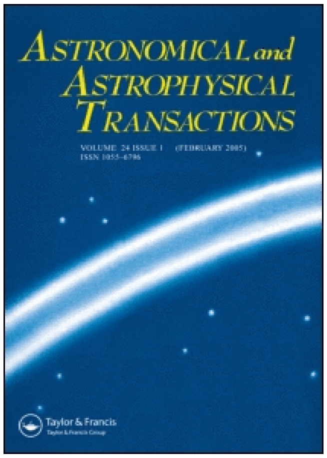

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THE MOTION OF THE EARTH'S PRINCIPAL AXES OF INERTIA, CAUSED BY TIDAL AND ROTATIONAL DEFORMATIONS

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Perturbations of the pole position of the Earth's axes of inertia due to its tidal and rotational deformations have been found.

KEY WORDS Earth's rotation, motion of the principal axes of inertia, tidal variation

1 INTRODUCTION

The Earth's mass redistribution generates temporal variations of the coefficients of geopotential, of the components of the tensor of inertia and as result leads to the variation of the orientation in its principal axis of inertia. The purpose of this paper is to study the motion of the Earth's principal axis of inertia $C\xi\eta\zeta$ due to tidal deformations from the Moon's and Sun's attraction, and also due to its rotational deformations. To obtain corresponding corrections to values of the geopotential, coefficients are determined with respect to the reference system $C\xi\eta\zeta$.

One investigation of this problem was made by Bursa (1983). In this paper we will use Ferrandiz and Getino's paper (1993) in which in details of the periodic tidal variations of the coefficients of the second harmonic of the gopotential were described. The authors use the classical model of the Earth with an elastic mantle (model Takeuchi 2). For a description of the rotational deformations of the second harmonic of the geopotential in this paper classical expressions for the rotational deformations of the component of the tensor inertia are used.

This investigation has an important significance for the determination of the main reference systems for different geodynamical studies, in satellite dynamics and for the theory of the rotation of the deformable Earth.

Table 1. Secular variations of the zonal coefficients of the geopotential ($\times 10^{-9}$ 1/cy)

Authors, years	*Data of observations*	$\dot{J}_2$	$\dot{J}_3$	$\dot{J}_4$
Yoder *et al.*, 1983	Lageos, 5,5y.	-3.0 ± 0.3	-1.0	–
Rubincam, 1984	Lageos, 6,0y.	-2.6 ± 0.6	–	–
Cheng *et al.*, 1989	Starllete, 3y.	-2.5 ± 0.1	-0.1 ± 0.3	0.3 ± 0.6
Schwintzer *et al.*,	27 sat. gravity	-6.7 ± 1.6	–	–
Marchenko, 1992	Etalon (21 months) (6 months)	-2.6 ± 0.4	–	–
Marchenko, 1992	Lageos (6y)	-2.7 ± 0.7	-0.1 ± 0.6	0.3 ± 0.8

2 TEMPORAL VARIATIONS OF THE GEOPOTENTIAL COEFFICIENTS

The Earth's dynamical structure essentialy changes in time. The Earth experiences tidal and centrifugal (rotational) deformations, different kinds of geophysical and geodynamical changes in the atmosphere and tectonosphere. These processes lead to temporal variations of the geopotential coefficients. The dependence of these variations an the character and nature of the Earth's mass redistribution can be secular, periodic or long-periodic.

Experimentally, on the basis of a long series of highly accurate satellite observations the secular variation of the J_2 parameter of the Earth's gravitational field was detected (Yoder *et al.*, 1983; Chang *et al.*, 1989 and others).

In Table 1 (Marchenko, 1992) modern data about the experimental definition of the secular variation of the some zonal harmonic coefficients of the geopotential are given (obtained by different authors in recent years). Theoretical explanation of the observed secular variations J_n and predictions of analogous variations of the other coefficients by the second and third harmonics of the geopotential is an important scientific problem. The principal role here is played by the following mechanisms: sea level change, ice rebound, the lithosphere plates motion, etc.

In the paper by Ferrandiz and Getino (1993) the periodical variations of the coefficients of the second harmonic of the geopotential due to tidal attraction of the Moon and Sun were studied. Their investigations have a theoretical character and use a classical Earth model with elastic mantle. The periodic variations of the standard coefficients J_2, C_{22}, S_{12}, C_{21}, S_{21} of the geopotential were presented in the following form:

$$\begin{aligned}
\delta J_2 &= \sum_i K_2(i) \cos \Theta_i \\
\delta C_{22} &= \sum_i K_{22a}(i) \cos(2\mu + 2\nu - \Theta_i) + \sum_i K_{22b}(i) \cos(2\mu + 2\nu + \Theta_i) \\
\delta S_{22} &= \sum_i -K_{22a}(i) \sin(2\mu + 2\nu - \Theta_i) + \sum_i -K_{22b}(i) \sin(2\mu + 2\nu + \Theta_i)
\end{aligned}$$

Table 2. Main tidal periodic variations of the geopotential coefficients ($\times 10^{-9}$)

