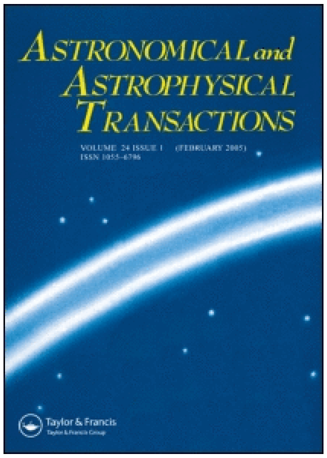

Astronomical & Astrophysical Transactions
The Journal of the Eurasian Astronomical Society
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713453505

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Online Publication Date: 01 July 1996
To cite this Article: Gavrilov, V. R., Ivashchuk, V. D. and Melnikov, V. N. (1996) 'Multidimensional cosmology and toda lattices', Astronomical & Astrophysical Transactions, 10:4, 333 - 339
To link to this article: DOI: 10.1080/10556799608205450
URL: http://dx.doi.org/10.1080/10556799608205450

Astronomical and Astrophysical Transactions, 1996,
Vol. 10, pp. 333–339
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MULTIDIMENSIONAL COSMOLOGY AND TODA LATTICES

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(Received October 19, 1994)

A multidimensional cosmological model describing the evolution of n Einstein spaces in the presence of a multicomponent perfect fluid is considered. When vectors corresponding to the equations of state of the components are orthogonal with respect to the minisuperspace metric, the Einstein equations are integrated and a Kasner-like form of the solutions is presented. For special sets of parameters the cosmological model is reduced to the Euclidean Toda-like system connected with some Lie algebra.

PACS numbers: 04.20.J, 04.60.+n, 03.65.Ge

1 INTRODUCTION

We study integrable pseudo-Euclidean Toda-like systems appearing in multidimensional cosmology [1, 2, 4–9, 14 and references therein]. This direction in the modern theoretical physics has appeared within the new paradigm based on unified theories and the hypothesis of additional space-time dimensions. According to this hypothesis, the physical space-time manifold has the topology $M^4 \times B$, where M^4 is a 4-dimensional manifold, and B is the so-called internal space (or spaces). Nonobservability of additional dimensions is attained in multidimensional cosmology by spontaneous or dynamical compactification of internal spaces to the Planck scale (10^{-33} cm). Integrable cosmological models are of great interest, because the exact solutions allow to study dynamical properties of the model, in particular the compactification of internal spaces, in detail.

In the Section 2, as in [9], we consider the Einstein equations for a cosmological model where the D-dimensional space-time manifold M is a direct product of the time axis R and n Einstein spaces $M_1, \ldots, M_n$. We recall that any manifold of constant curvature is an Einstein one. The source is chosen in the form of a multicomponent perfect fluid. To integrate the resulting equations we develop an n-dimensional vectors formalism in Section 3. Section 4 is devoted to explicit integration, in particular, by reduction to Toda lattices.

2 THE METRIC AND EQUATIONS OF MOTION

We consider Einstein equations $R_N^M - \frac{1}{2}\delta_N^M R = \kappa^2 T_N^M$ for the metric

$$g = -\exp[2\gamma(t)]\, dt \otimes dt + \sum_{n=1}^{n} \exp[2x^i(t)]\, g^i, \quad n \geq 2, \tag{1}$$

defined on the D-dimensional space-time manifold $M = R \times M_1 \times \ldots \times M_n$, where the manifold M_i is an Einstein space of the dimension N_i with the metric $g^{(i)}$, i.e.,

$$R_{m_i n_i}[g^{(i)}] = \lambda^i g^{(i)}_{m_i n_i}, \quad m_i, n_i = 1, \ldots, N_i. \tag{2}$$

The energy-momentum tensor is taken in the following form:

$$T_N^M = \sum_{\alpha=1}^{m} T_N^{M(\alpha)}, \tag{3}$$

$$(T_N^{M(\alpha)}) = \mathrm{diag}\,(\rho^{(\alpha)}(t), p_1^{(\alpha)}(t)\delta_{k_1}^{m_1}, \ldots, p_n^{(\alpha)}(t)\delta_{k_n}^{m_n}), \tag{4}$$

We assume that for any α-th component of the perfect fluid the pressures in all spaces are proportional to density

$$p_i^{(\alpha)}(t) = (1 - h_i^{(\alpha)})\rho^{(\alpha)}(t), \tag{5}$$

where $h_i^{(\alpha)} = \text{const}$.

