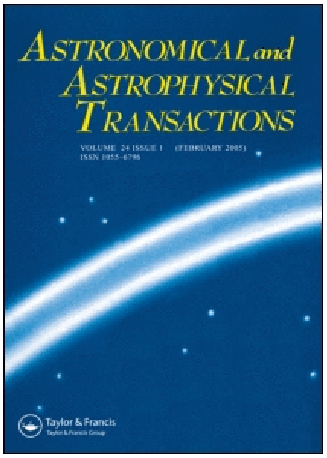

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Small-Radius perturbation of a self-gravitating gas with cylindrical symmetry†

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SMALL-RADIUS PERTURBATION OF A SELF-GRAVITATING GAS WITH CYLINDRICAL SYMMETRY†

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Self-consistent motion of an initial perturbation in density, velocity and gravitational potential on the background of a stationary cylindrical configuration of gas with gravitation and pressure in Lagrange variables is studied. A non-linear partial differential equation for the description of the radial motion is obtained. The linearization of this equation is reduced to a Klein–Gordon equation. Its analytical solution with initial condition in the form of a delta-function has a formal oscillation character. A set of parameters which satisfy the linear approximation is studied.

1 THE MODEL

Antonov and Chernin (1977) show that cylindrical symmetry admits an equilibrium state of a gas with self-gravitation and pressure. The cylinder is supposed to be infinitely long and all physical values are assumed to depend only on the radius of the cylinder. The equilibrium is provided by the equality of the gravitational force and the force of the gas pressure at every point. The quasi-two-dimensional modelling of the weak transverse non-homogeneity approximation for longitudinal motion is presented in Antonov and Chernin (1977) and Gromov and Perepelovsky (1996). In this article the model used by Antonov and Chernin (1977) and Gromov and Perepelovsky (1996) appears as the background for a small perturbations in velocity, density, pressure and gravitational potential. In the general case the perturbation may depend on three coordinates and may have a very complicated geometrical form. It is not possible to study three-dimensional non-linear motion in the general case, but the small deviation from the stationary state admits the principle of superposition. According to this it is possible to study radial and longitudinal

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motions independently and then to sum them, taking into consideration the fact that they are vectors. This article is dedicated to studying one-dimensional pure radial motion.

A model of perturbation depending only on the radius of the cylinder is chosen. This means that the cylinder moves (as a whole) along the radius under the influence of an initial perturbation of the stationary state. The perturbation is distributed along the axes of the cylinder, thus its value is dependent on the radial coordinate only. Two causes are able to generate the initial perturbation: (1) the velocity is not equal to zero for particles in the equilibrium state, or (2) the displacement of a particle from the equilibrium point is not equal to zero. Their combination is also possible.

The mathematical description of this model is represented by the Cauchy problem for three-dimensional non-stationary partial differential equations of motion, continuity, the Poisson equation and the algebraic equation of state:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla P}{\rho} - \nabla \Phi \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2}$$

$$\nabla^2 \Phi = 4\pi G \rho \tag{3}$$

$$P = A\rho^\gamma \tag{4}$$

with initial conditions

$$\left.\begin{array}{ll} \mathbf{v}|_{t=0} = \mathbf{v}(\mathbf{r},0) & P|_{t=0} = P(\mathbf{r},0) \\ \Phi\big|_{t=0} = \Phi(\mathbf{r},0) & \rho\big|_{t=0} = \rho(\mathbf{r},0) \\ \left.\dfrac{\partial \mathbf{v}(\mathbf{r},t)}{\partial t}\right|_{t=0} = \dot{\mathbf{v}}(\mathbf{r},0) & \left.\dfrac{\partial P(\mathbf{r},t)}{\partial t}\right|_{t=0} = \dot{P}(\mathbf{r},0) \\ \left.\dfrac{\partial \Phi(\mathbf{r},t)}{\partial t}\right|_{t=0} = \dot{\Phi}(\mathbf{r},0) & \left.\dfrac{\partial \rho(\mathbf{r},t)}{\partial t}\right|_{t=0} = \dot{\rho}(\mathbf{r},0) \end{array}\right\} \tag{5}$$

where P is the pressure, $\mathbf{v}$ is the speed, Φ is the gravitational potential, ρ is the gas density, t is time, $A = \text{const}$, $1 \le \gamma \le 2$. $\mathbf{v}(\mathbf{r},0)$, $P(\mathbf{r},0)$, $\Phi(\mathbf{r},0)$, $\rho(\mathbf{r},0)$, $\dot{\mathbf{v}}(\mathbf{r},0)$, $\dot{P}(\mathbf{r},0)$, $\dot{\Phi}(\mathbf{r},0)$, $\dot{\rho}(\mathbf{r},0)$ are specified functions. The system (1)–(5) describes the hydrodynamical motion of an ideal classical gas with self-gravitation and pressure.