Period	l_S	l_M	F	D	Ω	K_{22a}	K_{22b}	K_{21a}	K_{21b}	
-31.81	1	0	0	-2	0	0.2033	0.0053	0.0053	0.0486	0.0486
27.55	1	0	0	0	0	1.0631	0.0276	0.0276	0.2543	0.2543
14.77	0	0	0	2	0	0.1764	0.0046	0.0046	0.0422	0.0422
9.12	1	0	2	0	1	0.1597	-0.0278	0.0012	-0.1167	0.0171
13.63	0	0	2	0	1	0.8342	-0.1453	0.0063	-0.6099	0.0891
-6798.36	0	0	0	0	1	-0.8433	0.1469	-0.0063	0.6167	-0.0901
9.56	-1	0	2	2	2	0.0732	0.1417	0.0003	-0.1176	0.0051
27.09	-1	0	2	0	2	-0.0569	-0.1102	-0.0002	0.0914	-0.0039
9.13	1	0	2	0	2	0.3854	0.7462	0.0014	-0.6192	0.0266
7.10	0	0	2	2	2	0.0615	0.1192	0.0002	-0.0989	0.0043
13.66	0	0	2	0	2	2.0124	3.8976	0.0072	-3.2339	0.1391
365.26	0	1	0	0	0	0.1487	0.0039	0.0039	0.0356	0.0356
121.75	0	1	2	-2	2	0.0548	0.1060	0.0002	-0.0880	-0.0009
182.62	0	0	2	-2	2	0.9342	1.8090	0.0034	-1.5011	-0.0160

$$
\begin{aligned}
\delta C_{21} &= \sum_i K_{21a}(i)\sin(\mu+\nu-\Theta_i) + \sum_i K_{21b}(i)(\mu+\nu+\Theta) \\
\delta S_{21} &= \sum_i K_{21a}(i)\cos(\mu+\nu-\Theta_i) + \sum_i K_{21b}(i)(\mu+\nu+\Theta)
\end{aligned} \tag{1}
$$

where $K_{(\dots)}(i)$ are numerical coefficients (they are given in Table 2),

$$\Theta_i = m_1 m_l l_M + m_2 l_S + m_3 F + m_4 D + m_5 \Omega$$

is a linear combination with numerical coefficients of the classical arguments of the Moon's orbital theory:

$$F = l_M + g_M, \quad D = l_M + g_M + h_M - l_S - g_S - h_S,$$

and l_M, g_M, h_M and l_S, g_S, h_S are Delanay variables for the Moon's and Sun's orbital motions; these are linear functions of time. μ, ν are Andoyer's variables, describing the daily Earth rotation with frequency $n_\nu + n_\mu \approx \omega$ (ω is the mean value of the Earth's angular velocity).

Variations of the parameters (1) are considerable and characterized by definite periods. For the J_2 parameter these periods are given in Table 2 and are from a few days to 18.6 y. For others parameters (from formula (1)) these variations are quasidaily.

These variations of the parameters (1) were detected in the principal axes of inertia of the Earth in its undeformed position.

Redistribution of the Earth's mass leads to definite displacements of the Earth's principal axes of inertia (in the general case the orientation of these axes and position of the centre of mass of the Earth will by changed).

The Earth's deformations, caused by its rotation, also lead to additional variations of the geopotential coefficients. Let $\overline{\omega}$ is the vector of the angular velocity of the reference system $Cxyz$; p, q and r are the components of this vector in the axes Cx, Cy and Cz, respectively.

Using the classical expressions of the variations of the components of the tensor of inertia for an elastic body, caused by rotational deformation for corresponding variations of the coefficients of the Earth's second harmonic, we will have the following representations (Bursa, 1983; Ferrandiz and Getino, 1991):

$$\begin{aligned}
\delta C_{20} &= \frac{3D_1}{2mR^2}(2r^2 - p^2 - q^2) \\
\delta C_{22} &= \frac{3D_1}{4mR^2}(p^2 - q^2) \\
\delta S_{22} &= \frac{3D_1}{2mR^2}pq \\
\delta S_{21} &= \frac{3D_1}{mR^2}qr \\
\delta C_{21} &= \frac{3D_1}{mR^2}pr
\end{aligned} \tag{2}$$

where D_1 is a coefficient which characterizes the elastic properties of the Earth's mantle, and m and R are the mass and mean radius of the Earth. In (2) p, q, r are functions of time.

The values of the components of the inertia tensor and of the coefficients of the geopotential C_{nm}, S_{nm} depend on the reference system. The main interest in these values of present is for the principal axes of inertia; this is an important question for different dynamical investigations and for the correct interpretation of the corresponding results.

Therefore in the following paragraphs we will consider the questions of the principal moments of inertia of a deformable body, and the parameters C_{nm}, S_{nm} in the principal axes of inertia; we will also obtain some additional corrections to the values of the principal moments of inertia of the Earth and its parameters C_{20}, C_{22}.

