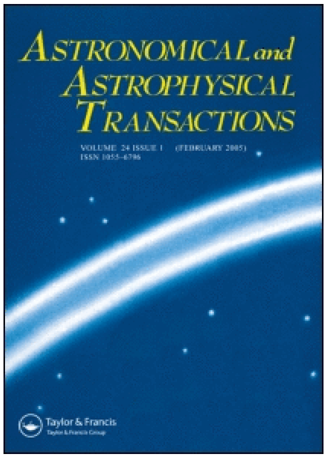

Astronomical & Astrophysical Transactions
The Journal of the Eurasian Astronomical Society
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713453505

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Online Publication Date: 01 January 1998
To cite this Article: Aquilano, R. O., Castagnino, M. A. and Neira, L. P. (1998) 'Spherical shell dynamics', Astronomical & Astrophysical Transactions, 17:1, 1 - 7
To link to this article: DOI: 10.1080/10556799808235418
URL: http://dx.doi.org/10.1080/10556799808235418

Astronomical and Astrophysical Transactions, 1998, Vol. 17, pp. 1–7
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SPHERICAL SHELL DYNAMICS

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(Received April 18, 1997)

A post–Newtonian approximation is used to find the equation of motion of a spherically symmetric radiating shell moving in a central gravitational potential. A Schwarzschild metric inside and a Vaidya metric outside is used.

KEY WORDS Spherical shells, dynamics, post–Newtonian corrections

1 INTRODUCTION

The radiating shell was used in previous papers (Hamity and Gleiser, 1978; Hamity and Spinosa, 1984; Castagnino and Umérez, 1983; Aquilano and Castagnino, 1985; Aquilano, Castagnino and Lara, 1987) as a model of a burster or a supernova explosion. A relativistic model was used, but the exact solution is very difficult to find (Aquilano, Barreto and Núñez, 1991); so in this paper we present a method, of computing the post–Newtonian corrections of a classical model, because it is generally a simpler problem to solve.

The usual post–Newtonian approximation only deals with free falling particles, so we must generalize the formalism to particles that move under the action of an external force.

We obtain the approximation of the shell with a Schwarzschild metric inside and a Vaidya metric outside, because for this case we can use the well-studied relativistic model.

2 THE EXACT EQUATION OF MOTION AND THE POST-NEWTONIAN ORDER

We consider a Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m^-}{r}\right) dt^2 - \left(1 - \frac{2m^-}{T}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{2.1}$$

where m is the mass of the central object. If $R(\tau)$ is the shell radius, the metric on the shell will be

$$ds^2 = \left(1 - \frac{2m^-}{R}\right) dt^2 - \left(1 - \frac{2m^-}{R}\right)^{-1} dR^2 - R^2 d^2\Omega. \tag{2.2}$$

As all the shell particles move radially and (as $c = 1$) $ds^2 = d\tau^2$ we have

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{2m^-}{R} - \frac{u^2}{1 - \frac{2m^-}{R}} = \frac{\left(1 - \frac{2m^-}{R}\right)^2 - u^2}{1 - \frac{2m^-}{R}} \tag{2.3}$$

where u is the usual velocity of the shell particles, i.e. $u = dR/dt$, while we will $\dot{R} = dR/d\tau$. Using $\dot{R}$, equation (2.3) reads

$$\left(\frac{d\tau}{dt}\right)^2 = \frac{\left(1 - \frac{2m^-}{R}\right)^2}{1 - \frac{2m^-}{R} + \dot{R}^2}. \tag{2.4}$$

Now let us introduce the exact equations of motion of a shell moving between a Schwarzschild metric and a Vaidya metric (Vaidya, 1951; Hamity and Gleiser, 1978; Hamity and Spinosa, 1984). They are

$$R(1 - B) = m_0$$

$$\frac{1}{B}\left[\ddot{R} + \frac{m^+}{R^2} - \frac{\dot{m}^+}{R(\dot{R} + B)}\right] - \frac{1}{A}\left[\ddot{R} + \frac{m^-}{R^2}\right] = \frac{m_0}{R^2} \tag{2.5}$$

where

$$A = \left(1 - \frac{2m^-}{R} + \dot{R}^2\right)^{1/2}$$

$$B = \left(1 - \frac{2m^+}{R} + \dot{R}^2\right)^{1/2} \tag{2.6}$$

is the proper mass of the shell and $m^+ = \hat{m} + m^-$, where $\hat{m}$ is the total mass (proper mass plus energy) of the shell. From these equations we can obtain

$$\frac{\ddot{R} + m^-/R^2}{A} + \frac{m_0}{2R^2} = -\frac{L_R}{m_0} \tag{2.7}$$

where $L_R = -\dot{m}^+(\dot{R}+B)^{-1}$ can be considered as the shell-radiated luminosity reaction force (Hamity and Gleiser, 1978; Hamity and Spinosa, 1984; Castagnino and Umérez, 1983). Therefore equation (2.1) reads

$$\frac{d\tau}{dt} = \frac{1-\frac{2m^-}{R}}{A}. \tag{2.8}$$

This equation can be used to change the derivative of the absolute velocity in equation (2.7) to a coordinate time derivative. We obtain