2 CHARACTERISTIC VALUES AND PARAMETERS

The characteristic values are needed to pass from dimensional to dimensionless equations. The choice of characteristic values depends on the physical model. Having been chosen, these values constitute dimensionless coefficients of the equations. Depending on the initial and/or boundary conditions these coefficients are able to play the role of small or large parameters. There are two kinds of characteristic dimensional values in this problem: one is connected with the stationary gas configuration (the background) and the second with perturbations propagating on the stationary background. The various perturbations dependent on the initial conditions differ by time characteristics (a period, for example). To compare them it is essential to have a time scale not depending on the perturbation. There is only one choice: to attach the time scale to the stationary configuration of the gas.

The stationary configuration of the gas will be described by a series of characteristic values – the Jeans length L_0, the gravitational potential Φ_0, the gas pressure P_0, the mass of the particle m, the concentration of the particles N_0, the density ρ_0, the temperature T_0, and the sound speed c_0:

$$L_0^2 = \frac{\pi c_0^2}{G\rho_0}, \quad \Phi_0 = \frac{Gm}{L_0}, \quad P_0 = N_0 k T_0, \quad c_0^2 = A\gamma\rho_0^{\gamma-1}. \tag{6}$$

Then

$$t_{\text{scale}} = \frac{1}{\sqrt{G\rho_0}} \tag{7}$$

The full energy of the gas consists of gravitational energy and the energy of thermal expansion of the gas. The full energy is distributed between these two components depending on the adiabatic index γ. The function $\mu(\gamma)$ describes this distribution as follows:

$$\mu(\gamma) = \frac{kT_0}{m\Phi_0}. \tag{8}$$

Equations (6) and (8) give the correlation between the number of particles in a cube with edge length equal to the Jeans length and the parameter $\mu(\gamma)$:

$$N_0 L_0^3 = \pi\gamma\mu(\gamma).$$

The perturbation will be characterized by the following values defined by the initial conditions: the wavelength λ_0, the period of perturbation t_0, the velocity v_0, the characteristic Mach number and the characteristic dimensionless wavelength. They are respectively:

$$\lambda_0 = v_0 t_0, \quad M_0 = \frac{v_0}{c_0}, \quad \kappa = \frac{\lambda_0}{L_0}. \tag{9}$$

A parameter α will also be used:

$$\alpha = \frac{t_0}{t_{\text{scale}}} \tag{10}$$

It follows from (6), (9) and (10) that

$$\alpha = \frac{\sqrt{\pi}\kappa_0}{M_0} \tag{11}$$

3 THE INITIAL SYSTEM OF EQUATIONS IN THE LAGRANGE VARIABLES

To convert the dimensional equations (1)–(5) to dimensionless form the dimensionless radius, density, pressure, gravitational potential and velocity are introduced according to the following rules:

$$\xi = \frac{r}{L_0}, \quad \delta = \frac{\rho}{\rho_0}, \quad p = \frac{P}{P_0}, \quad \varphi = \frac{\Phi}{\Phi_0}, \mathrm{v} = \frac{v}{v_0}. \tag{12}$$

In the cylindrical system of coordinates the dimensionless radius components of equations (1)–(5) are

$$\frac{\partial \mathrm{v}}{\partial \tau} + a_1 \mathrm{v}\frac{\partial \mathrm{v}}{\partial \tau} = -\frac{a_2}{\delta}\frac{\partial P}{\partial \xi} - a_3\frac{\partial \varphi}{\partial \xi} \tag{13}$$

$$\frac{\partial \rho}{\partial \tau} + \frac{a_1}{\xi}\frac{\partial(\xi\delta\mathrm{v})}{\partial \xi} = 0 \tag{14}$$

$$\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial\varphi}{\partial\xi}\right) = a_4\delta \tag{15}$$