3 VALUES OF THE EARTH'S GEOPOTENTIAL COEFFICIENTS IN ITS PRINCIPAL AXES OF INERTIA

The orientation of the principal axes of the Earth $C\xi\eta\zeta$ about the reference system of the Earth $Cxyz$ will be represented by Eulerian angles ψ, θ, φ and by the system of direction cosines:

$$\begin{aligned}
a_{11} &= \cos(\xi, x) = \cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta \\
a_{21} &= \cos(\xi, y) = \sin\psi\cos\varphi + \cos\psi\sin\varphi\cos\theta \\
a_{31} &= \cos(\xi, z) = \sin\varphi\sin\theta \\
a_{12} &= \cos(\eta, x) = -\cos\psi\sin\varphi - \sin\psi\cos\varphi\cos\theta \\
a_{22} &= \cos(\eta, y) = -\sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta
\end{aligned}$$

$$
\begin{aligned}
a_{32} &= \cos(\eta, z) = \cos\varphi \sin\theta \\
a_{13} &= \cos(\zeta, x) = \sin\psi \sin\theta \\
a_{23} &= \cos(\zeta, y) = -\cos\psi \sin\theta \\
a_{33} &= \cos(\zeta, z) = \cos\theta
\end{aligned} \tag{3}
$$

where ψ is the precession angle, θ is the angle of nutation, and φ is the angle of rotation.

The components of the inertia tensor of the Earth in the $Cxyz$ axes are defined by the formulae:

$$
\begin{aligned}
A &= \int\!\!\int\!\!\int \rho(y^2 + z^2)\, d\sigma \\
B &= \int\!\!\int\!\!\int \rho(x^2 + z^2)\, d\sigma \\
C &= \int\!\!\int\!\!\int \rho(x^2 + y^2)\, d\sigma \\
F &= \int\!\!\int\!\!\int \rho xy\, d\sigma \\
E &= \int\!\!\int\!\!\int \rho xz\, d\sigma \\
D &= \int\!\!\int\!\!\int \rho yz\, d\sigma
\end{aligned} \tag{4}
$$

where ρ is the Earth's density, $d\sigma$ is an elementary volume, and the integral is extended to the whole volume of the Earth.

The components of the inertia tensor (4) are connected with the coefficients of the second harmonic of the geopotential by the relations:

$$
\begin{aligned}
&J_2 = -C_{20} = \frac{2C - A - B}{2mR^2}, \quad C_{22} = \frac{B - A}{4mR^2} \\
&S_{21} = \frac{D}{mR^2}, \quad S_{22} = \frac{F}{2mR^2}, \quad C_{21} = \frac{E}{mR^2}
\end{aligned} \tag{5}
$$

where m and R is the mass and the radius of the Earth.

With respect to the principal axes of inertia the components of the inertia tensor are

$$
\begin{aligned}
A_p &= \int\!\!\int\!\!\int \rho(\eta^2 + \zeta^2)\, d\sigma \\
B_p &= \int\!\!\int\!\!\int \rho(\xi + \zeta^2)\, d\sigma \\
C_p &= \int\!\!\int\!\!\int \rho(\xi^2 + \eta^2)\, d\sigma \\
F_p &= \int\!\!\int\!\!\int \rho\xi\eta\, d\sigma
\end{aligned}
$$

$$
\begin{aligned}
E_p &= \int\int\int \rho\xi\zeta \, d\sigma \\
D_p &= \int\int\int \rho\eta\zeta \, d\sigma
\end{aligned}
\tag{6}
$$

so the axes $C\xi\eta\zeta$ are principal, as here

$$F_p = E_p = D_p = 0.$$

The parameters of the second harmonic of the geopotential of the Earth about the axes $C\xi\eta\zeta$ are defined by the formulae:

$$
\begin{aligned}
J_2^p &= -C_{20}^p = \frac{2C_p - A_p - B_p}{2mR^2} \\
C_{22}^p &= \frac{B_p - A_p}{4mR^2} \\
S_{22}^p &= \frac{F_p}{2mR^2} = 0 \\
C_{21}^p &= \frac{E_p}{mR^2} = 0 \\
S_{21}^p &= \frac{D_p}{mR^2} = 0.
\end{aligned}
\tag{7}
$$

The Cartesian coordinates (x, y, z) of the elementary volume of the body of the Earth $d\sigma$ about axes $Cxyz$ and $(\xi,\ \eta,\ \zeta)$ about axes $C\xi\eta\zeta$, which are in formulae (4) and (6), are connected by the following transformation formulae:

$$
\begin{aligned}
x &= a_{11}\xi + a_{12}\eta + a_{13}\zeta \\
y &= a_{21}\xi + a_{22}\eta + a_{23}\zeta \\
z &= a_{31}\xi + a_{32}\eta + a_{33}\zeta.
\end{aligned}
\tag{8}
$$

Substituting formulae (8) in to integrals (4), after some algebra, we obtain expressions for the components of the inertia tensor for non-principal axes through the principal moments (Tisserand, 1996):

$$
\begin{aligned}
A &= a_{11}^2 A_p + a_{12}^2 B_p + a_{13}^2 C_p \\
B &= a_{21}^2 A_p + a_{22}^2 B_p + a_{23}^2 C_p \\
C &= a_{31}^2 A_p + a_{32}^2 B_p + a_{33}^2 C_p \\
-F &= a_{11}a_{21} A_p + a_{12}a_{22} B_p + a_{13}a_{23} C_p \\
-E &= a_{31}a_{11} A_p + a_{32}a_{12} B_p + a_{33}a_{13} C_p \\
-D &= a_{21}a_{31} A_p + a_{22}a_{32} B_p + a_{23}a_{23} C_p
\end{aligned}
\tag{9}
$$

