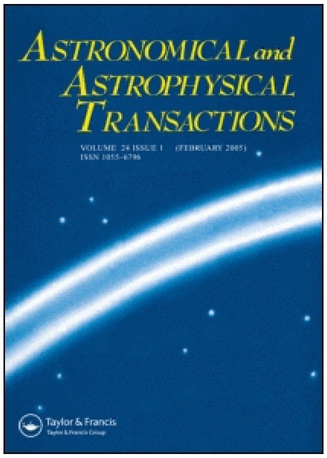

Astronomical & Astrophysical Transactions
The Journal of the Eurasian Astronomical Society
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713453505

Gravitational collapse of polytropic configurations

V. Ureche [a]
[a] Faculty of Mathematics and Computer Science, "Babes Bolyai" University of Cluj-Napoca, Cluj-Napoca, Romania

Online Publication Date: 01 December 1999
To cite this Article: Ureche, V. (1999) 'Gravitational collapse of polytropic configurations', Astronomical & Astrophysical Transactions, 18:3, 533 - 541
To link to this article: DOI: 10.1080/10556799908203011
URL: http://dx.doi.org/10.1080/10556799908203011

Astronomical and Astrophysical Transactions, 1999, Vol. 18, pp. 533–541
Reprints available directly from the publisher
Photocopying permitted by license only

GRAVITATIONAL COLLAPSE OF POLYTROPIC CONFIGURATIONS

V. URECHE

"Babes Bolyai" University of Cluj-Napoca, Faculty of Mathematics and Computer Science, Str. M. Kogălniceanu 1, 3400 Cluj-Napoca, Romania

(Received October 24, 1998)

The results of Goldreich and Weber (1980) for the gravitational collapse of a polytrope of index $n = 3$ are generalized for whatever polytropic configuration with $n \in (0, 5)$. The time dependence of the solution is analysed separately for $n \in (0, 1)$, $n = 1$ and $n \in (1, 5)$. The spatial dependence of the solution is given by a differential equation that generalizes the Lane–Emden equation. The polytropic collapsing time is defined and determined for the three considered cases of n. This gives a characteristic time-scale for the collapse, which can be compared with the free-fall time.

KEY WORDS Astrophysics, stellar evolution, gravitational collapse

1 INTRODUCTION

In the evolution of the Universe, the most important processes are the explosion and the collapse. The history of the Universe develops between the Big Bang and the Big Crunch.

At the stellar level, these two processes are also the most important. Gravitational collapse can be encountered in the early stellar evolution as well as in the final stages. There are many papers concerning this topic, which analyse the different aspects of the collapse, from the free-fall collapse of a homogeneous spherical cloud to the hydrodynamical approximation and numerical models (Kippenhahn and Weigert, 1991; Larson, 1969; Yahil, 1983).

Goldreich and Weber (1980) analysed the gravitational collapse of a polytrope of index $n = 3$. The obtained solution is useful for the study of the collapse of a white dwarf that reaches the Chandrasekhar limit or for the understanding of the collapse of a stellar core causing a supernova outburst.

In a recent paper we analysed the free-fall collapse of a homogeneous sphere in a Maneff gravitational field (Ureche, 1995). Here we shall generalize the results of Goldreich and Weber for a polytrope of a certain polytropic index $n \in (0, 5)$, this range being a realistic one from the physical point of view.

2 BASIC EQUATIONS OF COLLAPSE

The collapse will be described by the following equations:

(1) the equation of motion of an ideal fluid (equation of Euler)

$$\frac{d\boldsymbol{v}}{dt} = -\nabla\Phi - \frac{1}{\rho}\nabla P, \tag{1}$$

(2) the continuity equation,

$$\frac{d\rho}{dt} + \rho(\nabla\boldsymbol{v}) = 0, \tag{2}$$

(3) the equation of state (which for more generality can be considered as being barotropic),

$$P = P(\rho) \tag{3}$$

(4) and Poisson's equation:

$$\Delta\Phi = 4\pi G\rho, \tag{4}$$

where the notations are usual (Chandrasekhar, 1939; Cox and Giuli, 1968; Kippenhahn and Weigert, 1991).