$$p = \delta^\gamma \tag{16}$$

with initial conditions

$$\left.\begin{array}{ll} \mathrm{v}|_{\tau=0} = \mathrm{v}(\xi,0) & p|_{\tau=0} = p(\xi,0) \\ \phi|_{\tau=0} = \phi(\xi,0) & \delta|_{\tau=0} = \delta(\xi,0) \\ \left.\dfrac{\partial \mathrm{v}(\xi,\tau)}{\partial\tau}\right|_{\tau=0} = \dot{\mathrm{v}}(\xi,0) & \left.\dfrac{\partial p(\xi,\tau)}{\partial\tau}\right|_{\tau=0} = \dot{p}(\xi,0) \\ \left.\dfrac{\partial \phi(\xi,\tau)}{\partial\tau}\right|_{\tau=0} = \dot{\phi}(\xi,0) & \left.\dfrac{\partial \delta(\xi,\tau)}{\partial\tau}\right|_{\tau=0} = \dot{\delta}(\xi,0) \end{array}\right\} \tag{17}$$

where $\mathrm{v}(\xi,0)$, $p(\xi,0)$, $\phi(\xi,0)$, $\delta(\xi,0)$, $\dot{\mathrm{v}}(\xi,0)$, $\dot{p}(\xi,0)$, $\dot{\phi}(\xi,0)$, $\dot{\delta}(\xi,0)$ are specified functions and

$$a_1 = \frac{v_0 t_0}{L_0}, \quad a_2 = \frac{P_0 t_0}{\rho_0 L_0 v_0}, \quad a_3 = \frac{\Phi_0 t_0}{L_0 v_0}, \quad a_4 = \frac{4\pi G\rho_0 L_0^2}{\Phi_0}.$$

According to (6)–(12) the expressions for a_1–a_4 become:

$$a_1 = \frac{\alpha M_0}{\sqrt{\pi}}, \quad a_2 = \frac{\alpha M_0}{\gamma\sqrt{\pi}}, \quad a_3 = \frac{\alpha}{\sqrt{\pi}\gamma\mu(\gamma)M_0}, \quad a_4 = 4\pi^2\gamma\mu(\gamma). \tag{18}$$

Finally in Euler variables the equations being looked for are obtained by substituting α from (11) into (18) and substituting the result into (13)–(15):

$$\frac{1}{\kappa_0}\frac{\partial \mathrm{v}}{\partial \tau} + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \xi} = -\frac{1}{\gamma\delta}\frac{\partial P}{\partial \xi} - \frac{1}{\gamma\mu(\gamma)M_0^2}\frac{\partial\varphi}{\partial\xi} \tag{19}$$

$$\frac{\partial\delta}{\partial\tau}+\frac{\kappa_0}{\xi}\frac{\partial(\xi\delta v)}{\partial\xi}=0 \tag{20}$$

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\varphi}{\partial\xi}\right)=4\pi^2\gamma\mu(\gamma)\delta \tag{21}$$

with the initial conditions (17). However the problem under consideration may be solved in Lagrange coordinates only. Applying the transformation rules (Samarskiy and Popov, 1975):

$$\frac{\partial}{\partial\tau}\to\frac{\partial}{\partial\tau}-\xi\delta v\frac{\partial}{\partial\sigma},\quad \frac{\partial}{\partial\xi}\to\xi\delta\frac{\partial}{\partial\sigma}\quad \text{where } \sigma=\int_0^{\xi}\delta\xi\,d\xi \tag{22}$$

(σ is a dimensionless Lagrange mass variable) to equations (19)–(21) yields the system of equations describing the problem in Lagrange variables:

$$\frac{1}{\kappa_0}\frac{\partial v}{\partial\tau}=-\frac{\xi}{\gamma}\frac{\partial p}{\partial\sigma}-\frac{\delta\xi}{\gamma\mu(\gamma)M_0^2}\frac{\partial\varphi}{\partial\sigma} \tag{23}$$

$$\frac{\partial\delta}{\partial\tau}+\kappa_0\delta^2\frac{\partial(v\xi)}{\partial\sigma}=0 \tag{24}$$

$$\frac{\partial}{\partial\sigma}\left(\xi^2\delta\frac{\partial\varphi}{\partial\sigma}\right)=4\pi^2\gamma\mu(\gamma) \tag{25}$$

with initial conditions:

