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ON THE CALCULATION OF THE VERTICAL STRUCTURE OF ACCRETION DISCS

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A new method of calculating the vertical structure of optically thick accretion discs is proposed. In order to solve the problem, a search for the eigenvalues of the dimensionless parameters for a set of four ordinary differential equations with definite boundary conditions was undertaken. It is shown that the vertical structure of optically thick accretion discs can be satisfactorily described by polytropic models.

KEY WORDS Accretion discs, viscosity, opacities, polytropics

1 BASIC EQUATIONS

We consider the structure of geometrically thin accretion discs. To a first approximation the matter in such discs is moving along Keplerian orbits with tangential velocity

$$v_\varphi = \sqrt{\frac{GM}{r}} = \omega r.$$

The vertical structure of such accretion discs can be determined through solving a set of four ordinary differential equations with definite boundary conditions.

Along the normal to the disc plane (along the Z-coordinate) hydrostatic equilibrium takes place:

$$\frac{1}{\rho}\frac{\partial P}{\partial Z} = -\frac{GM}{r^3}Z = -\omega^2 Z. \qquad (1)$$

We investigate the structure of those regions of accretion discs where gaseous pressure dominates:

$$P = \rho\frac{\Re T}{\mu}.$$

The second differential equation is the equation of continuity:

$$\frac{dZ}{d\Sigma} = \frac{1}{\rho}, \tag{2}$$

connecting the Lagrangian mass-coordinate $\Sigma\ [\frac{g}{sm^2}]$ with the Euler space-coordinate Z. The total surface density of the matter in the disc at the given radius r is therefore

$$2\Sigma_0 = 2\int_0^{Z_0} \rho\, dZ$$

where Z_0 is the half-thickness of the disc.

Differential rotation with finite viscosity ν gives rise to the dissipation of the kinetic energy of the Keplerian motion with energy generation in the unit volume:

$$\varepsilon\left[\frac{\text{erg}}{\text{sm}^3\ \text{s}}\right] = -w_{r\varphi} r \frac{d\omega}{dr},$$

where

$$w_{r\varphi} = -\rho\nu r \frac{d\omega}{dr}$$

are the tangential viscous tensions.

It is well known that accretion with high-power energy generation in Keplerian discs is possible only in the presence of highly developed turbulence and/or small-scale magnetic fields. To describe the turbulence in accretion discs we introduce the dimensionless α-parameter in terms of which the viscous tensions can be written as (see Shakura, 1972; Shakura and Sunyaev, 1973):

$$w^t_{r\varphi} = \alpha P.$$

The α-parameter is proportional to the square of the turbulent Mach-number $M_t^2 = v_t^2/v_s^2$ and its values are limited to the interval $0 < \alpha \leq 1$.

In convectively stable regions of accretion discs the thermal energy generated by the shear tensions is transferred to the surface of the disc by the radiation flux Q:

$$\frac{dQ}{dZ} = \varepsilon. \tag{3}$$

To calculate the structure of optically thick accretion discs one can use the radiative transfer equation in the diffusive approximation:

$$\frac{1}{3\kappa\rho}\frac{d\varepsilon_r}{dZ} = -Q, \tag{4}$$

where $\varepsilon_r = aT^4$ is the density of the radiative energy and κ is the opacity (Rosseland's average). Consider the regions dominated by free-free and free-bound absorption for which Kramer's law $\kappa = \kappa_{ff} = \kappa_0\rho/T^{7/2}$ is valid, as well as the regions dominated by scattering by free electrons with $\kappa = \kappa_T = 0.38$.

If the dependence of the energy release and opacity on the density and temperature obeys a power law the set of equations (1)–(4) can be transformed into dimensionless equations and then solved as a problem to search for eigenvalues of the dimensionless parameters of the set of equations. Similar problems occur in the calculation of the structure of stellar interiors (see Dibay and Kaplan, 1976). It is convenient to use the dimensionless mass coordinate

$$\sigma = \frac{\Sigma}{\Sigma_0}$$

as the independent variable. Furthermore, let us introduce the dimensionless variables $p = \frac{P}{P_c}$, $t = \frac{T}{T_c}$, $j = \rho/\rho_c$ (P_c, T_c, ρ_c are the values in the equatorial plane where $Z = 0$), $z = \frac{Z}{Z_0}$, $q = \frac{Q}{Q_0}$ ($Q_0 = \left(\frac{ac}{4}\right) T_{ef}^4$ is the total radiative flux from unit disc surface).