For spherical-symmetric collapse, equations (1)–(4) become ($\boldsymbol{v} = (v_r, 0, 0)$):

$$\begin{aligned}
\frac{\partial v_r}{dt} + v_r\frac{\partial v_r}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{\partial\Phi}{\partial r} &= 0, \\
\frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) &= 0 \\
P &= K\rho^{1+\frac{1}{n}} \text{ (polytropic equation of state)} \\
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) &= 4\pi G\rho
\end{aligned} \tag{5}$$

In order to remain close to the polytropic formalism for static polytropes, Emden's variables (with the notation of Kippenhahn and Weigert, 1991) will be introduced:

$$r = \alpha z, \quad \rho = \rho_c w^n, \tag{6}$$

where

$$\alpha^2 = \frac{(n+1)K}{4\pi G}\rho_c^{\frac{1-n}{n}}, \quad \rho_c \text{ – central density.} \tag{7}$$

Here $r = r(t)$ and we shall consider (as Goldreich and Weber, 1980, for $n = 3$) that $\alpha = \alpha(t)$ (and $\rho_c = \rho_c(t)$), but dimensionless (Emden's) variables z and w are independent of time t. Then:

$$r = \alpha(t)z, \quad v_r = \dot{\alpha}z \tag{8}$$

give a homologous change.

Introducing a velocity potential ψ by:

$$v_r = \frac{\partial \psi}{\partial r} = \frac{1}{\alpha}\frac{\partial \psi}{\partial z}, \tag{9}$$

taking $\psi = 0$ at $z = 0$ we have:

$$\psi = \frac{1}{2}\alpha\dot{\alpha}z^2, \tag{10}$$

and

$$\frac{d\psi}{dt} = \frac{\partial \psi}{dt} + v_r\frac{\partial \psi}{\partial r} = \frac{\partial \psi}{\partial t} + (\dot{\alpha}z)^2. \tag{11}$$

In the new (Emden's) variables, Poisson's equation becomes:

$$\frac{1}{z^2}\frac{\partial}{\partial z}\left(z^2\frac{\partial \Phi}{\partial z}\right) = 4\pi G\rho\alpha^2, \tag{12}$$

while the continuity equation can be written:

$$\frac{1}{\rho}\frac{d\rho}{dt} + 3\frac{1}{\alpha}\frac{d\alpha}{dt} = 0, \tag{13}$$

that is, $\rho \sim \alpha^{-3}$, an obvious result.

From (6) and (7) we have:

$$\rho = \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{n}{n-1}} \frac{1}{\alpha^{\frac{2n}{n-1}}} w^n(z), \quad n \neq 1. \tag{14}$$

The case $n = 1$ will be analysed separately.

3 SEPARATION OF VARIABLES. TIME DEPENDENCE

The change (6), (7) with $\alpha = \alpha(t)$ allows the separation of the temporal dependence of the solution from its spatial dependence. For this, a new function h (the enthalpy) is defined by:

$$h = \int_0^\rho \frac{dP}{\rho} = K(n+1)\rho^{\frac{1}{n}}. \tag{15}$$

Introducing the functions ψ from (9), (10) and h from (15) into the equation of motion (first equation from (5)), we obtain:

$$\frac{\partial^2 \psi}{\partial r \partial t} + \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial r}\right)^2 + \frac{\partial h}{\partial r} + \frac{\partial \Phi}{\partial r} = 0. \tag{16}$$

Integrating this equation with respect to r, setting the integration constant to zero and taking into account (8), (9), (10) and (11) we find that:

$$\frac{d\psi}{dt} = \frac{1}{2}\dot{\alpha}^2 z^2 - \Phi - h. \tag{17}$$

Using here the expression (10) of ψ we have:

$$\frac{1}{2}\alpha\ddot{\alpha}z^2 = -\Phi - h. \tag{18}$$

With (14), (15) the expression for h becomes:

$$h = \left[\frac{(n+1)^n K^n}{4\pi G}\right]^{\frac{1}{n-1}} \frac{1}{\alpha^{\frac{2}{n-1}}} w(z). \tag{19}$$

Trying a similar dependence of Φ on t we can define another dimensionless function $g(z)$ by:

$$\Phi = \left[\frac{(n+1)^n K^n}{4\pi G}\right]^{\frac{1}{n-1}} \frac{1}{\alpha^{\frac{2}{n-1}}} g(z). \tag{20}$$

