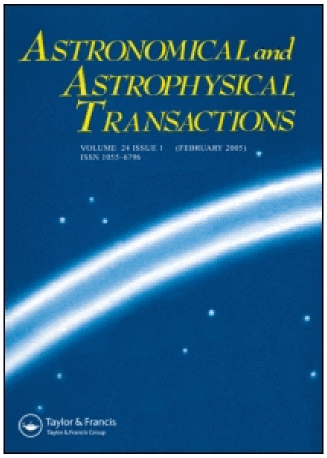

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NEW CONTINUED FRACTION FORM OF THE MAPPING FUNCTIONS OF ATMOSPHERIC REFRACTION CORRECTIONS

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As basic research problems in practical astronomy and geodesy, the formulation of the mapping functions both of the refractive delay and of astronomical refraction are studied in an analytical manner in this paper. First, we have proved that the complementary error function can be related to expressions for the atmospheric refraction integrals of the signal delay and also of the signal bending angle as well. An improved continued fraction expansion of the complementary error function is then derived to describe the new mapping functions of the refractive delay and astronomical refraction at radio and optical frequencies, respectively. Not only do the new mapping functions have good convergence at an elevation coverage near to the horizon, but they can also be applied to various atmospheric profiles at high accuracy, better than 1 cm for refraction delay and better than $0''.3$ for angular bending at an elevation angle near to $2^\circ.5$ or lower over a wide range of meteorological and geophysical conditions. Some extra corrections are also briefly summarized in this paper.

KEY WORDS Atmospheric refraction: mapping functions

1 INTRODUCTION

Astronomical refraction is one of the oldest topics in astronomy. As said by Newcomb almost 90 years ago, "There is perhaps no branch of practical astronomy on which so much has been written as on this, and which is still in so unsatisfactory a state" (Newcomb, 1906). In modern space techniques, the refractive delay is a critical topic in applications to astrometry, geodesy and geophysics. It is known that the accuracies of modern space techniques, such as Very Long Baseline Interferometry (VLBI), Satellite Laser Range (SLR) and Global Positioning System (GPS), have reached one centimetre or better; however the accuracy of atmospheric refraction is in the same magnitude of accuracy (Herring *et al.*, 1990). Much effort has been made in different aspects of this field; some progress was reported in the last few decades upon discussions of mathematical expressions and of physical properties for atmospheric refraction (Chao, 1972a, b; Davis *et al.*, 1985; Herring, 1992; Niell, 1996), but less progress has been made in the theory.

In mathematics atmospheric integrals are not analytically integrable, and in physics the atmosphere of the Earth is complicated and changeable with time, space and other geographic and geophysical parameters. But there is a critical situation met in refraction research: the formulas used for astronomical refraction in classical astronomical observations (the Astronomical Almanac, 1996) and for refractive delay in modern techniques are based on series expansions of refractive integrals (Marini and Murray, 1973) or on mathematical empirical expressions (Marini, 1972; Davis *et al.*, 1985; Herring, 1992); and the accuracies of these formulas for atmospheric refraction corrections are greatly limited for observational elevations near 10° for astronomical refraction and to 5° for refractive delay. The method of series expansion has been used for hundreds years in astronomical refraction (Woolard and Clemence, 1966) and was also recently considered in refractive delay. The accuracies of this method are limited, and some compensative terms or tables have to be added when higher accuracies are required (Saastamoinen, 1972). The empirical continued fraction form of the mapping function was first introduced by Marini (1972); it has contributed to the research of refractive delay. Because of the defaults in the mathematical form, the old continued fraction mapping function has poor behaviour when observational elevation is near the horizon. More sophisticated discussions on astronomical refraction and atmospheric profiles have been offered by Radau (1882), Danjon (1952) and Garfinkel (1967).

The main subject of this paper is the discussion of the theoretical derivation of the mapping function and to establishing new models of high accurate mapping functions of refractive delay and astronomical refraction for practical astronomy, geodesy and modern space technique facilities.