Using well-known properties of the direction cosines (3), we can describe formulae (9) in the following form:

$$
\begin{aligned}
A &= A_p + (B_p - A_p)a_{12}^2 + (C_p - A_p)a_{13}^2 \\
B &= B_p + (A_p - B_p)a_{21}^2 + (C_p - B_p)a_{23}^2
\end{aligned}
$$

$$\begin{aligned}
C &= C_p + (A_p - C_p)a_{31}^2 + (B_p - C_p)a_{32}^2 \\
D &= (A_p - B_p)a_{22}a_{32} + (A_p - C_p)a_{23}a_{33} \\
E &= (B_p - A_p)a_{31}a_{11} + (B_p - C_p)a_{33}a_{13} \\
F &= (C_p - A_p)a_{11}a_{21} + (C_p - B_p)a_{12}a_{22}
\end{aligned} \tag{10}$$

Substituting formulae (10) into (5) we obtain analogous relations between the geopotential coefficients (5) and (7):

$$\begin{aligned}
C_{20} &= \frac{1}{2}C_{20}^p(3a_{33}^2 - 1) + C_{22}^p[a_{13}^2 - a_{23}^2 + 2(a_{31}^2 - a_{32}^2 + a_{12}^2 - a_{21}^2)] \\
C_{22} &= \frac{1}{4}(a_{33}^2 - 1)C_{20}^p + C_{22}^p\left[\frac{1}{2}(1 + a_{33}^2) - a_{21}^2 - a_{12}^2\right] \\
S_{21} &= -4C_{22}^p a_{22}a_{32} + (C_{20}^p - 2C_{22}^p)a_{23}a_{13} \\
C_{21} &= 4C_{22}^p a_{31}a_{11}a_{32} + (C_{20}^p - 2C_{22}^p)a_{33}a_{23}a_{13} \\
S_{22} &= 2\left\{(C_{20}^p - 2C_{22}^p)a_{11}a_{21} - (C_{20}^p + 2C_{22}^p)a_{12}a_{22}\right\}.
\end{aligned} \tag{11}$$

The direction cosines of the principal axes of inertia are defined by the following system of linear homogeneous equations:

$$\begin{aligned}
(A - J_i)a_{1i} - Fa_{2i} - Ea_{3i} &= 0 \\
-Fa_{1i} + (B - J_i)a_{2i} - Da_{3i} &= 0 \\
-Ea_{1i} - Da_{2i} + (C - J_i)a_{3i} &= 0
\end{aligned} \tag{12}$$

where $J_i = (A_p, B_p, C_p)$ $(i = 1, 2, 3)$.

The equations (12) are linearly dependent and admit solutions only for three values of J, which are the principal moments of inertia, defined as the roots of the cubic equation:

$$J^3 + aJ^2 + bJ + cC = 0$$

where

$$\begin{aligned}
a &= -(A + B + C) \\
bD &= AB + AC + BC - F^2 - D^2 - E^2 \\
cC &= -ABC + 2FED + AD^2 + BE^2 + CF^2.
\end{aligned}$$

By Cordano's formula we find:

$$\begin{aligned}
A_p &= \frac{A + B + C}{3} - 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\gamma}{3} + \frac{\pi}{3}\right) \\
B_p &= \frac{A + B + C}{3} - 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\gamma}{3} - \frac{\pi}{3}\right) \\
C_p &= \frac{A + B + C}{3} + 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\gamma}{3}\right)
\end{aligned}$$

where

$$p = -\frac{1}{6}(A-C)^2 - \frac{1}{6}(A-B)^2 - \frac{1}{6}(C-B)^2 - (F^2 + E^2 + D^2)$$

and the value of the angle γ is defined by

$$\cos\gamma = \frac{-q}{2\sqrt{-\left(\frac{p}{3}\right)^3}}$$

where

$$\begin{aligned} q &= \frac{1}{27}\Big[A^2(2A-3B-3C) + B^2(2B-3A-3C) + C^2(2C-3A-3B) \\ &+ 12ABC - 54FED + 9A(F^2+E^2-2D^2) + 9B(D^2+F^2-2E^2) \\ &+ 9C(E^2+D^2-2F^2)\Big]. \end{aligned}$$

Using the natural assumption about small inclinations of the corresponding axes of the systems $C\xi\eta\zeta$ and $Cxyz$ we obtain approximate expressions for the principal moment of inertia. In this case the moments F, E, D are small (with respect to the values A, B, C) and we can obtain the solution of the cubic equation by the method of successive approximations. So with a third-order accuracy with respect the values F, E, D the expressions of the principal moments of inertia have the form:

$$\begin{aligned} A_p &= A + \frac{F^2}{A-B} + \frac{E^2}{A-C} - \frac{2FED}{(A-C)(A-B)} \\ B_p &= B + \frac{F^2}{B-A} + \frac{D^2}{B-C} - \frac{2FED}{(B-C)(B-A)} \\ C_p &= C + \frac{F^2}{C-A} + \frac{D^2}{C-B} - \frac{2FED}{(C-B)(C-A)}. \end{aligned} \tag{13}$$

It follows from Table 2 that tidal variations of the products of inertia for the Earth have order $\approx 10^{-9}C$. On other hand the differences of the axial moments of inertia of the Earth are of the order:

$$C - A \approx 10^{-3}\,C, \quad C - B \approx 10^{-3}\,C, \quad B - A \approx 10^{-6}\,C \tag{14}$$

These values follow from relations (5)

$$B - C = (C_{20} + 2C_{22})mR^2, \quad C - A = (2C_{22} - C_{20})mR^2, \quad B - A = 4C_{22}mR^2.$$

For example, in accordance with the model of the Earth's gravitational field GEM6

$$C_{20} = -1.0826 \times 10^{-3}, \quad C_{22} = 1.565 \times 10^{-6}$$

and evaluations (14) hold.