From (18), (19), (20)

$$\frac{1}{2}\alpha\ddot{\alpha} = -\left[\frac{(n+1)^n K^n}{4\pi G}\right]^{\frac{1}{n-1}} \frac{1}{\alpha^{\frac{2}{n-1}}} [g(z) + w(z)] \frac{1}{z^2}, \tag{21}$$

or

$$\left[\frac{4\pi G}{(n+1)^n K^n}\right]^{\frac{1}{n-1}} \frac{1}{2}\alpha^{\frac{n+1}{n-1}}\ddot{\alpha} = -\frac{g(z) + w(z)}{z^2}. \tag{22}$$

The left-hand side of this equation is a function of t only, while the right-hand side is a function of z only. Therefore, both sides must be constant and we shall take this constant equal to $-\lambda/6$ ($\lambda = \text{const}$). Then we have:

$$\left[\frac{4\pi G}{(n+1)^n K^n}\right]^{\frac{1}{n-1}} \alpha^{\frac{n+1}{n-1}}\ddot{\alpha} = -\frac{\lambda}{3} \tag{23}$$

$$6\frac{g(z) + w(z)}{z^2} = \lambda. \tag{24}$$

The equations (23) will be separately integrated for $n \in (1,5)$ and $n \in (0,1)$.

(i) **Case** $n \in (1,5)$. We shall take the following initial conditions for $t = 0$:

$$\alpha(0) = \alpha_0 \text{ - very large}, \quad \dot{\alpha}(0) = 0. \tag{25}$$

Approximating the integration constant by zero, the first integration of (23) gives:

$$\alpha^{\frac{1}{n-1}}\dot{\alpha} = \pm\left\{\frac{\lambda}{3}(n-1)\left[\frac{(n+1)^n K^n}{4\pi G}\right]^{\frac{1}{n-1}}\right\}^{1/2}, \quad \lambda > 0, \tag{26}$$

Taking the $(-)$ sign ($\dot{\alpha} < 0$ – collapse) and integrating (26) with (25) we obtain:

$$\alpha^{\frac{n}{n-1}}(t) = \alpha_0^{\frac{n}{n-1}} - \frac{n}{n-1}\left\{\frac{\lambda}{3}(n-1)\left[\frac{(n+1)^n K^n}{4\pi G}\right]^{\frac{1}{n-1}} \{\}\right\}^{1/2} t. \tag{27}$$

From (27), for $n = 3$ the result of Goldreich and Weber (1980) can be obtained.

(ii) **Case** $n \in (0,1)$. In this case $n - 1 < 0$, $1 - n > 0$ and we cannot take the constant of integration equal to zero. With the initial conditions (25), the first integration of (23) gives:

$$\dot{\alpha} = -\left\{\frac{\lambda}{3}(1-n)\left[\frac{4\pi G}{(n+1)^n K^n}\right]^{\frac{1}{1-n}}\right\}^{1/2}\left(\alpha_0^{\frac{2}{1-n}} - \alpha^{\frac{2}{1-n}}\right). \tag{28}$$

With a new integration we obtain:

$$\alpha(t) = \alpha_0\left\{1 - \frac{1}{4}\alpha_0^{\frac{2n}{1-n}}\frac{\lambda}{3}(1-n)\left[\frac{4\pi G}{(n+1)^n K^n}\right]^{\frac{1}{1-n}} t^2\right\}. \tag{29}$$

(iii) **Case** $n = 1$. From (6), (7), for $n = 1$, we have:

$$\alpha^2 = \frac{K}{2\pi G}, \quad \rho = \rho_c w(z). \tag{30}$$

This means that α is independent of ρ_c. The variation of α results from the variation of K. Instead of equation (22), we have:

$$\frac{\ddot{\alpha}}{8\pi G\rho_c\alpha} = -\frac{g(z) + w(z)}{z^2} \tag{31}$$

Here, again, both sides of the equation must be constant, that is to say, $-\lambda/6$. Then, instead of equation (23) we obtain:

$$\frac{\ddot{\alpha}}{4\pi G\rho_c\alpha} = -\frac{\lambda}{3} \tag{32}$$

and equation (24) is recovered.

A first integration of (32) with the conditions (25) gives:

$$\dot{\alpha}^2 = \frac{4\pi G\rho_c\lambda}{3}\left(\alpha_0^2 - \alpha^2\right) \tag{33}$$

and the solution is:

$$\alpha(t) = \alpha_0 \cos\left(\frac{4\pi G\rho_c\lambda}{3}\right)^{1/2} t. \tag{34}$$

From (27), (29) and (34) we have the expression of $\alpha(t)$ for whatever $n \in (0,5)$. From (5), (6) and (14) we have the time dependence of the density and pressure.