2 ATMOSPHERIC PROFILES

The atmosphere of the Earth is divided into a neutral layer near the Earth and an outer ionized layer. We will only deal with the former in this paper, for the effects of latter can be cancelled to some extent by multifrequency observations. Because of the complexity of the true Earth's atmospheric construction in space and of the variability in time, the research of atmospheric refraction becomes more difficult. The first step in refraction study is to use a rather simple mathematical atmospheric model to describe the true atmosphere.

An exponential atmospheric profile was derived from the static isotherm condition (Thayer, 1961; Rowlandson and Moldt, 1969). Although the exponential model is too simple for an accurate resolution of the mapping function, it has a special role in theoretical deduction of the new mapping function (Yan and Ping, 1995; Yan, 1996). Another commonly used atmospheric profile in refraction research is Hopfield's quartic refractivity profile (Hopfield, 1969; 1971), which is based on the assumption of an atmosphere with constant lapse rate of temperature. This model still has some use today in GPS applications. The model widely used in modern space techniques is a spherically symmetric, layered atmosphere constructed by the troposphere with a constant lapse rate of temperature near the Earth's ground and the stratosphere with constant temperature up the troposphere. The boundary between the troposphere and the stratosphere is the tropopause. It is commonly agreed that this atmospheric model is a good approximation to the true atmosphere (Allen, 1973). More detailed discussion about atmospheric profiles needs be related to local or global characteristics of the atmosphere of the Earth (Herring, 1992; Niell, 1996).

3 BASIC MATHEMATICAL PROBLEMS

In propagation of electromagnetic signals through the medium near the Earth, the Fermat principle holds: an electromagnetic wave travelling between two arbitrary points A and B takes the shortest path; and the travel time

$$T = \int_A^B dt \tag{1}$$

is a minimum. From the law of refraction the astronomical refraction Δz_0 defined as the difference between the true zenith distance ξ_0 and the arrival zenith distance z_0 at a station, see Figure 1 (here the source S is considered to be at infinity), can be expressed as an integral (Saastamoinen, 1972):

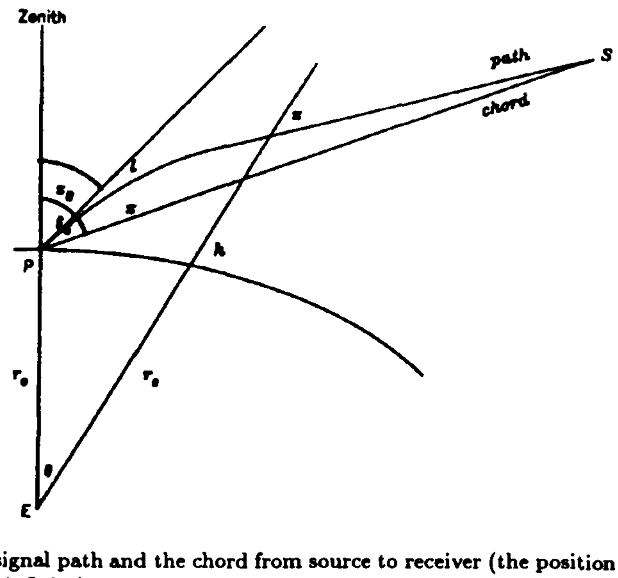

Figure 1 The signal path and the chord from source to receiver (the position of the source S is assumed to be at infinity).