So, the second terms in the expressions A_p, B_p (13) have order $\approx 10^{-12}mR^2$, the third have order $\approx 10^{-15}mR^2$, and the last have order $\approx 10^{-18}mR^2$.

The second and third terms in the expression for C_p have order $10^{-15}mR^2$, and the last has order $10^{-18}mR^2$.

As result here we have following approximate formulae for the principal moment of inertia:

$$A_p = A + \frac{F^2}{A-B}, \quad B_p = B + \frac{F^2}{B-A}, \quad C_p = C \tag{15}$$

or using formulae (5)

$$\begin{aligned} A_p &= A - \frac{S_{22}^2}{C_{22}} mR^2 \\ B_p &= B + \frac{S_{22}^2}{C_{22}} mR^2 \\ C_p &= C. \end{aligned}$$

Similar expressions for the parameters J_2^p, C_{22}^p are characterized by the following simple form:

$$J_2^p = J_2 = -C_{20}, \quad C_{22}^p = C_{22} + \frac{S_{22}^2}{2C_{22}}. \tag{16}$$

Formulae (16) allows us to obtain the temporal variation of the parameter C_{22}^p. Substituting expression δS_{22} (1) formula (16) we have:

$$\begin{aligned} \delta C_{22}^p = \delta C_{22} &+ \frac{1}{2C_{22}^0} \sum_i \sum_m \Bigg\{ \Bigg[K_{22a}(i)K_{22b}(i-m) + \frac{1}{2}K_{22a}(i)K_{22a}(i-m) \\ &+ \frac{1}{2}K_{22b}(i)K_{22b}(i-m) \Bigg] \cos\Theta_m \\ &+ \Bigg[-K_{22a}(i)K_{22b}(i+m) - \frac{1}{2}K_{22a}(i)K_{22a}(-i-m) \\ &- \frac{1}{2}K_{22b}(i)K_{22b}(m-i) \Bigg] \cos(4\mu + 4\nu + \Theta_m) \Bigg\}. \end{aligned} \tag{17}$$

From formula (22) we find that the variation of the parameter C_{22}^p contains additional terms: a constant (given by $m = 0$) and a periodic term with period $T = 2\pi/(4\omega + \dot{\Theta}_m)$. Here we also have terms with periods that don't depend on the Earth's rotation.

For the main terms in the variation δC_{22}^p (22) we obtain the following additional terms to δC_{22}:

$$\begin{aligned} (\delta C_{22})_{\mathrm{ad}} &= 0.0026 \times 10^{-9} + 0.0020 \times 10^{-9} \cos 2D \\ &- 0.0021 \times 10^{-9} \cos(4\mu + 4\nu - 4F - 4\Omega) \\ &- 0.0005 \times 10^{-9} \cos(4\mu + 4\nu - 4F + 4D - 4\Omega) \\ &- 0.0020 \times 10^{-9} \cos(4\mu + 4\nu - 4F - 4\Omega + 2D). \end{aligned} \tag{18}$$

The constant term in expression (18) gives the constant components of the differences of the principal and non-principal moments of inertia:

$$\begin{aligned} C_p - A_p &= C - A + 0.0052 \times 10^{-9} mR^2 \\ C_p - B_p &= C - B - 0.0052 \times 10^{-9} mR^2 \\ B_p - A_p &= B - A + 0.0104 \times 10^{-9} mR^2. \end{aligned}$$

4 THE EARTH'S POLE MOTION CAUSED BY ITS TIDAL DEFORMATIONS

Equations (12) are easy to solve with respect direction cosines:

$$\begin{aligned} a_{1i} &= \frac{DE + F(C - J_i)}{\Delta_i} \\ a_{2i} &= \frac{(ED + F(C - J_i))(D(A - J_i) + FE)}{(E(B - J_i) + FD)\Delta_i} \\ a_{3i} &= \frac{D(A - J_i) + FE}{\Delta_i} \end{aligned} \tag{19}$$

where

$$\begin{aligned} \Delta_i^2 &= D^2(A - J_i)^2 + E^2(B - J_i)^2 + F^2(C - J_i)^2 \\ &+ 2FED(A + B + C - 3J_i) + F^2E^2 + F^2D^2 + E^2D^2. \end{aligned} \tag{20}$$