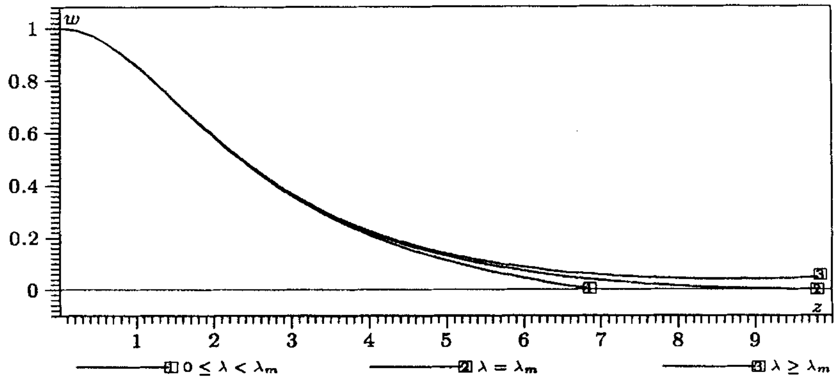

Figure 1: The solution of equation (35) for $n = 3$ and different values of λ

Figure 1 The solution of equation (35) for $n = 3$ and different values of λ.

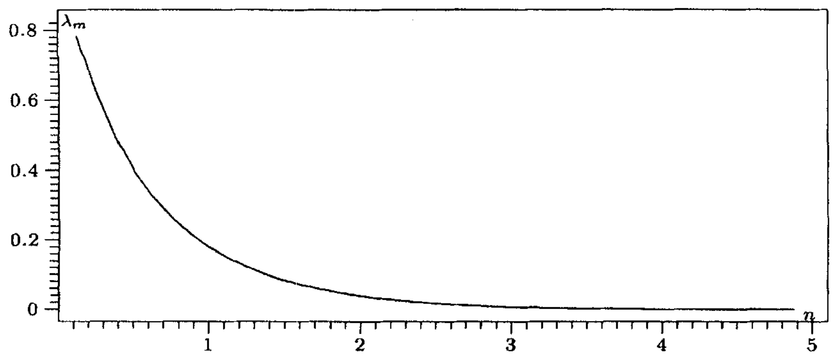

Figure 2: The maximum allowed value of λ as a function of the polytropic index n

Figure 2 The maximum allowed value of λ as a function of the polytropic index n.

4 SPATIAL DEPENDENCE

In order to obtain the spatial dependence of the solution, the function $w(z)$ from (14) or (30) will be determined. In the case $n \in (1,5)$, using the equations (14), (20), (24), from (12) we obtain:

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{dw}{dz}\right) + w^n = \lambda. \tag{35}$$

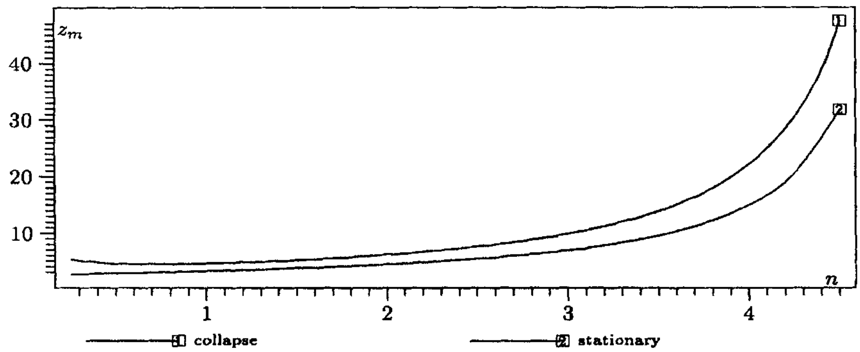

Figure 3 The non-dimensional radius of the configurations as a function of n.

For the cases $n \in (0,1)$ and $n = 1$ the same equation (35) is valid (in the last case we must put $n = 1$ in (35)). Taking $\lambda = 0$ in (35), the classical Lane–Emden differential equation is obtained. For $\lambda \neq 0$, the equation (35) describes the deviation from hydrostatic equilibrium. The maximum allowed value $\lambda_m > 0$ of λ is a measure of the deviation (Kippenhahn and Weigert, 1991).