$$\Delta z_0 = \xi_0 - z_0 = \int_1^{n_0} \frac{\tan z}{n} dn, \tag{2}$$

where n is the refractive index of the atmosphere, which is defined as the ratio of the constant light velocity in vacuum, c, to the instantaneous light velocity, v, at the field point, and n_0 is the value taken at the station; z is the arrival zenith distance of the signal taken at the field point on the path. The refraction correction of the signal delay in atmospheric propagation, $\Delta\sigma$, is then written as the difference between the optical distance σ and the geometrically straight distance X between the source S and receiver P (Figure 1);

$$\Delta\sigma = \sigma - X = \int_l n\, dl - \int_x dx, \tag{3}$$

which can be related to an integral along the vertical coordinate h:

$$\Delta\sigma = \int_{h_0}^{h_a} \frac{n-1}{\cos z} dh + \delta\sigma, \tag{4}$$

where dl and dx are the elements on the signal path and on the chord x between the source and receiver, respectively; h_0 and h_a are the heights of the station and the top of the atmosphere, respectively; $\delta\sigma$ is called the ray-bending correction term in refractive delay, which originates from the difference of geometric lengths between the path of the signal l and the straight line x within the source and receiver, and is much smaller than the first main term.

Equations (2) and (4) are the basic mathematical equations for studying refractive corrections of the signal delay and bending.

It was a revolutionary step in atmospheric refraction research to divide the refractive delay into the mapping function and the zenith delay, as made by Marini (1972). Since the zenith delay can be described fairly well by ground meteorological measurements at a station, the study of the mapping function becomes the basic problem in atmospheric refraction. We then write the atmospheric delay $\Delta\sigma$ as (Marini, 1972):

$$\Delta\sigma = \Delta\sigma_z m(E, \mathbf{p}), \tag{5}$$

in which $\Delta\sigma_z$ is the zenith delay; $m(E, \mathbf{p})$ is the mapping function; here E, in contrast to Marini's expression, is the true elevation; and $\mathbf{p}$ represent the meteorological and geophysical variables, such as temperature, pressure, humidity, height of the tropopause etc. Some continued fraction forms of the mapping function can be found in Marini (1972), Davis *et al.* (1985) and Herring (1992), but all of them are mathematically deficient in rigorous derivations and physically less connected to the geophysical characteristics.

4 GENERATOR FUNCTION METHOD

Rowlandson and Moldt (1969) demonstrated that refractive features of an exponential atmospheric profile could be approximately expressed by a complementary error function. Yan and Ping (1995) and Yan (1996) further proved that expansion forms of the complementary error function could offer appropriate forms of the mapping functions to both refractive delay and astronomical refraction.

In an exponential atmospheric profile, the distribution function of refractivity, $N(h)$, is written as a function of the vertical coordinate h of the field point (Froome and Essen, 1969):

$$N(h) = N_0 e^{-\frac{(h-h_0)}{H}}, \tag{6}$$

where the refractivity N is related to the refractive index n by

$$N = 10^6(n-1);$$

and N_0 is the value measured at a station of heigh h_0; the parameter

$$H = \frac{RT}{Mg} \tag{7}$$

in equation (6) is defined as the effective height of the atmosphere of exponential profile, in which R is the universal gas constant, M the molar mass of the atmosphere, T the temperature of the atmosphere in Kelvin and g is the gravitation acceleration constant taken at the centre of the vertical air column (Saastamoinen, 1972). The effective height H of the atmosphere can also be written as a vertical integration function of the refractivity N:

$$H = \frac{1}{N_0} \int_{h_0}^{\infty} N(h)\, dh. \tag{8}$$

For convenience, we set $h_0 = 0$; this will have no influence on our discussion. If we ignore the ray-bending term in equation (4), then the refractive delay $\Delta\sigma$ in the Earth's atmosphere can be approximately expressed by the integrals along the path of the signal l:

$$\Delta\sigma \approx 10^{-6} \int_{l} N(l)\, dl. \tag{9}$$

This equation is widely used in the literature. If we further use an integral along the chord x connecting the receiver P and the source S instead of integrating along the path, and note the approximate relationship between h and x:

$$h \approx x \cos \xi_0 + \frac{x^2 \sin^2 \xi_0}{2r_0}, \tag{10}$$

we can deduce the mapping function in equation (5) related to the complementary error function (Yan and Ping, 1995):

$$m(\xi_0) \approx \frac{\sqrt{\pi}}{\sin \xi_0} \sqrt{\frac{r_0}{2H}} e^{I^2} erfc(I). \tag{11}$$