$$\frac{du}{dt} = \frac{m^-}{R^2}\frac{(1-2m^-/R)^2-3u^2}{(1-2m^-/R)} + \frac{(1-2m^-/R)^3}{A^3}\tilde{F} = \left(\frac{d\tau}{dt}\right)^3\tilde{F} \tag{2.9}$$

where

$$\tilde{F} = -\frac{m_0}{2R^2} - \frac{L_R}{m_0}. \tag{2.10}$$

Now we expand equation (2.9) up to order $\overline{u}^4/F$, taking into account that up to this order

$$\begin{aligned} m_0 &= R(A-B) \cong R\left[1+\frac{1}{2}\left(\dot{R}^2-\frac{2m^-}{R}\right)-1-\frac{1}{2}\left(\dot{R}^2-\frac{2m^+}{R}\right)\right] \\ &= m^+ - m^- = \hat{m}. \end{aligned} \tag{2.11}$$

We obtain

$$\frac{du}{dt} = -\frac{m^-}{R^2} + \frac{2m^{-2}}{R^3} + 3u^2\frac{m^-}{R^2} + \overset{(4)}{\left(\frac{d\tau}{dt}\right)^3}\left[-\frac{\hat{m}}{2R^2}-\frac{L_R}{\hat{m}}\right]. \tag{2.12}$$

3 POST–NEWTONIAN EQUATION OF MOTION

In order to perform this programme we must compute the potentials Φ, Ψ and $\overline{\zeta}$ from the ordinary post–Newtonian approximation for our particular case; we compute the absolute force f^μ in the radial motion.

In general the potentials Φ, Ψ and $\overline{\zeta}$ are obtained as

$$\Phi(t,\bar{x}) = -G\int d^3x'\frac{\overset{(0)}{T^{00}}(t,\bar{x}')}{|\bar{x}-\bar{x}'|} \tag{3.1}$$

$$\Psi(t,\bar{x}) = -\int\frac{d^3x'}{|\bar{x}-\bar{x}'|}\left[\frac{1}{4\pi}\frac{\partial^2\Phi(t,\bar{x}')}{\partial t^2} + G\overset{(2)}{T^{00}}(t,\bar{x}') + G\overset{(2)}{T^{il}}(t,\bar{x}')\right] \tag{3.2}$$

$$\zeta_i(t,\bar{x}) = -hG\int\frac{d^3x'\overset{(1)}{T^{i0}}(t,\bar{x}')}{|\bar{x}-\bar{x}'|} \tag{3.3}$$

where G is Newton's constant and $\overset{(n)}{T}{}^{\mu\nu}$ is the order n of the expansion of the energy-momentum tensor (Weinberg, 1972). In our radial model the potential Φ is simply

$$\Phi = -\frac{m^-}{R}. \tag{3.4}$$

The other two potentials can be computed indirectly. In general we have that the ratio between the proper time and the coordinate time up to fourth order is (Weinberg, 1972)

$$\overset{(4)}{\left(\frac{d\tau}{dt}\right)^2} = 1 + [2\Phi - u^2] + 2[\Phi^2 + \Psi - \overline{\zeta}\cdot\overline{u} + \Phi\overline{u}^2]. \tag{3.5}$$

In our particular case we can expand equation (2.3) up to order $\overline{u}^4$, and we obtain

$$\overset{(4)}{\left(\frac{d\tau}{dt}\right)^2} = 1 - \frac{2m^-}{R} - u^2 - \frac{2m^-}{R}u^2 \tag{3.6}$$

or using equation (3.4)

$$\overset{(4)}{\left(\frac{d\tau}{dt}\right)^2} = 1 + 2\Phi - u^2 + 2\Phi u^2. \tag{3.7}$$

Comparing this equation with equation (3.5) we have

$$\Phi^2 + \Psi - \overline{\zeta}\cdot\overline{u} = 0. \tag{3.8}$$

Let us consider a rotating spherically symmetric body that in the rest reference system has an angular velocity ω. We know that

$$\overline{\zeta}(t,\overline{x}) = \frac{2G}{r^3}(\overline{x}\times\overline{J}) \tag{3.9}$$

where $\overline{J}$ is the angular mementum. In our case we deal with a static central body; then $\overline{J}_{m^-} = \overline{0}$ and therefore $\overline{\zeta} = \overline{0}$. Thus

$$\Psi = -\Phi^2 = -\frac{m^{-2}}{R^2}. \tag{3.10}$$

Using equations (3.4), (3.10) and $\overline{\zeta} = \overline{0}$, the equation of motion, up to fourth order, is

$$\frac{d\overline{u}}{dt} = -\frac{m^-}{R^2} + \frac{2m^{-2}}{R^3} + \frac{3m^-}{R^2}u^2 + \left\{\left(\frac{d\tau}{dt}\right)^2\left[\frac{f^R}{m} - \frac{f^0}{m}u\right]\right\}^{(4)} \tag{3.11}$$