While for the time dependence an analytic (exact) solution was obtained, for the spatial dependence we must integrate equation (35) numerically. The results are given in Figure 1, for $n = 3$. The values of λ_m as a function of n are plotted in Figure 2. In Figure 3, the non-dimensional radius z_m of the configuration is given as a function of n, both for the stationary case and for the collapse. In Table 1, the values of λ_m and z_m (for collapse) are given as function of n.

5 POLYTROPIC COLLAPSING TIME

Now, let us define a characteristic time-scale for the gravitational collapse of the polytropic configurations.

It is well known that the time-scale for stellar collapse is the stellar free-fall time:

$$t_{ff}^{s} = \left(\frac{3}{4\pi G\bar{\rho}}\right)^{\frac{1}{2}}, \tag{36}$$

where $\bar{\rho}$ is the mean density of the star. The time-scales for the collapse of the spherical homogeneous cloud is:

$$t_{ff}^{cl} = \left(\frac{3\pi}{32G\rho_0}\right)^{\frac{1}{2}}, \tag{37}$$

where ρ_0 is the initial density of the cloud.

Table 1. The values of λ_m and z_m as functions of n.

n	λ_m	z_m	n	λ_m	z_m
0.125	0.7821	6.65	2.625	0.0132	8.05
0.250	0.6195	5.21	2.750	0.0106	8.59
0.375	0.4958	4.70	2.875	0.0083	9.20
0.500	0.4000	6.25	3.000	0.0065	9.89
0.625	0.3247	4.38	3.125	0.0051	10.68
0.750	0.2650	4.37	3.250	0.00389	11.59
0.875	0.2172	4.42	3.375	0.00292	12.64
1.000	0.1785	4.50	3.500	0.00216	13.88
1.125	0.1470	4.61	3.625	0.00156	15.36
1.250	0.1212	4.74	3.750	0.00111	17.14
1.375	0.1000	4.90	3.875	0.00076	19.34
1.500	0.0825	5.09	4.000	0.00050	22.10
1.625	0.0681	5.30	4.125	0.00032	25.68
1.750	0.0561	5.53	4.250	0.00019	30.49
1.875	0.0461	5.79	4.375	0.00010	37.27
2.000	0.0378	6.07	4.500	0.000487	47.54
2.125	0.0309	6.39	4.625	0.000191	64.77
2.250	0.0252	6.74	4.750	0.000052	99.51
2.375	0.0204	7.13	4.875	0.000006	197.76
2.500	0.0165	7.56			

The characteristic time-scale for the collapse of the polytropes can be obtained from (27), (29) or (34). The function $\alpha(t)$ has the dimension of length. At the beginning of the collapse it has the value α_0. At the end of the collapse $\alpha \simeq 0$ and the time so defined will be called the polytropic collapsing time $t_{pc}(n)$. For the three cases considered, we have:

$$t_{pc}(n) = \frac{1}{2n}\left[\frac{3(n-1)}{\pi G\lambda_m(n)\rho_c(n)}\right]^{1/2}, \quad n \in (1,5), \tag{38}$$

$$t_{pc}(n) = \left[\frac{3}{(1-n)\pi G\lambda_m(n)\rho_c(n)}\right]^{1/2}, \quad n \in (0,1), \tag{39}$$

and

$$t_{pc}(1) = \left[\frac{3\pi}{16G\lambda_m(1)\rho_c(1)}\right]^{1/2}, \quad \text{for } n = 1, \tag{40}$$

where $\rho_c(n)$ is the central density of the configuration at the beginning of the collapse, which for whatever n can be expressed with the mean density $\bar{\rho}$ (for the same time).

References

Chandrasekhar (1939) *An Introduction to the Study of Stellar Structure*, University of Chicago Press, Chicago.

Cox and Giuli (1968) *Principles of Stellar Structure*, Gordon and Breach, New York.
Goldreich and Weber (1980) *Astrophys. J.* **238**, 991.
Kippenhahn and Weigert (1991) *Stellar Structure and Evolution*, Springer–Verlag, Berlin.
Larson (1969) *Mon. Not. R. Astron. Soc.* **145**, 271.
Ureche (1995) *Rom. Astron. J.* **5**, No. 2, 145.
Yahil (1983) *Astrophys. J.* **265**, 1047.