Substituting in formulae (19), (20) the values of the principal moments of inertia $J_i = (A_p, B_p, C_p)$ defined by the approximate formulae (15), after some reduction we obtain the following approximate expressions for the direction cosines:

$$\begin{array}{lll} a_{11} \cong 1, & a_{21} \cong \dfrac{F}{B_0 - A_0}, & a_{31} \cong \dfrac{E}{C_0 - A_0} \\ a_{12} \cong \dfrac{F}{A_0 - B_0}, & a_{22} \cong 1, & a_{32} \cong \dfrac{D}{C_0 - B_0} \\ a_{13} \cong \dfrac{E}{A_0 - C_0}, & a_{23} \cong \dfrac{D}{B_0 - C_0}, & a_{33} \cong 1. \end{array} \tag{21}$$

Now on the basis of formulae (8), (21) we find the following approximate values of the Cartesian coordinates of the Earth's poles of the inertia axes $C\xi$, $C\eta$ and $C\zeta$ in the coordinate system $Cxyz$:

$$\begin{array}{lll} x_\xi \cong R, & y_\xi \cong \dfrac{RF}{B_0 - A_0}, & z_\xi \cong \dfrac{RE}{C_0 - A_0} \\ x_\eta \cong \dfrac{RF}{A_0 - B_0}, & y_\eta \cong R, & z_\eta \cong \dfrac{RD}{C_0 - B_0} \\ x_\zeta \cong \dfrac{RE}{A_0 - C_0}, & y_\zeta \cong \dfrac{RD}{B_0 - C_0}, & z_\zeta \cong R. \end{array} \tag{22}$$

Let us now give a description in detail of the poles of inertia P_ζ and P_ξ, caused by tidal deformations of the Earth, with respect to the reference system $Cxyz$, which is the system of principal axes in the undeformed state of the Earth.

Using the relations (5) we describe the formulae for the coordinates of the poles P_ζ and P_ξ (22) in the following form:

$$\begin{aligned} x_\zeta &= \frac{-R\delta C_{21}}{2C_{22}^0 - C_{20}^0}, \quad y_\zeta = \frac{R\delta S_{21}}{C_{20}^0 + 2C_{22}^0} \\ y_\zeta &= \frac{R\delta S_{22}}{2C_{22}^0}, \qquad z_\zeta = \frac{R\delta C_{21}}{2C_{22}^0 - C_{20}^0} \end{aligned} \tag{23}$$

where C_{20}^0, C_{22}^0 are constant values of the parameters C_{20}, C_{22} for the undeformed Earth, and δC_{21}, δS_{21}, δS_{22} are tidal variation, defined by formulae (1).

We will use the parameters of the Earth's gravitational field from model SEIII and reduce it to their principal axes of the undeformed Earth. The Cx axis forms an angle of 14°5 with the corresponding Greenwich axis and displaced to the West. In the reference system $Cxyz$, corresponding values of the parameters C_{20} and C_{22} are:

$$C_{20}^0 = -1082.6370 \times 10^{-6}, \quad C_{22}^0 = 1.7711 \times 10^{-6}.$$

Let us now substitute formulae (1) into (23). As a result we obtain the final formulae for the coordinates of the poles of the polar and equatorial axes of inertia:

$$\begin{aligned} x_\zeta &= \sum_i x_{21a}^\zeta(i)\sin(\mu+\nu-\Theta_i) + \sum_i x_{21b}^\zeta(i)\sin(\mu+\nu+\Theta_i) \\ y_\zeta &= \sum_i y_{21a}^\zeta(i)\cos(\mu+\nu-\Theta_i) + \sum_i y_{21b}^\zeta(i)\cos(\mu+\nu+\Theta_i) \\ y_\xi &= \sum_i y_{21a}^\xi(i)\sin(2\mu+2\nu-\Theta_i) + \sum_i y_{21b}^\xi(i)\sin(2\mu+2\nu+\Theta_i) \\ z_\xi &= \sum_i z_{21a}^\xi(i)\sin(\mu+\nu-\Theta_i) + \sum_i z_{21b}^\xi(i)\sin(\mu+\nu+\Theta_i) \end{aligned} \tag{24}$$

where the coefficients of the series are defined in the following way:

$$\begin{aligned} x_{21a}^\zeta(i) &= -z_{21a}^\xi(i) = -\frac{RK_{21a}(i)}{2C_{22}^0 - C_{20}^0} \\ x_{21b}^\zeta(i) &= -z_{21b}^\xi(i) = -\frac{RK_{21b}(i)}{2C_{22}^0 - C_{20}^0} \\ y_{21a}^\zeta(i) &= \frac{RK_{21a}(i)}{2C_{22}^0 + C_{22}^0}, \quad y_{21b}^\zeta(i) = \frac{RK_{21b}(i)}{C_{20}^0 + 2C_{22}^0}, \\ y_{22a}^\xi(i) &= -\frac{RK_{22a}(i)}{2C_{22}^0}, \quad y_{22b}^\xi(i) = -\frac{RK_{22b}(i)}{2C_{22}^0}. \end{aligned} \tag{25}$$

The numerical values of the coefficients (25), which define the dependence of the pole coordinates on time are presented in Table 3.