In the above equation, the complementary error function $erfc(x)$ is defined as:

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2}\, dy, \tag{12}$$

which is taken to be the *generator function* of the mapping function of the refractive delay; r_0 is the radius of the Earth and ξ_0 is the true zenith distance at the station; and the parameter

$$I = \sqrt{\frac{r_0}{2H}} \cot \xi_0, \tag{13}$$

is defined as the *normalized effective zenith argument.*

From equation (2) and the differential relation

$$dn = -10^{-6} \frac{N(h)}{H} dh, \tag{14}$$

we can rewrite the astronomical refraction Δz_0 as the approximation:

$$\Delta z_0 \approx 10^{-6} \frac{\sin z}{H} \int_{h_0}^{\infty} \frac{N(h)}{n \cos z} dh. \tag{15}$$

According to the definition of refractive delay, the above expression can be formally written as (Yan, 1996)

$$\Delta z_0 = \sin \xi_0 \frac{\Delta \sigma_z}{H} m'(\xi_0) = 10^{-6} N_0 \sin \xi_0 m'(\xi_0), \tag{16}$$

where the function $m'(\xi_0)$ can be defined as the *mapping function of astronomical refraction*, which can also be related to the complementary error function similar to the case of refractive delay in equation (11).

The errors in the above approximate procedures applied in this section can be reduced to some extent by the proper selection of the coefficients of expansion of the complementary error function as shown in the following sections.

5 ZENITH DELAY

The value of the zenith delay $\Delta \sigma_z$ in equation (5) can be easily calculated from the integral

$$\Delta \sigma_z = 10^{-6} \int_{h_0}^{h_p} N\, dh, \tag{17}$$

in which the refractivity N at radio frequencies can be obtained from the Smith–Weintraub equation (Smith and Weintraub, 1953):

$$N = 77.6\frac{P}{T} - 12.8\frac{e}{T} + 3.776 \times 10^5 \frac{e}{T^2}, \tag{18}$$

in which P is the total pressure in mbars, T the temperature in Kelvin and e the wet partial in mbars. At optical frequencies, we have to take the dispersive effect into account (Gardner and Rowlett, 1976). We can write the frequency-related group refractivity N_g as (Yan, 1996):

$$N_g = 82.4148 f(\lambda)\frac{P}{T} - 11.268\frac{e}{T}, \tag{19}$$

in which the frequency correction term

$$f(\lambda) = 0.94075 + \frac{0.01598}{\lambda^2} + \frac{0.0002224}{\lambda^4}, \tag{20}$$

is written as a function of wavelength λ. In contrast to the expression given by Marini and Murray (1973), in the above expressions the relation

$$f(0.5320) = 1$$

holds.

The zenith delay can be proved to have a similar relation to the exponential atmosphere for our atmospheric model mentioned in Section 2 (Appendix B, Yan *et al.*, 1996):

$$\Delta\sigma_z = 10^{-6} N_0 \frac{T_0 R}{g}, \tag{21}$$

where the gravitation acceleration constant g is represented by (Saastamoinen, 1972)

$$g = 9.784 W(\phi, h_0), \tag{22}$$

in which the correction term for latitude ϕ and height h_0 (in kilometres) of a station is written as:

$$W(\phi, h_0) = 1 - 0.00266 \cos 2\phi - 0.00028 h_0. \tag{23}$$

At optical frequencies, the zenith delay can be simply obtained by

$$\begin{aligned} \Delta\sigma_z &= \Delta\sigma^{0.532} f(\lambda) \\ &= \frac{f(\lambda)}{W(\phi, h_0)}(0.0024178 P_0 + 0.00014586 e_0), \end{aligned} \tag{24}$$

in which $\Delta\sigma_z^{0.532}$ means the corresponding value for a YAG laser, $\lambda = 0.532\ \mu$m.