The conservation law constraints, $\nabla_M T_N^{M(\beta)} = 0$, lead to the relations

$$\rho^{(\alpha)}(t) = A^{(\alpha)} \exp[-2\gamma_0 + \sum_{i=1}^{n} N_i h_i^{(\alpha)} x_i], \tag{6}$$

where $A^\alpha = \text{const}$ and $\gamma_0 = \sum_{i=1}^{n} N_i x^i$.

Einstein equations for the scale factors $exp\,[x^i]$ are equivalent to the Langrange–Euler equations for the Lagrangian

$$L = \frac{1}{2} \exp[-\gamma + \gamma_0] \sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j - \exp[\gamma - \gamma_0] V \tag{7}$$

with the zero-energy constraint imposed. Here $G_{ij} = N_i \delta_{ij} - N_i N_j$ are the components of the minisuperspace metric. This metric has a pseudo-Euclidean signature $(-, +, \ldots, +)$ [5, 6]. The potential V contains the terms induced by the curvature, perfect fluid and cosmological constant Λ:

$$
\begin{aligned}
V &= \sum_{k=1}^{n}(-\frac{1}{2}\lambda^k N_k)\exp[\sum_{i,j=1}^{n} G_{ij}v^i_{(k)}x^j \\
&+ \sum_{\alpha=1}^{m}\kappa^2 A^{(\alpha)}\exp[\sum_{i,j=1}^{n} G_{ij}u^i_{(\alpha)}x^j] + \Lambda\exp[\sum_{i,j=1}^{n} G_{ij}u^i x^j], \qquad (8)
\end{aligned}
$$

where $G^{ij} = \delta^{ij}/N_i + 1/(2-D)$ are the components of the matrix inverse to (G_{ij}). We also introduce the following notation:

$$
v^k_{(i)} = -2\frac{\delta^k_i}{N_i}, \quad u^i = \frac{2}{2-D}, \qquad (9)
$$

$$
u^i_{(\alpha)} = h^{(\alpha)}_i + \frac{1}{2-D}\sum_{j=1}^{n} N_j h^{(\alpha)}_j. \qquad (10)
$$

We usually employ the harmonic time gauge: $\gamma \equiv \gamma_0$.

3 THE N-DIMENSIONAL VECTOR FORMALISM

To develop the integration procedure for the equations of motion following from the Lagrangian (7), we introduce the n-dimensional real vector space R^n. By $e_1, \ldots, e_n$ we denote the canonical basis in $R^n (e_1 = (1,0,\ldots,0)$, etc.).

Let $\langle.,.\rangle$ be a symmetrical bilinear form on R^n such that $\langle e_i, e_j\rangle = G_{ij}$. We recall that this form has a pseudo-Euclidean signature $(-,+,\ldots,+)$. Then, by definition,

$$
\begin{aligned}
x &\equiv x^1 e_1 + \ldots + x^n e_n, \qquad v_i \equiv v^1_{(i)}e_1 + \ldots + v^n_{(i)}e_n, \qquad (11)\\
u_\alpha &\equiv u^1_{(\alpha)}e_1 + \ldots + u^n_{(\alpha)}e_n, \quad u \equiv u^1 e_1 + \ldots + u^n e_n, \qquad (12)
\end{aligned}
$$

and we present the Lagrangian (7) (for the harmonic time) in the form

$$
\begin{aligned}
L &= \frac{1}{2}\langle\dot{x},\dot{x}\rangle - \sum_{k=1}^{n}(-\frac{1}{2}\lambda^k N_k)\exp[\langle v_k, x\rangle] \\
&- \sum_{\alpha=1}^{m}\kappa^2 A^{(\alpha)}\exp[\langle u_\alpha, x\rangle] - \Lambda\exp[\langle u, x\rangle]. \qquad (13)
\end{aligned}
$$

A vector $y \in R^n$ is called time-like, space-like or isotropic, if $\langle y,y\rangle$ is negative, positive or zero, respectively. Vector y and z are called orthogonal if $\langle y,z\rangle = 0$.

In Table 1 we present the values of the bilinear form $\langle.,.\rangle$ for vectors v_i, u_α and u induced by the curvature of the space M_i, α-th component of the perfect fluid and Λ-term, respectively.