Table 3. Coefficients of the tidal periodic perturbations in the polar motion of the P_ζ (polar) and of the P_ξ (equatorial) axes of inertia of the Earth

No.	l_S	l_M	F	D	Ω	$y^{\xi}_{22a}(i)$ (km)	$y^{\xi}_{22b}(i)$ (km)	$z^{\xi}_{21a}(i)$ (m)	$z^{\xi}_{21b}(i)$ (m)	$x^{\zeta}_{21a}(i)$ (m)	$x^{\zeta}_{21b}(i)$ (m)	$y^{\zeta}_{21a}(i)$ (m)	$y^{\zeta}_{21b}(i)$ (m)
1.	1	0	0	−2	0	−0.0095	−0.0095	0.2854	0.2854	−0.2854	−0.2854	−0.2873	−0.2873
2.	1	0	0	0	0	−0.0497	−0.0497	1.4932	1.4932	−1.4932	−1.4932	−1.5030	−1.5030
3.	0	0	0	2	0	−0.0083	−0.0083	0.2478	0.2478	−0.2478	−0.2478	−0.2494	−0.2494
4.	1	0	2	0	1	0.0501	−0.0022	−0.6853	0.1004	0.6853	−0.1004	0.6898	−0.1011
5.	0	0	2	0	1	0.2616	−0.0113	−3.5813	0.5232	3.5813	−0.5232	3.6048	−0.5266
6.	0	0	0	0	1	−0.2645	0.0113	3.6212	−0.5291	−3.6212	0.5291	−3.6450	0.5325
7.	−1	0	2	2	2	−0.2551	−0.0005	−0.6905	0.0300	0.6905	−0.0300	0.6951	−0.0301
8.	−1	0	2	0	2	0.1984	0.0004	0.5367	−0.0229	−0.5367	0.0229	−0.5402	0.0231
9.	1	0	2	0	2	−1.3436	−0.0025	−3.6359	0.1562	3.6359	−0.1562	3.6598	−0.1572
10.	0	0	2	2	2	−0.2146	−0.0004	−0.5807	0.0253	0.5807	−0.0253	0.5846	−0.0254
11.	0	0	2	0	2	−7.0179	−0.0130	−18.9893	0.8168	18.9893	−0.8168	19.1140	−0.8222
12.	0	1	0	0	0	−0.0070	−0.0070	0.2090	0.2090	−0.2090	−0.2090	−0.2104	−0.2104
13.	0	1	2	−2	2	−0.1909	−0.0004	−0.5167	−0.0053	0.5167	0.0053	0.5201	0.0053
14.	0	0	2	−2	2	−3.2572	−0.0061	−8.8144	−0.0940	8.8144	0.0940	8.8723	0.0946

Let us also present brief formulae for the coordinates of the pole P_ζ saving only the main terms in (24) on the basis of Table 3:

$$\begin{aligned} x_\zeta &= -18.99\sin(\mu+\nu-2F-2\Omega) \\ &- 8.81\sin(\mu+\nu-2F+2D-2\Omega) \\ &- 3.64\sin(\mu+\nu-l_M-2F-2\Omega) \\ &+ 3.62\sin(\mu+\nu-\Omega) \\ &- 3.58\sin(\mu+\nu-2F-\Omega) \\ &+ 1.49\sin(\mu+\nu-l_M) \\ &+ 1.49\sin(\mu+\nu+l_M) \\ y_\zeta &= -19.11\cos(\mu+\nu-2F-2\Omega) \\ &- 8.87\cos(\mu+\nu-2F+2D-2\Omega) \\ &- 3.66\cos(\mu+\nu-l_M-2F-2\Omega) \\ &+ 3.65\cos(\mu+\nu-\Omega) \\ &- 3.61\cos(\mu+\nu-2F-\Omega) \\ &+ 1.50\cos(\mu+\nu-l_M) \\ &+ 1.50\cos(\mu+\nu+l_M). \end{aligned} \tag{26}$$

Here the amplitudes are given in metres. For the main motion of the pole P_ζ (26) we have an ellipse with semiaxes $a_\zeta = 19.11m$ and $b_\zeta = 18.99m$ (with eccentricity $e_\zeta = 0.112$) and with a period close to one day.

The displacements of the equatorial axis of inertia $C\xi$ achieve big values. We describe the main kinematical effects in the motion of this axis by the formulae:

$$\begin{aligned} y_\xi &= 7.02\times10^3(m)\sin(2\mu+2\nu-2F-2\Omega) \\ z_\xi &= 18.98(m)\sin(\mu+\nu-2F-2\Omega). \end{aligned} \tag{27}$$

Thus the displacements in the equatorial plane for the $C\xi$ axis are kilometres.

Let us discuss the mechanical context of the kinematical effects obtained above in the Earth's polar axes of inertia.

The solar and lunar tidal influence sufficiently changes the geometry of the Earth's mass and generates the remarkable displacements of the poles of the Earth's principal axes of inertia on its surface. The large amplitudes of the perturbations (26), (27) (in particular for the equatorial axis of inertia) are due to the small ellipticity of the Earth's ellipsoid of inertia. Then nearer to the body by its dynamical structure to the body with concentric distribution of density then more "fill" the orientation of the axes of inertia to the changes of densities.

The trajectory of the poles of the axes of inertia of the Earth with respect to the axes $Cxyz$ is conditionally periodic and is characterized by the frequency of the Earth's rotation and by set frequencies of the theory of the orbital motion of the Moon.

