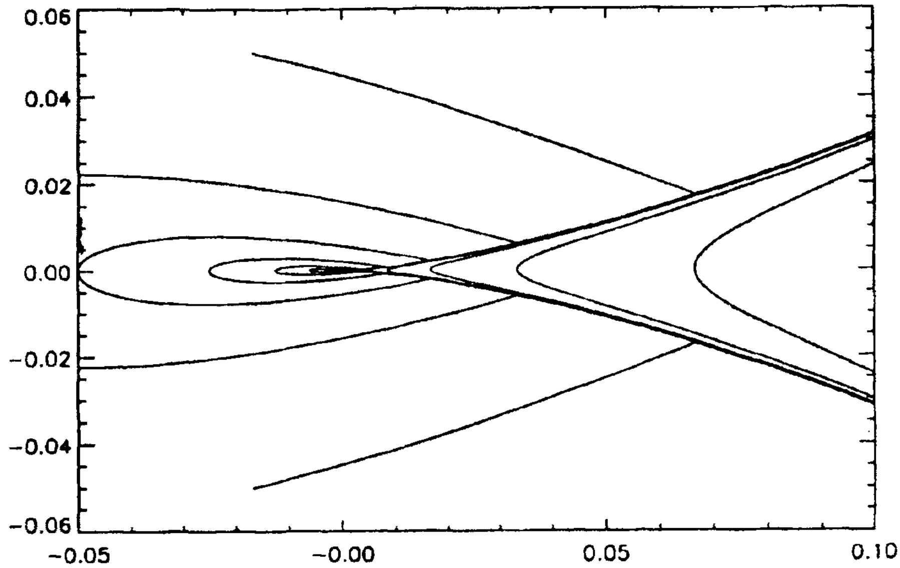

Figure 1 Magnification contours around a cusp for the sum of absolute values of magnifications of all images. The contour level range 10×2^i, $\forall_i \in [-1, 14]$, $i \in Z$.

and for points outside the cusp:

$$\tilde{\mu}^{(1)} = \left[\sqrt[3]{\sqrt{\tilde{y}_2^2 - \tilde{y}_1^3} + \tilde{y}_2} + \sqrt[3]{\sqrt{\tilde{y}_2^2 - \tilde{y}_1^3} - \tilde{y}_2}\right] / \sqrt{\tilde{y}_2^2 - \tilde{y}_1^3}. \qquad (33)$$

5 DISCUSSION

In Figure 1 we present the distribution of lines of equal magnification around a cusp for different images and the sum of absolute values of magnifications of images (we recall that the algebraic sum of the values is zero). We obtained the contours using expressions (30–33). It is possible to compare the figures with a similar figure from Schneider and Weiss (1992). We recall that Figure 5 in their paper was obtained using the ray-shooting method (see, for example, Wambsganss, 1990). As expected, we see from Figure 1 that the magnification near the cusp point increases much more strongly than near the fold singularities (this follows, of course, from expressions (30–33)).

Acknowledgements

We would like to express our gratitude to the Organizing Committee of the Conference on Current Problems in Astrophysics in memory of three distingushed astrophysicists: Prof. I. S. Shklovsky, Prof. S. A. Kaplan and Prof. S. B. Pikelner for their invitation to present the paper at the Conference.

This research has been supported in part by the Russian Foundation of Fundamental Research (grant No. 96–02–17434).

References

Arnold, V. I. (1972) *Funct. Anal. Appl.* **6**, 619 (in Russian).

Arnold, V. I. (1976) *Comm. Pure Appl. Math.* **29**, 557.

Arnold, V. I. (1979) *Mathematical Methods of Classical Mechanics*, Nauka, Moscow (in Russian).

Arnold, V. I. (1983) *Uspehi mat. nauk.* **38**, 77 (in Russian).

Bronstein, I. N. and Semendjajew, K. A. (1980) *Taschenbuch der Mathematik*, Verlag Harri Deutsch Thun und Frankfurt/Main.

Bruno, A. D. (1989) *Local Methods in Nonlinear Differential Equations*, Springer Verlag, Berlin.

Mandzhos, A. V. (1995) *Astron. Zh.* **72**, 146.

Schneider, P., Ehlers, J., and Falco, E. (1992) *Gravitational Lenses*, Springer, New York.

Schneider, P. and Weiss, A. (1986) *Astron. Astrophys.* **164**, 237.

Schneider, P. and Weiss, A. (1992) *Astron. Astrophys.* **260**, 1.

Wambsganss, J. (1990) *Gravitational Microlensing*, Dissertation der Fakultät für Physik der Ludwig – Maximilians – Universität, München.