$$\left.\begin{array}{lll}
v|_{\tau=0}=v(\sigma,0) & p|_{\tau=0}=p(\sigma,0) & \phi|_{\tau=0}=\phi(\sigma,0)\\
\delta|_{\tau=0}=\delta(\sigma,0) & \xi|_{\tau=0}=\xi(\sigma,0) & \delta(0,0)=1\\
\left.\dfrac{\partial v(\sigma,\tau)}{\partial\tau}\right|_{\tau=0}=\dot v(\sigma,0) & \left.\dfrac{\partial p(\sigma,\tau)}{\partial\tau}\right|_{\tau=0}=\dot p(\sigma,0) & \\
\left.\dfrac{\partial\phi(\sigma,\tau)}{\partial\tau}\right|_{\tau=0}=\dot\phi(\sigma,0) & \left.\dfrac{\partial\delta(\sigma,\tau)}{\partial\tau}\right|_{\tau=0}=\dot\delta(\sigma,0) &
\end{array}\right\} \tag{26}$$

where $v(\sigma,0)$, $p(\sigma,0)$, $\phi(\sigma,0)$, $\delta(\sigma,0)$, $\xi(\sigma,0)$, $\dot v(\sigma,0)$, $\dot p(\sigma,0)$, $\dot\phi(\sigma,0)$, $\dot\delta(\sigma,0)$ are specified functions. The explicit form of the dependence of the function $\xi(\sigma,0)$ on σ will not be used in this article and so is not studied.

4 THE DEVELOPMENT OF THE RADIAL MOTION EQUATION

In this section the system of equations (23)–(25) will be reduced to one equation for the Euler coordinate $\xi(\sigma,\tau)$. For all functions the dependence on (σ,τ) will be omitted except the case $\tau=0$. This means that δ signifies $\delta(\sigma,\tau)$, but for $\tau=0$ we will write $\delta(\sigma, 0)$. The Poisson equation (25) may be integrated directly with initial condition:

$$\left.\frac{\partial \varphi}{\partial \sigma}\right|_{\sigma=0} = \text{const}.$$

According to the structure of equation (25), const should be finite but the value is not essential and it does not matter what the constant is. The first integral of the Poisson equation is:

$$\xi^2 \delta \frac{\partial \phi}{\partial \sigma} = 4\pi\gamma\mu(\gamma)\sigma - L(\tau) \tag{27}$$

where $L(\tau)$ is some function which will be defined later. Excluding $\xi\delta\frac{\partial \varphi}{\partial \sigma}$ from (23) and (27), the equation

$$\frac{\xi}{\kappa_0}\frac{\partial \mathrm{v}}{\partial \tau} = -\frac{\xi^2}{\gamma}\frac{\partial p}{\partial \sigma} - \frac{4\pi^2\gamma\mu(\gamma)\sigma - L(\tau)}{\gamma\mu(\gamma)M_0^2} \tag{28}$$

is obtained.

The following transformation is connected with the equation of continuity (24). Using the definition of velocity in the Lagrange variables:

$$\mathrm{v} = \frac{\partial \xi}{\partial \tau}$$

and dividing (24) by δ^2 the equation

$$\frac{\partial}{\partial \tau}\frac{1}{\delta} = \frac{\kappa_0}{2}\frac{\partial^2 \xi^2}{\partial \tau \partial \sigma}$$

is obtained. Its integral is

$$\frac{1}{\delta} - \frac{\kappa_0}{2}\frac{\partial \xi^2}{\partial \sigma} = G(\sigma) \tag{29}$$

where $G(\sigma)$ is some function which will be defined later. The next step is substituting ρ from (29) into the equation of state (16) and equation (28):

$$\frac{\xi}{\kappa_0}\frac{\partial^2 \xi}{\partial \tau^2} = -\frac{\xi^2}{\gamma}\frac{\partial}{\partial \sigma}\left(\frac{\kappa_0}{2}\frac{\partial \xi^2}{\partial \sigma} + G(\sigma)\right)^{-\gamma} - \frac{4\pi^2\gamma\mu(\gamma)\sigma - L(\tau)}{\gamma\mu(\gamma)M_0^2}. \tag{30}$$