After introducing the new variables, the set of equations looks like:

$$\begin{aligned}
&\frac{dp}{d\sigma} = -\Pi_1 \Pi_2 z; && \Pi_1 = \frac{\omega^2 Z_0^2 \mu}{\Re T_c} \\
&\frac{dz}{d\sigma} = \Pi_2 \frac{t}{p}; && \Pi_2 = \frac{\Sigma_0}{Z_0 \rho_c} \\
&\frac{dq}{d\sigma} = \Pi_3 t; && \Pi_3 = \frac{3}{2} \frac{\alpha\omega\Sigma_0 \Re T_c}{Q_0 \mu} \\
&\frac{dt}{d\sigma} = -\Pi_4 \frac{qp}{t^{15/2}}; && \Pi_4 = \frac{3}{16}\left(\frac{T_{ef}}{T_c}\right)^4 \frac{\Sigma_0 \kappa_0 \rho_c}{T_c^{7/2}} \quad (\kappa_T \ll \kappa_{ff}) \\
&\text{or} && \\
&\frac{dt}{d\sigma} = -\Pi_4 \frac{q}{t^3}; && \Pi_4 = \frac{3}{16}\left(\frac{T_{ef}}{T_c}\right)^4 \Sigma_0 \kappa_T \quad (\kappa_T \gg \kappa_{ff}).
\end{aligned}$$

2 KRAMER'S OPACITY

Firstly let us consider the situation with $\kappa_{ff} \gg \kappa_T$.

2.1 Boundary Conditions

In the plane of the equator we clearly have

$$p(0) = 1; \quad z(0) = 0; \quad q(0) = 0; \quad t(0) = 1.$$

Let us find the solution of the obtained set of equations, together with the photospheric solution at the point $\sigma = 1$ near the surface. In the main part of the disc body with its significant optical thickness we can neglect the energy release in the photospheric layers when the optical thickness is $\tau \sim 1$, so we can use the well-known approximate solution for the temperature (see Sobolev, 1985)

$$\frac{T}{T_{ef}} = \left(\frac{1 + \frac{3}{2}\tau}{2}\right)^{1/4},$$

where τ is the optical thickness for the absorption measured from the photospheric surface toward the disc interior. Let the point $\sigma = 1$ be identified with the point $\tau = 2/3$, where $T = T_{ef}$. So, for the dimensionless variable t the boundary condition is

$$t(\sigma = 1) = \left[\frac{16}{3}\frac{\Pi_4}{\tau_0}\right]^{1/4},$$

where

$$\tau_0 = \frac{\Sigma_0 \kappa_0 \rho_c}{T_c^{7/2}}$$

is the dimensionless value proportional to the total optical thickness of the disc. This value is a free parameter varying within a wide range.

In order to find the boundary condition for the dimensionless value p we divide both parts of equation (1) by the opacity κ_{ff} and introduce a new variable τ for $d\tau = -\kappa_{ff}\rho dZ$. Then equation (1) can be written as which

$$\frac{1}{2}\frac{dP^2}{d\tau} = \frac{\omega^2 Z_0 \Re T^{9/2}}{\kappa_0 \mu}.$$

Within the photospheric layers the Z-coordinate is believed to be practically constant and equal to the half-thickness of the disc Z_0. Integrating this equation up to the point $\tau = \frac{2}{3}$ we can find the boundary condition for the dimensionless value p:

$$p(\sigma = 1) = \left[\frac{3}{16 \cdot 2^{1/8}}\frac{\Pi_1 \Pi_2}{\Pi_4}\left(\frac{16}{3}\frac{\Pi_4}{\tau_0}\right)^{17/8} f\left(\tau = \frac{2}{3}\right)\right]^{1/2},$$

where

$$f(\tau) = \int_0^\tau \left(1 + \frac{3}{2}\tau\right)^{9/8} d\tau.$$

The function $f(\tau) \approx 1$ if $\tau = \frac{2}{3}$.