6 MAPPING FUNCTION

From the continued fraction expression of the incomplete Γ-function (Press *et al.*, 1991), the complementary error function $erfc(z)$ can be readily written as a continued fraction

$$erfc(z) = \frac{1}{\sqrt{\pi}} e^{-z^2} \frac{1}{z + \frac{0.5}{z + \frac{1}{z + \frac{1.5}{z + \frac{2}{z + \dots}}}}} \tag{25}$$

It is therefore appropriate to rewrite the mapping function of equation (11) as an improved continued fraction form

$$m(\xi_0) = \frac{1}{\cos\xi_0 + \frac{D_1}{I^2 \sec\xi_0 + \frac{D_2}{\cos\xi_0 + \frac{D_3}{I^2 \sec\xi_0 + D_4}}}}, \tag{26}$$

in which at radio frequencies the coefficients D_i can be obtained under the atmospheric model mentioned in Section 2 by means of a least squares adjustment for integrated values along the paths of signals:

$$\begin{aligned}
D_1 &= 0.4613983 + 2.864 \times 10^{-5}(P_0 - 1013.25) \\
&+ 8.99 \times 10^{-6} e_0 - 6.98 \times 10^{-6} e_0^2 \\
&- 1.0914 \times 10^{-4}(T_0 - 15) + 1.30 \times 10^{-6}(T_0 - 15)^2 \\
&+ 9.4694 \times 10^{-3}(\beta + 6.5) \\
&- 2.4946 \times 10^{-3}(h_t - 11.231) + 1.8072 \times 10^{-4}(h_t - 11.231)^2, \\
D_2 &= 0.8276476 + 2.056 \times 10^{-5}(P_0 - 1013.25) \\
&+ 2.3820 \times 10^{-4} e_0 - 4.76 \times 10^{-6} e_0^2 \\
&+ 5.1125 \times 10^{-4}(T_0 - 15) + 1.23 \times 10^{-6}(T_0 - 15)^2 \\
&+ 3.6479 \times 10^{-2}(\beta + 6.5) \\
&- 1.5321 \times 10^{-2}(h_t - 11.231) + 9.4802 \times 10^{-4}(h_t - 11.231)^2, \\
D_3 &= 2.531492 + 1.093 \times 10^{-4}(P_0 - 1013.25) \\
&+ 2.6179 \times 10^{-3} e_0 + 1.33 \times 10^{-5} e_0^2 \\
&+ 3.7103 \times 10^{-3}(T_0 - 15) + 4.95 \times 10^{-6}(T_0 - 15)^2 \\
&+ 1.6022 \times 10^{-1}(\beta + 6.5) \\
&- 8.9980 \times 10^{-2}(h_t - 11.231) + 4.9496 \times 10^{-3}(h_t - 11.231)^2, \\
D_4 &= 47.07844 + 1.595 \times 10^{-3}(P_0 - 1013.25) \\
&+ 3.9026 \times 10^{-2} e_0 + 2.41 \times 10^{-4} e_0^2 \\
&- 4.1713 \times 10^{-2}(T_0 - 15) + 2.16 \times 10^{-4}(T_0 - 15)^2 \\
&+ 1.6313(\beta + 6.5) \\
&- 9.9757 \times 10^{-1}(h_t - 11.231) + 4.4528 \times 10^{-2}(h_t - 11.231)^2,
\end{aligned} \tag{27}$$