5 THE MOTION OF THE PRINCIPAL AXES OF INERTIA OF THE EARTH, CAUSED BY ITS ROTATIONAL DEFORMATION

Let us now consider the motion of the Earth's principal axes of inertia $C\xi\eta\zeta$ relative to the reference system $Cxyz$, caused by rotational deformation of the Earth's elastic mantle. The axes $Cxyz$ correspond to the principal axes of inertia of the Earth in the undeformed position of its mantle.

The rotational deformation of the Earth gives the following variations of the components of the Earth's inertia tensor (Getino and Ferrandiz, 1991):

$$\begin{aligned} &A = A_0 + \delta A, \quad B = B_0 + \delta B, C = C_0 + \delta C \\ &F = \delta F, \quad E = \delta E, \quad D = \delta D \end{aligned} \tag{28}$$

where

$$\begin{aligned} \delta A &= D_0\omega^2 + D_1(\omega^2 - 3p^2), \quad \delta F = 3D_1pq \\ \delta B &= D_0\omega^2 + D_1(\omega^2 - 3q^2), \quad \delta E = 3D_1pr \\ \delta C &= D_0\omega^2 + D_1(\omega^2 - 3r^2), \quad \delta E = 3D_1qr. \end{aligned} \tag{29}$$

Here ω is the modulus of the vector of the angular velocity $\overline{\omega}$ of the reference system $Cxyz$; and p, q and r are components of this vector with respect to the axes Cx, Cy and Cz. D_0 and D_1 in (34) are small constant coefficients, characterized by the elastic properties of the mantle.

So the coefficient D_1 gives an increasing ΔC of the polar moment of inertia of the Earth due to its rotational deformation (Getino and Ferrandiz 1991)

$$\Delta C = -2D_1\omega_0^2, \quad \frac{\Delta C}{mR^2} = 0.2358204 \times 10^{-3}.$$

Variations of the components of the Earth's inertia tensor (28), (29) define the corresponding variations of the geopotential coefficients (from (2)). Therefore for a description of the pole's motion (for principal axes of inertia $C\xi$ and $C\zeta$) we will again use formulae (22). We have:

$$x_\zeta \cong \frac{3RD_1pr}{A_0 - C_0}, \quad y_\zeta = \frac{3RD_1qr}{B_0 - C_0}$$

$$y_\xi \cong \frac{3RD_1pq}{B_0 - C_0}, \quad z_\xi = \frac{3RD_1pr}{C_0 - A_0}$$

where A_0, B_0 and C_0 are constant values of the moments inertia in the undeformed position of the Earth.

For the parameters used in this work we obtain:

$$\begin{aligned} x_\zeta &= R \cdot 0.3258 \left(\frac{p}{\omega}\right), \quad y_\zeta = R \cdot 0.3277 \left(\frac{q}{\omega}\right) \\ y_\xi &= R \cdot 49.9296 \left(\frac{pq}{\omega^2}\right), \quad z_\xi = -R \cdot 0.3258 \left(\frac{p}{\omega}\right). \end{aligned} \tag{30}$$

Formulae (30) allow us to give a simple geometrical interpretation. The pole of the principal axis of inertia of the Earth P_ζ due to its rotational deformation is a displacement about the position for the undeformed Earth and describes a trajectory similar to the trajectory of the pole P_ω of the Earth's rotation (reduced approximately by three times, $x_\zeta = R \cdot 0.33\frac{p}{\omega}$, $y_\zeta = R \cdot 0.33\frac{q}{\omega}$) (Bursa, 1983).

The vector $\bar{\omega}$ is situated in the plane which is formed by axes Cz and C_ζ. For example, for Chandler's pole motion in a circle with angular radius $0.''3$ a similar P_ζ pole motion will be in a circle with angular radius $0.''1$. Some perturbations of the poles of principal axes of inertia of the Earth due to its rotational deformations are discussed in an another paper (Barkin, 1997).

For example, the variations of the angular velocity components for the model of the Chandler motion of the Earth's pole (Ferrandiz, Getino, Barkin, 1995) are given by the following elliptical functions:

$$\frac{p}{\omega} = 1.2231\times10^{-6} cnu, \quad \frac{q}{\omega} = -1.2289\times10^{-6} snu, \quad \frac{\delta r}{\omega} = 0.7212\times10^{-14} sn2u \quad (31)$$

where $u = \frac{2\pi}{T_{\rm CH}}(t - t_0)$, $T_{\rm CH} = 449d$ is Chandler's period, $t_0 = 0h15Sept.1990$ is a initial moment of the time.

Formulae (30), (31) now permit to obtain the corresponding variations (or perturbations) in the motion of the principal axes poles of the Earth (also in the elliptical functions):

$$\begin{aligned} \frac{x_\zeta}{R} &= 0.''08219 \cdot cnu, & \frac{y_\zeta}{R} &= -0.''08307 \cdot snu, \\ \frac{y_\zeta}{R} &= 0.''000016 \cdot snu \cdot cnu, & \frac{z_\xi}{R} &= -0.''08219 \cdot cnu. \end{aligned}$$

Analogous perturbations take place in the motion of the Earth principal axes of inertia due to rotational deformations with annual, semiannual and other periods.

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