Table 1.

	v_j	u_β	u
v_i	$4(\frac{\delta_{ij}}{N_i}-1)$	$-2h_i^{(\beta)}$	-4
u_α	$-2h_j^{(\alpha)}$	$\sum_{i=1}^n h_i^{(\alpha)}h_i^{(\beta)}N_i+$ $\frac{1}{2-D}\left[\sum_{i=1}^n h_i^{(\alpha)}N_i\right]\left[\sum_{j=1}^n h_j^{(\beta)}N_j\right]$	$\frac{2}{2-D}\sum_{i=1}^n h_i^{(\alpha)}N_i$
u	-4	$\frac{2}{2-D}\sum_{i=1}^n h_i^{(\beta)}N_i$	$-4\frac{D-1}{D-2}$

4 INTEGRABLE MODELS

We are interested in the integrability of dynamical systems described by the Lagrangian of the form

$$L=\frac{1}{2}\langle\dot{x},\dot{x}\rangle-\sum_{s=1}^{m}a^{(s)}\exp\left[\langle b_s,x\rangle\right], \tag{14}$$

where $b_s=b^1_{(s)}e_1+\ldots+B^n_{(s)}e_n\in R^n$. Such systems are algebraic generalizations of the well-known Toda lattices [13] in the case of indefinite bilinear form of the kinetic energy. We call such systems pseudo-Euclidean Toda-like systems.

It is quite obvious that integrability depends on the set of vectors $b_1,\ldots,b_m\in R^n$. For $m=1$ the system with the Lagrangian (14) is always integrable [5–9, 14]. In the present paper we consider, as in [4], a multicomponent case: $m\geq 2$.

4.1 The orthogonal set of vectors

We obtain the class of the exact solutions provided vectors $b_1,\ldots,b_m$ satisfy the following conditions:

(1) they are linearly independent; (2) $\langle b_\alpha,b_\beta\rangle=0$ for all $\alpha\neq\beta$, i.e., the set of vectors is orthogonal.

It is not difficult to prove

Proposition 1. The set of vectors $b_1,...,b_m$ may contain at most one isotropic vector.

Proposition 2. The set of vectors $b_1,...,b_m$ may contain at most one time-like vector and, if it holds, the other vectors must be space-like.

These propositions allow to divide the class of exact solutions under consideration into the following subclasses:

A. There are one time-like vector and at most $(n-1)$ space-like vectors.

B. There are at most $(n-1)$ space-like vectors.

C. There are one isotropic vector and at most $(n-2)$ space-like vectors (this subclass arises for $n\geq 3$).

To integrate the equations of motion for all subclasses, we construct an orthonormal basis $e'_1, \ldots, e'_n$ using the vectors $b_1, \ldots, b_m$. These vectors are such that $\langle e'_i, e'_j \rangle = \eta_{ij}$, where η_{ij} are components of the matrix $(\eta_{ij}) = \mathrm{diag}(-1, +1, \ldots, +1)$. It is clear that the Lagrangian (14) has a more convenient form, in particular, with diagonalized kinetic energy form $\langle \dot{x}, \dot{x} \rangle$ in terms of coordinates X^i of the vector x in such a basis ($x = X^1 e'_1 + \ldots + X^n e'_n$).

For the subclass A when, for instance, b_1 is a time-like vector and other vectors are space-like, we choose the orthogonal basis as $e'_s = b_s / |\langle b_s, b_s \rangle|^{1/2}$, $s = 1, \ldots, m$ (if $m < n$, then the necessary number of vectors are added). It is not hard to check that the equations of motion are easily integrable for the new coordinates X^i. After the inverse linear transformation, we obtain the coordinates x^i in the canonical basis and present the exact solution in the Kasner-like form:

$$\exp[x^i] = \prod_{s=1}^{m} [F_s^2(t - t_{0s})]^{-b^i_{(s)}/\langle b_s, b_s \rangle} \exp[\alpha^i t + \beta^i], \quad i = 1, \ldots, n, \tag{15}$$

where we have denoted

$$F_s(t - t_{0s}) = \sqrt{|a^{(s)}/E_s|} \cosh[\sqrt{|E_s \langle b_s, b_s \rangle|/2}(t - t_{0s})], \ \eta_{ss} a^{(s)} > 0, \ \eta_{ss} E_s > 0, \tag{16}$$

$$= \sqrt{|a^{(s)}/E_s|} \sin[\sqrt{|E_s \langle b_s, b_s \rangle|/2}(t - t_{0s})], \ \eta_{ss} a^{(s)} > 0, \ \eta_{ss} E_s > 0, \tag{17}$$

$$= \sqrt{|a^{(s)}/E_s|} \sinh[\sqrt{|E_s \langle b_s, b_s \rangle|/2}(t - t_{0s})], \ \eta_{ss} a^{(s)} > 0, \ \eta_{ss} E_s > 0, \tag{18}$$

$$= \sqrt{|\langle b_s, b_s \rangle a^{(s)}|/2}(t - t_{0s}), \ \eta_{ss} a^{(s)} < 0, \ E_s = 0, \tag{19}$$

and t_{0s} and E_{0s} ($s = 1, \ldots, m$) denote arbitrary integration constant. The Kasner-like parameters satisfy the relations

$$\sum_{i,j=1}^{n} G_{ij} \alpha^i \beta^j = 2(E_0 - E_1 - \ldots - E_m) \geq 0, \tag{20}$$

$$\sum_{i,j=1}^{n} G_{ij} \alpha^i b^j_{(s)} = \sum_{i,j=1}^{n} G_{ij} \alpha^i b^j_{(s)} = 0, \quad s = 1, \ldots, m, \tag{21}$$

where E_0 is the total energy of the system (14).