in which P_0 is the total ground pressure in millibars; e_0 the wet partial pressure in millibars; T_0 the ground temperature in Celsius; β the tropospheric temperature lapse rate in K km^{-1}, and h_t the height of the tropopause in kilometres. The above version is named UNSW931. The accuracy of the UNSW931 model is better than 1 cm to an elevation lower than 2°5 over a wide range of meteorological conditions. The constant system of geophysical parameters in this paper is taken as: the radius of the Earth is $r_0 = 6378$ km; the gravitational accelaration is $g = 978.40$ cm s^{-2} (Saastamoinen, 1972); $T(\mathrm{K}) = T(°\mathrm{C}) + 273.15$; the universal gas constant is $R = 8314.34$ Joules kmol^{-1} K^{-1}; and the molecular weight of the atmosphere is $M_0 = 28.970$ kg kmol^{-1}. It should be noticed that apart from the definition of the effective height of the atmosphere in the exponential model of equation (7), the parameter H of a non-exponential atmospheric profile is re-defined as the *normalized effective height* of the atmosphere and represented by equation (8).

For optical frequencies we have the coefficients D_i as follows:

$$
\begin{aligned}
D_1 &= 0.463184 + 3.019 \times 10^{-5}(P_0 - 1013.25) \\
&- 1.222 \times 10^{-4}(T_0 - 15) + 1.1 \times 10^{-6}(T_0 - 15)^2 \\
&- 9.122 \times 10^{-3}(\lambda - 0.532) + 2.74 \times 10^{-2}(\lambda - 0.532)^2 \\
D_2 &= 0.828752 + 1.905 \times 10^{-5}(P_0 - 1013.25) \\
&+ 5.203 \times 10^{-4}(T_0 - 15) + 0.6 \times 10^{-6}(T_0 - 15)^2 \\
&- 5.887 \times 10^{-3}(\lambda - 0.532) + 1.82 \times 10^{-2}(\lambda - 0.532)^2 \\
D_3 &= 2.53662 + 0.9095 \times 10^{-4}(P_0 - 1013.25) \\
&+ 3.869 \times 10^{-3}(T_0 - 15) + 0.3 \times 10^{-6}(T_0 - 15)^2 \\
&- 2.787 \times 10^{-2}(\lambda - 0.532) + 8.76 \times 10^{-2}(\lambda - 0.532)^2 \\
D_4 &= 47.1584 + 1.377 \times 10^{-3}(P_0 - 1013.25) \\
&- 3.584 \times 10^{-2}(T_0 - 15) + 1.1 \times 10^{-4}(T_0 - 15)^2 \\
&- 4.291 \times 10^{-1}(\lambda - 0.532) + 1.34 \times 10^{-4}(\lambda - 0.532)^2, \qquad (28)
\end{aligned}
$$

where λ is the wavelength of the signal in micrometres. Equations (24) and (28) can be used in place of the formulas given by Marini and Murray (1973) for low elevations of the SLR facility with higher accuracies. The formulas of Marini and Murray have been used for more than twenty years and are used in the elevation range above 10°.

For astronomical refraction, the mapping function $m'(\xi_0)$ in equation (16) can be written as (Yan, 1996):

$$
m'(\xi_0) = \frac{1}{\cos\xi_0 + \frac{A_1}{I^2 \sec\xi_0 + \frac{A_2}{\cos\xi_0 + \frac{13.24969}{I^2 \sec\xi_0 + 173.4233}}}}, \qquad (29)
$$

where the normalized effective zenith argument I was defined in equation (13), and H is the normalized effective height of the atmosphere given by equation (8).

The meteorological parameters in A_1 and A_2 of the above equation also come from the least squares adjustment procedure of the numerical integrals of equation (2) along the paths of signals for different ground meteorological parameters and various elevations. At radio frequencies, these coefficients can be written as:

$$\begin{aligned} A_1 &= 0.5753868 + 0.5291\times 10^{-4}(P_0 - 1013.25) - 0.2819\times 10^{-4}e_0 \\ &- 0.9381\times 10^{-6}e_0^2 - 0.5958\times 10^{-3}(T_0 - 15) + 0.2657\times 10^{-5}(T_0 - 15)^2 \\ A_2 &= 1.301211 + 0.2003\times 10^{-4}(P_0 - 1013.25) - 0.7285\times 10^{-4}e_0 \\ &+ 0.2579\times 10^{-5}e_0^2 - 0.2595\times 10^{-2}(T_0 - 15) + 0.8509\times 10^{-5}(T_0 - 15)^2. \end{aligned} \tag{30}$$