where f^R, f^0 and m are the absolute radial force, the time component of the absolute force and the rest mass of a small piece of the shell, respectively. Now, let us introduce a vierbein field e^α_μ tangent to the coordinates curves at each point of the shell. The small increment of the coordinates measured in the vierbein will be

$$dx^\alpha_a = e^\alpha_\mu dx^\mu \tag{3.12}$$

where the subindex a means anholonomic coordinates. We are only interested in the radial and time–like components and as

$$e^0_0 = \left(1 - \frac{2m^-}{R}\right)^{1/2}, \quad e^R_R = \left(1 - \frac{2m^-}{R}\right)^{-1/2} \tag{3.13}$$

the relevant components of equation (3.12) read

$$\begin{aligned} dt_a &= \left(1 - \frac{2m^-}{R}\right)^{1/2} dt \\ dr_a &= \left(1 - \frac{2m^-}{R}\right)^{-1/2} dr. \end{aligned} \tag{3.14}$$

In fact

$$ds^2 = \left(1 - \frac{2m^-}{R}\right) dt^2 - \left(1 - \frac{2m^-}{R}\right)^{-1} dr^2 = dt_a^2 - dr_a^2 \tag{3.15}$$

and also

$$f^0_a = \left(1 - \frac{2m^-}{R}\right)^{1/2} f^0, \quad f^R_a = \left(1 - \frac{2m^-}{R}\right)^{-1/2} f^R \tag{3.16}$$

and using equation (3.13) we have

$$f^R = \frac{dt}{d\tau}\left(1 - \frac{2m^-}{R}\right) F^R_c, \quad f^0 = \frac{dt}{d\tau} F^0_c \tag{3.17}$$

where F^R_c and $F^0_c = F^R_c u^R_c$ are the ordinary classical radial force and the ordinary classical energy of the small piece of the shell, and u^R_c is the ordinary classical velocity measured in the vierbein, related to $u = dR/dt$, by

$$u^R_c = \frac{dR_a}{dt_a} = \left(1 - \frac{2m^-}{R}\right)^{-1} \frac{dR}{dt} = \left(1 - \frac{2m^-}{R}\right)^{-1} u. \tag{3.18}$$

Thus, from equation (3.17) we have

$$f^R = \left(1 - \frac{2m^-}{R}\right) \frac{dt}{d\tau} F^R_c, \quad f^0 = \left(1 - \frac{2m^-}{R}\right)^{-1} \frac{dt}{d\tau} F^R_c u. \tag{3.19}$$

Let us now analyse the term $m^{-1}(f^R - f^0\nu)$ of equation (3.11). Using equation (3.19) we have:

$$\frac{f^R}{m} - \frac{f^0}{m} u = \frac{1}{m}\frac{dt}{d\tau} F^R_c \left[\frac{\left(1 - \frac{2m^-}{R}\right)^2 - u^2}{1 - \frac{2m^-}{R}}\right] \tag{3.20}$$

which, taking into account equation (2.3), reads

$$\frac{f^R}{m} - \frac{f^0}{m}u = \frac{1}{m}\frac{d\tau}{dt}F_c^R. \tag{3.21}$$

With equation (3.11), we finally have

$$\frac{du}{dt} = -\frac{m^-}{R^2} + \frac{2m^{-2}}{R^3} + \frac{3m^-}{R^2}u^2 + \overset{(4)}{\left(\frac{d\tau}{dt}\right)}\frac{F_c^R}{m}. \tag{3.22}$$

This is the equation of a small piece of the shell. If we now consider the whole shell the equation will not change because the ratio F_c^R/m is the same for every part of a homogeneous shell. Finally if we realize that three forces are acting on the shell, the central body attraction, the exterior radiation force $(-L_R)$ and the shell self–gravity force,

$$-\frac{m\hat{m}}{2R^2} \quad \text{(see Castagnino and Umérez, 1983, equation 3–K)}$$

and that the first force is already taken into account by the Schwarzschild metric, we then have

$$F_c^R = -L_R - \frac{m\hat{m}}{2R^2}. \tag{3.23}$$

Thus, our post–Newtonian approximation method yields the equation of motion

$$\frac{du}{dt} = -\frac{m^-}{R^2} + \frac{2m^{-2}}{R^3} + \frac{3m^-}{R^2}u^2 + \overset{(4)}{\left(\frac{d\tau}{dt}\right)^3}\left[-\frac{1}{2}\frac{\hat{m}}{R^2} - \frac{L_R}{\hat{m}}\right] \tag{3.24}$$

which coincides with equation (3.11), which was obtained by expanding the exact equation of motion up to order $\overline{u}^4/\overline{r}$, as we have promised to demonstrate.

4 CONCLUSION

As all of this procedure is logical and straightforward, and yields the correct result in the case we have studied, we believe it can be successfully used in other more complicated cases.

In a recent paper (Aquilano *et al.*, 1995) we observed, in a practical case, that the observed values in X-ray novae disappear if we do not use the post–Newtonian corrections.

Acknowledgements

This work was partially supported by the following grants: CI1*–CT94–0004 from the European Community, PEI-0126/97 and PID–0150 of CONICET (National Research Council of Argentina).

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