To find the function $L(\tau)$ we calculate (30) for $\sigma = \xi = 0$:

$$L(\tau) \equiv 0.$$

To find the function $G(\sigma)$ we introduce a parameter $\ddot{\xi}_0(\sigma)$:

$$\left.\frac{\partial^2 \xi}{\partial \tau^2}\right|_{(\sigma,0)} = \ddot{\xi}_0(\sigma), \tag{31}$$

whose physical sense is the acceleration of the particle with coordinate σ at time $\tau = 0$. Calculating (30) for $\tau = 0$ an ordinary differential equation for the function $G(\sigma)$ is obtained:

$$G'(\sigma) + \kappa_0 \frac{\partial \xi}{\partial \sigma} \left(\xi \frac{\partial \xi}{\partial \sigma} \right) \bigg|_{\tau=0} = \left[\frac{\ddot{\xi}_0}{\kappa_0 \xi_0} + \frac{4\pi^2 \sigma}{M_0 \xi_0^2} \right] (\kappa_0 \xi_0 \dot{\xi}_0 + G(\sigma))^{\gamma+1} \tag{32}$$

where $' = \frac{d}{d\sigma}$. To find the initial condition for this equation we calculate equation (29) at the point $\sigma = 0$ and obtain

$$G(0) = 0. \tag{33}$$

So, the equation of radial motion which should be obtained in this section is

$$\frac{\xi}{\kappa_0} \frac{\partial^2 \xi}{\partial \tau^2} = -\frac{\xi^2}{\gamma} \frac{\partial}{\partial \sigma} \left(\frac{\kappa_0}{2} \frac{\partial \xi^2}{\partial \sigma} + G(\sigma) \right)^{-\gamma} - \frac{4\pi^2 \sigma}{M_0^2} \tag{34}$$

with the initial conditions

$$\xi|_{\tau=0} = \xi(\sigma, 0), \quad \frac{\partial \xi}{\partial \tau}\bigg|_{\tau=0} = \dot{\xi}(\sigma, 0) \tag{35}$$

where $\delta(\sigma, 0)$, $\xi(\sigma, 0)$ and $\dot{\xi}(\sigma, 0)$ are specified functions. The function $G(\sigma)$ is defined by equation (32) with initial condition (33) (according to (26)). Equation (34) is a non-linear non-stationary partial differential equation which cannot be solved analytically in the general case.

5 THE LINEARIZATION OF THE RADIAL MOTION EQUATION

Sections 5 and 6 are devoted to studying small perturbations of the gas. Arbitrary radial motions are described by equation (34) with initial conditions (35). According to the model the transformation of (34) and (35) to the linear approximation will now be made. Let us assume that a particle is in equilibrium at the point with Lagrange coordinate σ_0. Since the deviation of the particle from the point of equilibrium is small, the Euler coordinate of the particle may be presented as a sum:

$$\xi(\sigma, \tau) = \xi_0 + \psi(\sigma, \tau) \text{ where } |\psi(\sigma, \tau)| \ll \xi_0 \text{ and } \xi_0 = \xi(\sigma_0, 0). \tag{36}$$

To study the general case let us assume that the particle is displaced at the moment of time $\tau = 0$ from the point of equilibrium σ_0 to a new point with Lagrange coordinate $\sigma = \sigma_0 + \Delta\sigma$ and has a velocity equal to $\mathrm{v}(\sigma_0 + \Delta\sigma)$, where $\Delta\sigma$ is the particle displacement from the equilibrium point σ_0. So, the initial conditions for this model are

$$\xi(\sigma, 0) = \xi(\sigma_0, 0) + \psi(\sigma, 0), \quad \frac{\partial \xi(\sigma, \tau)}{\partial \tau}\bigg|_{\tau=0} = \dot{\psi}(\sigma, 0) \tag{37}$$

where

$$\left.\begin{aligned}\psi(\sigma,0) &= \Psi \int\limits_{0}^{+\infty} \delta[\sigma - (\sigma_0 + \Delta\sigma)]\,d\sigma \\ \dot{\psi}(\sigma,0) &= \dot{\Psi} \int\limits_{0}^{+\infty} \delta[\sigma - (\sigma_0 + \Delta\sigma)]\,d\sigma\end{aligned}\right\} \tag{38}$$

and Ψ, $\dot{\Psi}$ and $\Delta\sigma$ are the parameters of the problem, and $\delta[\sigma - (\sigma_0 + \Delta\sigma)]$ is a delta-function. Denote

$$\bar{\sigma} = \sigma_0 + \Delta\sigma.$$

In addition to (36) the condition

$$\left|\frac{\kappa_0}{2}\frac{\partial\xi^2}{\partial\sigma}\right| \ll G(\sigma) \tag{39}$$

will be used for the linearization of equation (34). The characteristic values used in (12) are now physical values calculated at the point σ_0. Equation (39) allows us to simplify the expression