The parameters T_0, P_0 and e_0 are the same as before. The atmospheric profile is the same as in Section 2. The accuracy of equations (29) and (30) is better than $0.''3$ in the elevation range above 2° for a wide range of meteorological conditions. For radio frequencies, the wet partial pressure distribution is based on the exponential distribution model given by Allen (1973) and incorporated with the Magnus empirical formula (Marini and Murray, 1973).

At optical frequencies, the appropriate coefficients of equation (29) are:

$$\begin{aligned} A_1 &= 0.5787089 + 0.5609\times 10^{-4}(P_0 - 1013.25) \\ &- 0.6229\times 10^{-3}(T_0 - 15) + 0.2824\times 10^{-5}(T_0 - 15)^2 \\ &+ 0.5177\times 10^{-3}e_0 + 0.29\times 10^{-6}e_0^2 \\ &- 0.1644\times 10^{-1}(\lambda - 0.532) + 0.491\times 10^{-1}(\lambda - 0.532)^2 \\ A_2 &= 1.302474 + 0.2142\times 10^{-4}(P_0 - 1013.25) \\ &+ 0.1287\times 10^{-2}e_0 + 0.65\times 10^{-6}e_0^2 \\ &- 0.6298\times 10^{-2}(\lambda - 0.532) + 0.189\times 10^{-1}(\lambda - 0.532)^2 \\ &- 0.2623\times 10^{-2}(T_0 - 15) + 0.8776\times 10^{-5}(T_0 - 15)^2, \end{aligned} \tag{31}$$

in which the dispersion is compensated these can be used for the polychrome laser technique. In equations (28), (30) and (31), we do not take the tropospheric temperature lapse rate β and the height of the tropopause h_t as variables of the mapping function, because they are usually not available for normal stations; so we take $\beta = -6.5$ km and $h_t = 11.23$ km in the above equations.

7 CORRECTION TERMS

In Yan and Ping (1995) and Yan (1996), some necessary correction terms in atmospheric refraction have been discussed. The correction related to the position of a station is given by equation (23) which originates from the correction of the gravitation acceleration constant. The wavelength correction is involved in the mapping functions by equations (28) and (31) for refractive delay and astronomical refraction, respectively; and the influence on the zenith delay is described by equation (24). For a finite-distance object observed at low elevation the correction formulas can be found in Yan (1996).

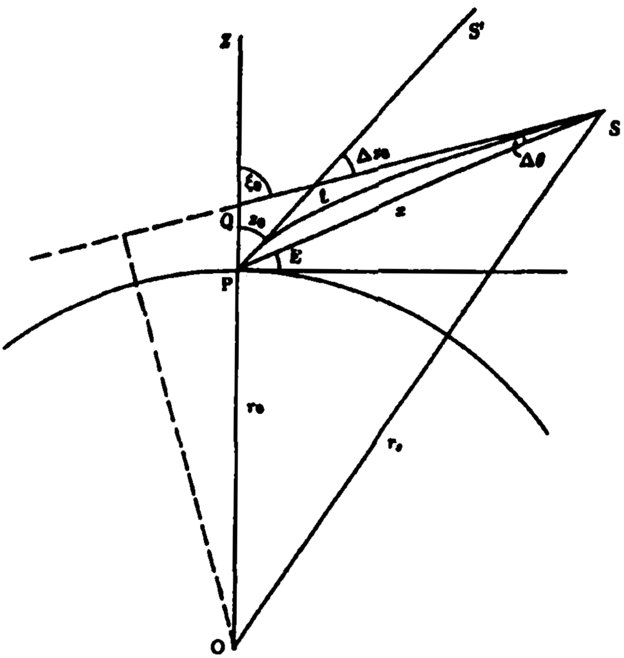

Figure 2 Geometry of a finite-distance object.