Clearly for the variables z and q we have

$$z(\sigma = 1) = 1; \quad q(\sigma = 1) = 1.$$

Thus for the set of four ordinary differential equations of first order we have eight boundary conditions. It is possible to satisfy all of these conditions only if the four dimensionless parameters Π_1, Π_2, Π_3, Π_4 have definite values. After numerical integration of the set of equations for the given value of the free parameter τ_0 we can find Π_1, Π_2, Π_3, Π_4.

2.2 The Results of Calculations

The results of numerical calculations are shown in Table 1a and in the illustrations (see Figure 1*a*).

Table 1a. $\tau = \int_{z_0}^{0} \kappa_R \rho dZ$ **is the real optical depth**

$\log \tau_0$	Π_1	Π_2	Π_3	Π_4	n'	n''	$\log \tau$
6.00	7.75	0.465	1.131	0.399	2.873	2.868	6.46
5.80	7.71	0.466	1.131	0.399	2.854	2.850	6.25
5.60	7.67	0.468	1.131	0.399	2.833	2.830	6.04
5.40	7.62	0.469	1.131	0.399	2.809	2.808	5.82
5.20	7.56	0.471	1.131	0.399	2.782	2.783	5.61
5.00	7.50	0.473	1.131	0.399	2.752	2.754	5.40
4.80	7.44	0.475	1.131	0.399	2.718	2.723	5.18
4.60	7.36	0.477	1.131	0.399	2.680	2.687	4.97
4.40	7.27	0.480	1.131	0.399	2.637	2.647	4.76
4.20	7.18	0.483	1.131	0.399	2.590	2.602	4.54
4.00	7.07	0.487	1.131	0.399	2.536	2.552	4.33
3.80	6.95	0.491	1.131	0.399	2.476	2.496	4.11
3.60	6.82	0.496	1.131	0.399	2.409	2.434	3.90
3.40	6.67	0.501	1.131	0.399	2.334	2.364	3.68
3.20	6.50	0.508	1.131	0.398	2.250	2.285	3.47
3.00	6.31	0.515	1.131	0.398	2.156	2.198	3.25
2.80	6.10	0.524	1.130	0.398	2.051	2.101	3.04
2.60	5.87	0.534	1.130	0.398	1.933	1.993	2.82
2.40	5.60	0.546	1.129	0.397	1.802	1.874	2.60
2.20	5.31	0.560	1.128	0.397	1.655	1.743	2.38
2.00	4.98	0.576	1.126	0.395	1.492	1.600	2.16
1.80	4.62	0.596	1.124	0.393	1.312	1.446	1.94
1.60	4.23	0.619	1.120	0.389	1.113	1.282	1.72
1.40	3.79	0.647	1.114	0.383	0.896	1.110	1.50
1.20	3.33	0.679	1.106	0.371	0.663	0.936	1.28
1.00	2.83	0.716	1.095	0.354	0.417	0.764	1.05
0.80	2.34	0.756	1.081	0.326	0.168	0.604	0.83
0.60	1.86	0.798	1.065	0.286	-0.072	0.461	0.61
0.40	1.42	0.838	1.050	0.237	-0.290	0.341	0.40
0.20	1.05	0.876	1.036	0.185	-0.476	0.245	0.19
0.00	0.75	0.908	1.025	0.136	-0.626	0.172	-0.01

Since we have no complete turbulence theory it is reasonable to consider constant energy release from unit mass in the accretion disc. Then, obviously

$$q = \sigma \quad \text{and} \quad \Pi_3 = 1.$$

The results of the calculation of equations (1), (2) and (4) are in Table 1b and in Figure 1*b*. Comparing Table 1a with Table 1b one can evaluate the uncertainty which always exists in a phenomenological theory of disc accretion.