$$\left(\frac{\kappa_0}{2}\frac{\partial\xi^2}{\partial\sigma} + G(\sigma)\right)^{\gamma+1} \approx G^{\gamma+1}(\sigma) + (\gamma+1)\frac{\kappa_0}{2}\frac{\partial\xi^2}{\partial\sigma}G^{\gamma}(\sigma) \tag{40}$$

and after substitution of (40) into (34) the following equation is obtained:

$$\left(\frac{1}{\kappa_0\xi}\frac{\partial^2\xi}{\partial\tau^2} + \frac{4\pi^2}{M_0^2}\frac{\sigma}{\xi^2}\right)\left(G^{\gamma+1} + \kappa_0\frac{\gamma+1}{2}G^{\gamma}\frac{\partial\xi^2}{\partial\sigma}\right) = \frac{\kappa_0}{2}\frac{\partial^2\xi^2}{\partial\sigma^2} + G' \tag{41}$$

where $G' = \frac{dG}{d\sigma}$ and $G_0' = \frac{dG}{d\sigma}\big|_{\sigma_0}$.

To linearize (41) use (36). This substitution gives the equation

$$\frac{\partial^2\psi}{\partial\tau^2} - \frac{\kappa_0^2\xi_0^2}{G_0^{\gamma+1}}\frac{\partial^2\psi}{\partial\sigma^2} + \frac{4\pi^2\kappa_0^2(\gamma+1)\bar{\sigma}}{G_0M_0^2}\frac{\partial\psi}{\partial\sigma} + \frac{4\pi^2\kappa_0\bar{\sigma}}{M_0^2\xi_0} - \frac{\kappa_0\xi_0G_0'}{G_0^{\gamma+1}} = 0. \tag{42}$$

To simplify this equation denote

$$w_0^2 = \frac{\kappa_0^2\xi_0^2}{G_0^{\gamma+1}}, \quad B = \frac{4\pi^2(\gamma+1)\kappa_0^2\bar{\sigma}}{G_0M_0^2}, \quad C = \kappa_0\xi_0\frac{G_0'}{G_0^{\gamma+1}} - \frac{4\pi^2\kappa_0\bar{\sigma}}{M_0^2\xi_0} \tag{43}$$

Then equation (42) reads:

$$\frac{\partial^2\psi}{\partial\tau^2} - w_0^2\frac{\partial^2\psi}{\partial\sigma^2} + B\frac{\partial\psi}{\partial\sigma} - C = 0 \tag{44}$$

with initial conditions (38). B has the meaning of acceleration. According to this, two new characteristic values are introduced:

$$\sigma^* = \frac{w_0^2}{B} \text{ and } \tau^* = \frac{\sigma^*}{w_0}. \tag{45}$$

Equation (44) will be transformed now by the substitution

$$\psi(\sigma,\tau) = u(\sigma,\tau)\exp\frac{\sigma-\bar{\sigma}}{2\sigma^*} + C(\tau^*)^2\frac{\sigma-\bar{\sigma}}{\sigma^*} \tag{46}$$

where $u(\sigma,\tau)$ is a new function. The result of the substitution (46) into (44) is:

$$\frac{\partial^2 u}{\partial\tau^2} - w_0^2\frac{\partial^2 u}{\partial\sigma^2} + \frac{u}{(2\sigma^*)^2} = 0. \tag{47}$$

For this equation the Cauchy problem is calculated with the following initial conditions which are obtained by transformation according to (46):

$$u(\sigma,0) = \psi(\sigma,0), \quad \left.\frac{\partial u(\sigma,\tau)}{\partial\tau}\right|_{\tau=0} = \dot{\psi}(\sigma,0) \tag{48}$$

where $\psi(\sigma,0)$ and $\dot{\psi}(\sigma,0)$ are defined by equation (38). Equation (47) is the Klein–Gordon equation (Zaslavskij and Sagdeev, 1988).