Marini (1972) first noticed the correction of a finite distance object in refractive delay, but his results had less accuracy. As shown in Figure 2, Murray (1983) defined the arrival direction z_0 as the tangent to the light trajectory at the observer, the true direction $\zeta_0 = 90° - E$ as the direction connecting the observer P to the source S, and the proper direction ξ_0 as parallel to the direction tangent to the light trajectory at the source, and Q the equivalent point of observation; PQ is the equivalent height for the source S. The proper direction is that direction in which a hypothetical observer would see the source if there were no atmosphere. From Section 6 we can see that the proper direction corresponds to the angular argument of the mapping function for a finite-distance object (Yan, 1996), and it is obvious that for an objectal infinity the true direction is in coincidence with the proper direction.

From Figure 2, if the refractive index at the source S is unity, $n(r_s) = 1$, we have the equivalent height:

$$QP = \left(\frac{n_0 \sin z_0}{\sin \xi_0} - 1 \right) r_0, \tag{32}$$

and the difference $\Delta\theta$ is written as:

$$\Delta\theta = \xi_0 - \zeta_0, \tag{33}$$

which describes the correction of the angular argument in the mapping function.

The true zenith distance can be accurately calculated by the ephemeris of the object and the geophysical parameters. Then the correction for the angular argument is written as (Yan, 1996):

$$\Delta\theta = \frac{r_0}{x}(n_0 \sin z_0 - \sin \xi_0). \tag{34}$$

Here x is the geometric distance between station and source, and n_0 is the value of the refractive index at the station. In the above expression the arrival zenith distance can be computed from the astronomical refraction formulas mentioned in the previous sections.

8 DISCUSSIONS

As one of the basic problems in practical astronomy and modern space techniques, atmospheric refraction corrections both in delay and bending of signals through the medium near the Earth are discussed in this paper in the manner of the generator function of atmospheric refractive integrals. It is a serious problem in astronomy and geodesy that the observational accuracy of modern space techniques has almost reached the same magnitude as that of the atmospheric delay correction (Herring *et al.*, 1990), and the routine formulas for astronomical refraction are only suitable for elevations above 10° and the accuracy is worse than $0\overset{'}{.}3$ in the range of usage (the Astronomical Almanac, 1996). The improved continued fraction form of the mapping function deduced from the complementary error function provides an appropriate construction of the mapping functions not only for refractive delay and but also for astronomical refraction at radio and optical frequencies, respectively. The old continued fraction form of the mapping function (Marini, 1972) is an approximation from the angle of the generator function method. The introduction of the normalized effective zenith argument I to the mapping function by equation (13) has changed a parallel plate atmosphere to a spherical model, and has also related the mapping function to the interior of the atmosphere as well. The angular argument in the mapping function is better to be taken as the true (or the proper for a finite-object) elevation rather than the arrival direction. The true direction can be more accurately obtained by calculation than the arrival direction which is usually read by measurements.

In comparison with the routine formulas, our new correction formulas have increased the accuracies of computation by about one order of magnitude for delay and by about two orders for bending, especially for low-elevation observations. From the results of integrals along ray tracking, the formulas for refractive delay have a theoretical accuracy better than 1 cm to an elevation $2\overset{\circ}{.}5$ and for astronomical refraction have better than $0\overset{''}{.}3$ to an elevation of about 2°. This improvement certainly promotes the development of astronomical refraction research which has been using series expansions for centuries. The frequency-related mapping function also includes the dispersion of the atmosphere at optical frequencies. Some further correction terms are listed for high-accuracy ranging and angle measurements in practical astronomy and surveying.

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