2.3 The Polytropic Approximation

In a number of works on accretion discs the vertical disc structure is approximated by the polytropic equation of state

$$P = K\rho^{1+\frac{1}{n}}.$$

Table 1b. $\tau = \int_{z_0}^{0} \kappa_{\pi} \rho dZ$ is the real optical depth

$\log \tau_0$	Π_1	Π_2	Π_3	Π_4	n'	n''	$\log \tau$
6.00	7.80	0.463	0.437	0.399	2.902	2.901	6.44
5.80	7.77	0.464	0.437	0.399	2.883	2.883	6.23
5.60	7.72	0.466	0.437	0.399	2.861	2.862	6.01
5.40	7.67	0.467	0.437	0.399	2.836	2.839	5.80
5.20	7.62	0.469	0.437	0.399	2.809	2.813	5.59
5.00	7.56	0.471	0.437	0.399	2.778	2.784	5.37
4.80	7.49	0.473	0.437	0.399	2.743	2.752	5.16
4.60	7.41	0.475	0.437	0.399	2.704	2.715	4.95
4.40	7.32	0.478	0.437	0.399	2.661	2.674	4.73
4.20	7.22	0.481	0.437	0.399	2.612	2.629	4.52
4.00	7.12	0.485	0.437	0.399	2.558	2.577	4.31
3.80	6.99	0.489	0.437	0.399	2.496	2.520	4.09
3.60	6.86	0.494	0.437	0.399	2.428	2.456	3.88
3.40	6.70	0.500	0.437	0.399	2.351	2.384	3.66
3.20	6.53	0.506	0.436	0.398	2.265	2.304	3.45
3.00	6.34	0.514	0.436	0.398	2.169	2.215	3.23
2.80	6.12	0.523	0.436	0.398	2.061	2.115	3.01
2.60	5.88	0.533	0.435	0.398	1.941	2.005	2.80
2.40	5.61	0.545	0.435	0.397	1.806	1.883	2.58
2.20	5.31	0.559	0.433	0.397	1.657	1.750	2.36
2.00	4.98	0.576	0.431	0.395	1.491	1.604	2.14
1.80	4.61	0.596	0.427	0.393	1.307	1.447	1.92
1.60	4.21	0.620	0.421	0.389	1.106	1.281	1.70
1.40	3.77	0.647	0.412	0.383	0.886	1.108	1.48
1.20	3.30	0.680	0.396	0.371	0.651	0.932	1.26
1.00	2.81	0.716	0.373	0.354	0.406	0.761	1.04
0.80	2.32	0.756	0.339	0.326	0.159	0.601	0.82
0.60	1.84	0.798	0.294	0.286	−0.079	0.460	0.60
0.40	1.41	0.839	0.241	0.237	−0.294	0.341	0.39
0.20	1.04	0.876	0.187	0.185	−0.478	0.246	0.19
0.00	0.75	0.908	0.137	0.136	−0.627	0.173	−0.01

In this case the last two equations of energy balance can be removed from the set of equations while the solution of the two remaining equations can be written as

$$p = (1 - z^2)^{n+1}; \quad t = 1 - z^2; \quad j = \frac{p}{t} = (1 - z^2)^n$$

$$\sigma = \frac{\Sigma}{\Sigma_0} = \frac{\rho_c Z_0}{\Sigma_0} \int_0^z (1 - z^2)^n dz.$$

In doing so, on the one hand the dimensionless parameter Π_1 is connected with the polytropic index (written as n') by the relation

$$\Pi_1 = 2(n' + 1),$$

and on another hand there is an obvious connection of the polytropic index (written as n'') with the dimensionless parameter Π_2