6 THE SOLUTION OF THE KLEIN–GORDON EQUATION

To solve the Klein–Gordon equation the Riemann method of solving the Cauchy problem will be used (Koshl'akov *et al.*, 1962). Equation (47) is a hyperbolic equation. In the canonical coordinates (ν,η):

$$\nu = \frac{\sigma + w_0\tau}{2\sigma^* w_0}, \quad \eta = \frac{\sigma - w_0\tau}{2\sigma^* w_0} \tag{49}$$

equation (47) is reduced to the form:

$$\frac{\partial^2 u}{\partial\nu\partial\eta} - \frac{u}{4} = 0 \tag{50}$$

with initial conditions (48) transformed according to the rules (49):

$$\left.\begin{aligned} u[\sigma^* w_0(\nu+\eta)]|_{\nu=\eta} &= u(\sigma,0) \\ \frac{1}{2\sigma^*}\left.\left(\frac{\partial u}{\partial\nu} - \frac{\partial u}{\partial\eta}\right)\right|_{\nu=\eta} &= \left.\frac{\partial u(\sigma,\tau)}{\partial\tau}\right|_{\tau=0} \end{aligned}\right\} \tag{51}$$

In the Riemann method the solution of equation (50) is determined by (51) and the Riemann function $V(\nu,\eta)$.

To describe the Riemann function two kinds of canonical variables are introduced as well: (ν,μ) are the coordinates of the point where the solution is found and $(\hat{\nu},\hat{\eta})$ belong to the line in which the initial conditions are specified. The function V satisfies the conjugate of equation (50):

$$\frac{\partial^2 V}{\partial\nu\partial\eta} - \frac{V}{4} = 0 \tag{52}$$

with the initial conditions:

$$V(\nu,\hat{\eta}) = 1, \quad V(\hat{\nu},\eta) = 1. \tag{53}$$

Let us find the Riemann function V in the form

$$V = N(p) \text{ where } p = \sqrt{(\nu-\hat{\nu})(\hat{\eta}-\eta)}. \tag{54}$$

The substitution of (54) into (52) gives the ordinary differential equation

$$N'' + \frac{1}{p}N' + N = 0$$

which has finite solution

$$N(p) = J_0\left(\sqrt{(\nu-\hat{\nu})(\hat{\eta}-\eta)}\right)$$

where J_0 is a Bessel function of null order. The initial conditions (53) are satisfied. Let us return to equation (50). According to the Riemann method the solution of equation (50) satisfying the conditions (51) has the form:

$$\begin{aligned} u(\nu,\eta) &= \frac{1}{2}\{u[2\sigma^* w_0\nu] + u[2\sigma^* w_0\eta]\} \\ &+ \frac{1}{2}\int_\eta^\nu J_0\left(\sqrt{(\nu-\hat{\nu})(\hat{\nu}-\eta)}\right)\left(\frac{\partial u}{\partial\nu} - \frac{\partial u}{\partial\eta}\right)\Big|_{\hat{\nu}=\hat{\eta}} d\hat{\nu} \\ &- \frac{\nu-\eta}{4}\int_\eta^\nu \frac{J_0'(\sqrt{(\nu-\hat{\nu})(\hat{\nu}-\eta)})}{\sqrt{(\nu-\hat{\nu})(\hat{\nu}-\eta)}} u[\sigma^* w_0(\hat{\nu}+\hat{\eta})\Big|_{\hat{\nu}=\hat{\eta}}]d\hat{\nu}. \end{aligned} \tag{55}$$

After substituting (51) and

$$\nu-\eta = \frac{\tau}{\sigma^*}, \quad (\nu-\hat{\nu})(\hat{\nu}-\eta) = \frac{(w_0\tau)^2 - (\sigma-\bar{\sigma})^2}{(2\sigma^* w_0)^2}$$

into (55) the solution of equation (50) with initial conditions (48) is

$$\begin{aligned} u(\sigma,\tau) &= \frac{\Psi}{2}\int_0^{+\infty}\{\delta[w_0\tau + (\sigma-\bar{\sigma})] + \delta[w_0\tau - (\sigma-\bar{\sigma})]\}d\sigma \\ &+ \dot{\Psi}\sigma^* J_0\left(\frac{\sqrt{(w_0\tau)^2-(\sigma-\bar{\sigma})^2}}{2\sigma^* w_0}\right) \\ &+ \frac{\tau\Psi w_0}{4}\frac{J_1\left(\frac{\sqrt{(w_0\tau)^2-(\sigma-\bar{\sigma})^2}}{2\sigma^* w_0}\right)}{\sqrt{(w_0\tau)^2-(\sigma-\bar{\sigma})^2}} \end{aligned} \tag{56}$$

where J_1 is the Bessel function of first order. This formula together with (46) defines the solution of the linear problem.