In the same manner we obtain the exact solutions in the subclasses B and C [4].

Let us consider cosmological models corresponding to the integrable pseudo-Euclidean Toda-like systems considered. As follows from the Table 1, the vectors v_i and u induced by the curvature of the space M_i and Λ-term are time- like. So, the subclasses B and C correspond to the models with all Ricci-flat spaces without the Λ-term for an m-component perfect fluid, when the components induce an orthogonal set of the vectors.

Within the subclass A, we are able to construct a model with one Einstein space of non-zero curvature. Let $(n-1)$ Einstein spaces be Ricci-flat and one, for instance

M_1, have a non-zero Ricci tensor. Then we put $b_1 \equiv v_1$. To get the orthogonality with b_1 for at most $(n-1)$ available components of the perfect fluid ($b_{(\alpha+1)} \equiv u_{(\alpha)}$ for $\alpha \leq n-1$) we put: $h_1^{(\alpha)} = 0$ (see Table 1). Then these components appear to be in the manifold M_1 in the Zeldovich matter form (or, equivalently in the form of a minimally coupled real scalar field [12]). In the same way a model with all Ricci- flat spaces and Λ-term arises. In this case we put $b_1 \equiv u$. The orthogonality condition reads $\sum_{i=1}^{n} h_i^{(\alpha)} N_i = 0$ for all $\alpha \leq n-1$.

4.2 *The set of collinear vectors*

It is not difficult to prove that, if all of the vectors $b_1, \ldots, b_m$ are collinear, the system with the Lagrangian (14) is integrable by quadratures. We obtain a two-component model of such type when Ricci-flat spaces $M_2, \ldots, M_n$ are filled with superradiation (in the equations of state, we have $h_i = 1 - 1/N_i$) and the space M_1 with a non-zero Ricci tensor is filled with dust ($h_1 = 1$). This model was integrated in [1]. Another two-component model with collinear vectors arises for all Ricci-flat spaces in the presence of the Λ-term and dust ($h_i = 1, i = 1, \ldots, n$) as a source.

The subclasses A, B and C can be easily extended. It is clear that the addition of a new component inducing a vector collinear to that from the orthogonal set $b_1, \ldots, b_m$ leads to a model integrable by quadratures [4].

4.3 *Models reducible to Toda Lattices*

Let us assume that the vector $b_1, \ldots, b_m$ themselves and their arbitrary combinations are space-like vectors. Then the first components of these vectors must be zero in a suitably chosen orthonormal basis, i.e., $b_s = 0e'_1 + B^2_{(s)}e'_2 + \ldots + B^n_{(s)}e'_n$. This implies that, in this basis, we obtain the Lagrangian (14) in the following form:

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \eta_{ij}\dot{X}^i\dot{X}^j - \sum_{s=1}^{m} a(s)\exp\left[\sum_{k,l=2}^{n} \delta_{kl}B^k_{(s)}X^l\right]. \tag{22}$$

The coordinate variable X^1 satisfied the following equation: $\ddot{X}^1 = 0$. The equations of motion for $X^2, \ldots, X^n$ follow from the Euclidean Toda-like Lagrangian

$$L_E = \frac{1}{2}\sum_{k,l=1}^{n} \delta_{kl}\dot{X}^k\dot{X}^l - \sum_{s=1}^{m} a(s)\exp\left[\sum_{k,l=2}^{n} \delta_{kl}B^k_{(s)}X^l\right]. \tag{23}$$

Thus, a pseudo-Euclidean Toda-like system has been reduced to an Euclidean one.

Nearly nothing is known about Einstein Toda-like systems with arbitrary sets of vectors $(B_2^{(s)}, \ldots, B_n^{(s)})(s = 1, \ldots, m)$. But, if they form the set of admissible roots of a simple complex Lie algebra, then the system is completely integrable and possesses a Lax representation [3]. An explicit integration procedure of the equations of motion was developed in [10,11]. In [4] we presented an explicit solution for the two-component cosmological model in the case when it is reducible to an open Toda lattice connected with the Lie algebra A_2.

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