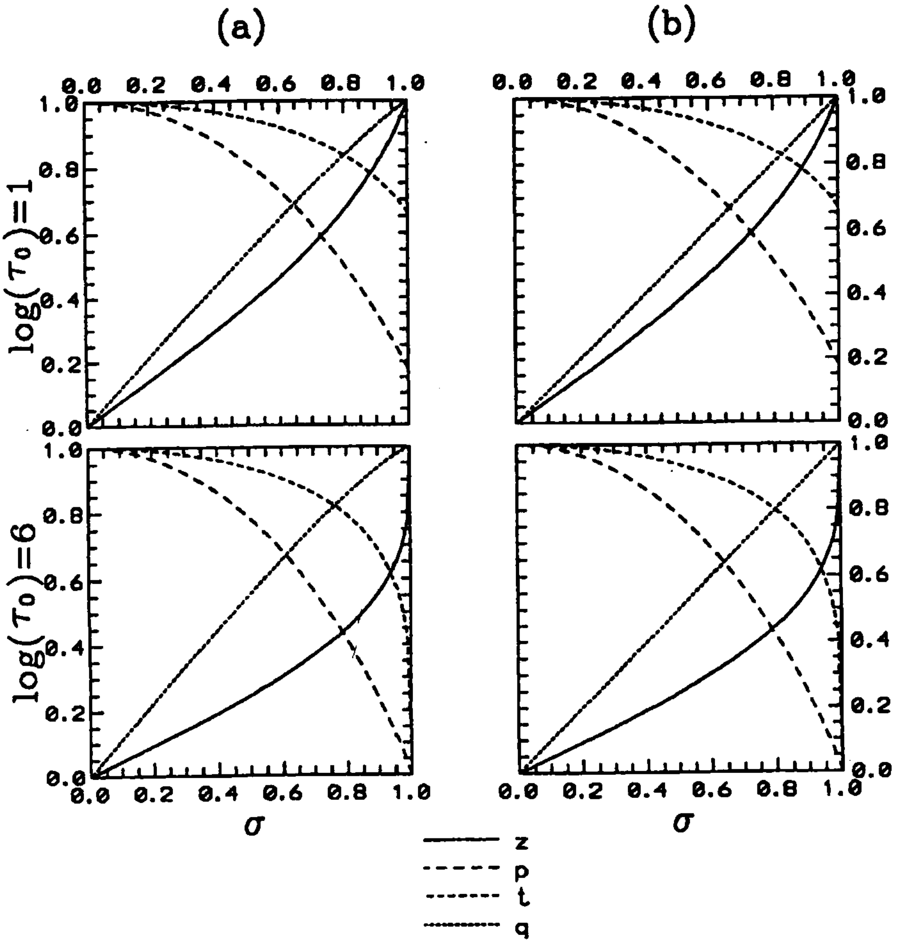

Figure 1 The dependence of the dimensionless physical variables z, p, t and q on the mass-coordinate σ for the case $\kappa_T \ll \kappa_{ff}$.

$$\Pi_2 = \frac{\Sigma_0}{\rho_c Z_0} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(n''+1)}{(\frac{1}{2}+n'')\Gamma(\frac{1}{2}+n'')}.$$

The values of the polytropic indexes calculated using the earlier parameters Π_1 and Π_2 are shown in Tables 1a and 1b. Notice the closeness of the values n' and n'' for accretion discs of large optical thickness; it is an indication of the possibility of using the polytropic approximations for such discs. However, the quality of the approximate polytropic solutions worsens with decreasing optical thickness (see the bottom of Tables 1a and 1b). In particular, the polytropic approximation is unjustified in the surface layers of the disc.

Table 2a.

$log\,\delta$	Π_1	Π_2	Π_3	Π_4	n'	n''
6.00	6.99	0.492	1.150	0.460	2.497	2.484
5.80	6.96	0.493	1.150	0.460	2.479	2.468
5.60	6.92	0.495	1.150	0.460	2.459	2.449
5.40	6.87	0.496	1.150	0.460	2.437	2.428
5.20	6.82	0.498	1.150	0.460	2.412	2.405
5.00	6.77	0.500	1.150	0.460	2.384	2.379
4.80	6.70	0.503	1.150	0.460	2.352	2.349
4.60	6.63	0.505	1.150	0.460	2.317	2.317
4.40	6.55	0.508	1.150	0.460	2.277	2.280
4.20	6.47	0.512	1.150	0.460	2.233	2.239
4.00	6.37	0.516	1.150	0.460	2.183	2.193
3.80	6.26	0.520	1.149	0.460	2.128	2.142
3.60	6.13	0.525	1.149	0.460	2.066	2.084
3.40	5.99	0.531	1.149	0.460	1.997	2.021
3.20	5.84	0.538	1.149	0.460	1.920	1.950
3.00	5.67	0.546	1.149	0.459	1.834	1.872
2.80	5.48	0.555	1.148	0.459	1.738	1.785
2.60	5.26	0.566	1.147	0.458	1.631	1.689
2.40	5.02	0.578	1.146	0.458	1.512	1.585
2.20	4.76	0.593	1.145	0.456	1.381	1.472
2.00	4.47	0.610	1.142	0.454	1.236	1.349
1.80	4.15	0.629	1.138	0.450	1.077	1.219
1.60	3.81	0.652	1.133	0.444	0.904	1.083
1.40	3.43	0.678	1.126	0.435	0.716	0.943
1.20	3.03	0.707	1.117	0.420	0.516	0.801
1.00	2.61	0.740	1.105	0.398	0.307	0.663
0.80	2.19	0.776	1.091	0.366	0.094	0.533
0.60	1.77	0.813	1.075	0.324	−0.115	0.415
0.40	1.38	0.849	1.059	0.274	−0.312	0.312
0.20	1.03	0.884	1.044	0.219	−0.485	0.226
0.00	0.74	0.914	1.032	0.166	−0.630	0.159