7 RESULTS AND DISCUSSION

This article studies the pure radial motion of a gas with pressure and self-gravitation. We start from the Euler equations in the cylindrical coordinates (19)–(21) and (16) and by the transformation rules (22) the Lagrange equation (23)–(25) are obtained. This system is reduced to one partial non-linear non-stationary differential equation (34) for the Euler coordinate $\xi(\sigma, \tau)$. The linear approximation (44) of this equation is obtained with the extra conditions (36) and (39). Equation (44) is reduced to the Klein–Gordon equation (47) using the substitution (46).

The cylindrical symmetry model of perturbation is used in this article. To study small perturbations of the stationary state the simple initial conditions (37), (38) are chosen. These conditions describe the model of perturbation analogous to the movement of weight on a string. The analytical solution of the linear problem (44), (37), (38) is represented by equations (56) and (46).

The solution obtained has the formal character of oscillations, but there are two reasons for which this conclusion should be treated critically: first, this is the solution of a linear equation; the area of the initial conditions where the linear solution differs slightly from the non-linear one is not discussed in this article; second, the solution is obtained in Lagrange variables but when we are speaking about oscillation, Euler variables are meant. The transformation from Lagrange variables to Euler variables is not performed in this article.

The solution obtained is non-symmetrical with respect to changing the σ direction around the point $\bar{\sigma}$ (see (46)).

The solution obtained depends on three parameters v_0, λ_0, t_0 introduced in (9). v_0 characterized the initial perturbation, not depending on local properties of the medium and specified by the initial conditions, but λ_0 and t_0 should be defined by the other parameters of the problem. There are two problems corresponding to two kinds of inital condition of the Cauchy problem.

First, initial conditions are delta-functions and do not include characteristic length and time as in our case. The values λ_0 and t_0 should be defined by the interaction between the perturbation and the medium. To find them two extra equation are needed. The initial perturbation is approximately constant in the small environment of the initial point. So, λ and t_0 may be defined in this environment by a solution of the dispersion equation corresponding to the Klein–Gordon equation. The second equation is the approximation of the initial condition (37).

The second kind of initial condition is not a delta-function. Two characteristic length scales correspond to this problem: the length scale of the initial perturbation (initial impulse range, for example) and the length scale connected with the interaction between the perturbation and the medium. The time scale may be defined by this interaction or from the approximation of the initial conditions. The area of the initial conditions where the linear solution differs slightly from the non-linear one is not discussed in this article, but we will now use the approximation of the initial conditions (48) to find λ_0 and τ_0. Let the solution of (47) be represented by the plane wave around the point $\bar{\sigma}$

$$u \sim \exp 1(\kappa_0(\sigma - \bar{\sigma}) - w_0 \tau).$$

where

$$w_0 = \frac{1}{\tau_0}, \quad \tau_0 = \frac{t_0}{\sqrt{G\rho_0}}.$$

The dispersion equation corresponding to the Klein–Gordon equation (47) is

$$w_0^2 = (w_0 \kappa_0)^2 + \frac{1}{(2\sigma^*)^2}. \tag{57}$$

The approximation of the initial condition (48) gives

$$\frac{\Delta u(\bar{\sigma}, 0)}{\tau_0} = \dot{\Psi}. \tag{58}$$

Substituting (58) into the transformation rules (46) and using (37) we obtain the definition of τ_0:

$$\tau_0 = \frac{\Psi}{\dot{\Psi}}.$$

This together with (57) defines κ_0 as well:

$$\left(\frac{\dot{\Psi}}{\Psi}\right)^2 = (w_0 \kappa_0)^2 + \frac{1}{(2\sigma^*)^2}. \tag{59}$$

Replacing the expression of w_0 from (43) and σ^* from (45) by the initial conditions and parameters of the problem the following equation is obtained:

$$\left(\frac{\dot{\Psi}}{\Psi}\right) = B_1 \frac{\kappa_0^4}{G_0^{\gamma+1}} + B_2 G_0^{2\gamma} \tag{60}$$

where $B_1 = \xi_0^2$ and $B_2 = \frac{\pi^2(\gamma+1)\bar{\sigma}}{\xi_0^2 M_0^2}$. This solution shows a set of parameters of this problem, which satisfies the linear approximation, defines κ_0 and gives the correlation between the parameters of the problem. The condition (39) is carried out in case $\kappa_0 \to 0$. Assuming $\kappa_0 = 0$ we obtain the main contribution in the solution of the equation (60):

$$\left.\frac{\dot{\Psi}}{\Psi}\right|_{\kappa_0=0} = \pm\sqrt{B_2} G^{\gamma}.$$

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