3 THOMPSON OPACITY

Let us consider the situation when $\kappa_T \gg \kappa_{ff}$.

3.1 The Boundary Conditions

If $\sigma = 0$ the boundary conditions remain the same:

$$p(0) = 1; \quad z(0) = 0; \quad q(0) = 0; \quad t(0) = 1.$$

As is well known, in photospheres whose opacity is dominated by Thompson scattering, thermalization takes place at the effective optical depth (see for instance

Table 2b.

$log\, \delta$	Π_1	Π_2	Π_4	n'	n''
6.00	7.10	0.488	0.500	2.549	2.542
5.80	7.06	0.489	0.500	2.531	2.525
5.60	7.02	0.490	0.500	2.510	2.506
5.40	6.97	0.492	0.500	2.487	2.485
5.20	6.92	0.494	0.500	2.462	2.461
5.00	6.87	0.496	0.500	2.433	2.434
4.80	6.80	0.498	0.500	2.401	2.404
4.60	6.73	0.501	0.500	2.365	2.371
4.40	6.65	0.504	0.500	2.324	2.333
4.20	6.56	0.507	0.500	2.279	2.291
4.00	6.46	0.511	0.500	2.229	2.244
3.80	6.34	0.516	0.500	2.172	2.191
3.60	6.22	0.521	0.500	2.109	2.133
3.40	6.08	0.527	0.500	2.038	2.068
3.20	5.92	0.534	0.499	1.959	1.995
3.00	5.74	0.542	0.499	1.871	1.915
2.80	5.55	0.551	0.498	1.773	1.826
2.60	5.33	0.562	0.498	1.664	1.728
2.40	5.09	0.574	0.496	1.544	1.622
2.20	4.82	0.588	0.494	1.410	1.505
2.00	4.52	0.605	0.490	1.262	1.380
1.80	4.20	0.625	0.485	1.099	1.247
1.60	3.85	0.648	0.476	0.923	1.107
1.40	3.46	0.674	0.463	0.731	0.963
1.20	3.06	0.704	0.444	0.528	0.818
1.00	2.63	0.737	0.417	0.315	0.676
0.80	2.20	0.773	0.380	0.099	0.542
0.60	1.78	0.811	0.333	–0.112	0.421
0.40	1.38	0.848	0.278	–0.310	0.316
0.20	1.03	0.883	0.221	–0.485	0.228
0.00	0.74	0.914	0.167	–0.629	0.160

Zeldovich and Shakura (1969), Mihalas (1978))

$$\tau^* = -\int_{Z_0}^{Z^*} \sqrt{\kappa_{ff}\kappa_T}\rho dZ \approx 1.$$

At the depth where $\tau^* \simeq 1$ the Thompson scattering optical depth is

$$\tau_T = -\int_{Z_0}^{Z^*} \kappa_T \rho dZ \gg 1.$$

Therefore, the boundary condition for the dimensionless temperature can be written as:

$$t(\sigma = 1) = \left[\frac{8}{3}\frac{\Pi_4}{\kappa_T\Sigma_0}\left(1 + \frac{3}{2}\tau\right)\right]^{1/4} \simeq \left[\frac{4\Pi_4\tau_T(\tau^* = 1)}{\kappa_T\Sigma_0}\right]^{1/4}$$

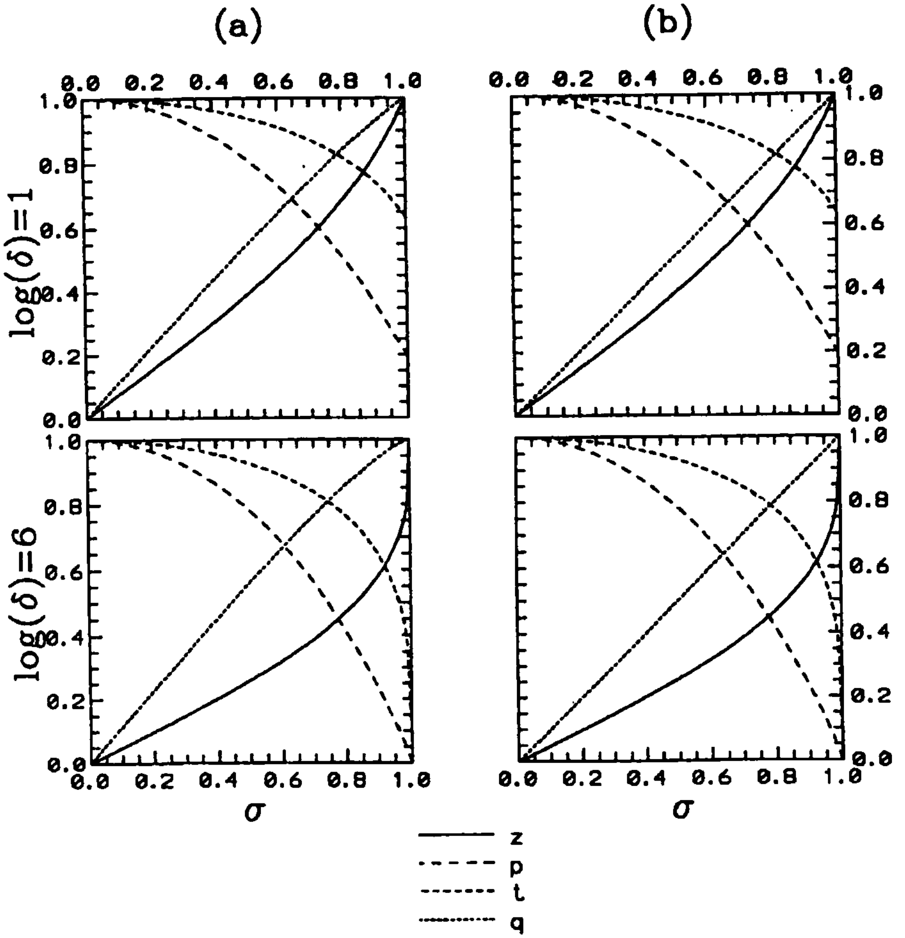

Figure 2 The dependence of the dimensionless physical variables z, p, t and q on the mass-coordinate σ for case $\kappa_T \gg \kappa_{ff}$.

Correspondingly, for the pressure one can obtain:

$$p(\sigma = 1) = \Pi_1 \Pi_2 \frac{\tau_T(\tau^* = 1)}{\kappa_T \Sigma_0}.$$

Besides that we have two obvious boundary conditions

$$z(\sigma = 1) = 1; \quad q(\sigma = 1) = 1.$$

Thus, here convenient free parameter is the ratio of the total optical depth $\kappa_T \Sigma_0$ to the optical depth of scattering occurring at the thermalization depth

$$\delta = \frac{\kappa_T \Sigma_0}{\tau_T(\tau^* = 1)}.$$

The subsequent solution of the set of ordinary differential equations is similar to that for the disc regions with Kramer opacity. The results obtained together with the polytropic approximation, are shown in Tables 2a and 2b and illustrated in Figure 2(*a*,*b*).

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References

Dibay, E. A. and Kaplan, S. A. (1976) In *Razmernosti i Podobie Astrofizicheskih Velichin*, Nauka, Moscow (in Russian).
Mihalas, D. (1978) In *Stellar Atmospheres*, W. H. Freeman (ed.), San Francisco.
Shakura, N. I. (1972) *Astron. Zh.* **49**, 921.
Shakura, N. I. and Sunyaev, R. A. (1973) *Astron. Astrophys.* **24**, 337.
Sobolev, V. V. (1985) In *Kurs Teoreticheskoy Astrofiziki*, Nauka, Moscow (in Russian).
Zeldovich, Ya. B. and Shakura, N. I. (1969) *Astron. Zh.* **46**, 225.