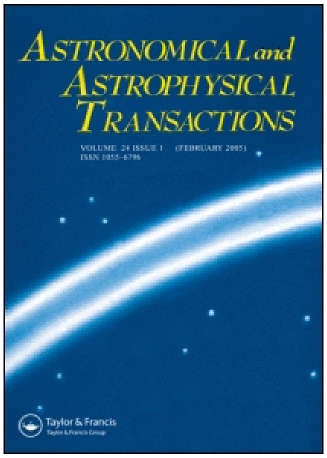

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A. A. Kirillov [a]
[a] Institute for Applied Mathematics and Cybernetics, Nizhny Novgorod, Russia

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COSMOLOGY

ON THE DESCRIPTION OF SPATIAL TOPOLOGIES IN QUANTUM GRAVITY

A. A. KIRILLOV

Institute for Applied Mathematics and Cybernetics, 10 Uljanova str., Nizhny Novgorod, 603005, Russia

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A new approach which allows to describe phenomenologically arbitrary topologies of the Universe is suggested. It consists in a generalization of the third quantization. This quantization is carried out for the case of asymptotic closeness to a cosmological singularity. It is also pointed out that this approach leads to a modification of the ordinary quantum field theory. In order to show this modification we consider example of a free massless scalar field.

1 INTRODUCTION

It is widely accepted that quantum fluctuations of metric at small-scale distances can change spatial topology of the Universe [1], [2]. Effects connected with topology changes were considered in Refs. [3–6]. Nevertheless, an adequate mathematical scheme for the description of such processes is still absent. Here we suggest an approach which, as we hope, give a possibility for at least a phenomenological description of arbitrary spatial topologies. To this end we use a generalization of the third quantization.

Third quantization has been already used in quantum cosmology for description of "wormholes" and "baby universes" [3–6] (which was shown to lead to the loss of quantum coherence) as well as for description of "spontaneous quantum creation of a universe from nothing" [7], [8] proposed earlier in Ref.[9]. Note however that in all of the considered cases a number of small closed universes (which branch off our large Universe) is a variable quantity but the topology of each closed universe turns out to be fixed. In order to describe different possible spatial topologies, it is necessary to modify the procedure of the third quantization. The simplest way to do this is to make it local. Such a possibility follows from the fact that the Wheeler–DeWitt equation which governs the evolution of a wave function of the Universe consists of an infinite set of Klein–Gordon type equations (one local Wheeler–DeWitt equation at each point x of the three-space S). We note that this is in accordance with the fact that time in General Relativity has only a local meaning. Therefore, one may attempt to quantize every local Wheeler–DeWitt equation independently. We call such a procedure the local third quantization (LTQ).

There is one problem with the LTQ procedure in the general case. The fact is that all local Wheeler–DeWitt equations are strongly coupled to each other and so it is very difficult to carry out such a procedure. Nevertheless, there is a situation when the connection between the local Wheeler–DeWitt equations disappears, at least in the leading order. It is just the case when one considers gravitational field near the cosmological singularity.

As was shown in Refs. [10, 11] that the general inhomogeneous gravitational field at the singularity can be considered as a continuum of uncoupled homogeneous fields (of IX type). Indeed, near the singularity it is always possible to choose an "elementary" volume ΔV in which the gravitational field is homogeneous in the leading order (for the sake of simplicity, we do not take into account the presence of matter; furthermore, it does not affect the properties of the gravitational field). In the vicinity of the singularity, the horizon size tends to zero ($l_h \to 0$) and different "elementary" domains $\Delta V(x)$ of the three-space S do not affect each other and can be considered independently (for validity of that, it is necessary to fulfill the following condition: $(\Delta V)^{1/3} \ll l_h$). The LTQ procedure consists then of the assumption that third quantization is carried out independently for each elementary spatial domain $\Delta V(x)$. Furthermore, one may assume that all these domains are indistinguishable. Localization of third quantization is achieved in the limit $\Delta V \to 0$ only. We note that, in this limit, every such "elementary" domain contains only one "physical point" of space and, therefore, the "elementary" domain will be understood as an isolate point of the physical continuum.

The local third quantization leads to a modification of the ordinary quantum field theory. Indeed, in the limit $\Delta V \to 0$ an "elementary" domain $\Delta V(x)$ contains a finite number of physical degrees of freedom which coincides with the number of physically arbitrary functions determining distributions of matter and gravitational field. At each point of S, these degrees of freedom form a set. Thus, the local third quantization can be understood as an independent third quantization of all such sets. In ordinary field theory, it is more convenient to use Fourier transformation for physical degrees of freedom, that is expansion in modes. So in the case of free fields local third quantization consists in third quantization of the field modes.

2 GRAVITATIONAL FIELD IN THE VICINITY OF THE COSMOLOGICAL SINGULARITY

As pointed out above, the problem of the local third quantization of the gravitational field at the singularity can be reduced to the third quantization of homogeneous field. In the vicinity of the singularity states of the homogeneous mixmaster field (or, in our case, of an "elementary" spatial domain $\Delta V(x)$) was shown to be classified by some quantum number $n (n = 0, 1 \ldots)$ [8, 11, 12]. When third quantization is imposed, the wave function becomes a field operator and can be expanded in the form (here we assume for simplicity that Ψ is a real scalar function)

$$\Psi = \sum (C_n U_n + C_n^+ U_n^*), \tag{2.1}$$

where $\{U_n, U_n^*\}$ is an arbitrary complete orthonormal set of solutions of the Wheeler-DeWitt equation:

$$(\Delta + V) U_n = 0; \tag{2.2}$$

here V is the potential, $\Delta = \frac{1}{\sqrt{-G}} \partial_A \sqrt{-G} G^{AB} \partial_B$ and G_{AB} is the metric on a minisuperspace W (for more detail see Ref. [11]). The operators C_n and C_n^+ satisfy the standard (anti) commutation relations

$$[C_n, C_m^+]_\pm = \delta_{n,m}, \tag{2.3}$$

where $\pm$ is related to the two possible statistics of the wave function.

In the case of an inhomogencous field, the Wheeler–DeWitt equation splits into a set of uncoupled equation of the (2.2) type, each containing variables describing a gravitational field at a particular point x of the three-manifold S:

$$(\Delta(x) + V_x)\Psi = 0. \tag{2.4}$$

The space H of solutions of this Wheeler–DeWitt equation has the form of a tensor product of "homogeneous" spaces H_x (written as $H = \prod_{x \in S} H_x$), where H_x is the space of solutions to Eq. (2.2). Then one can introduce a set of wave function Ψ_x describing quantum states of the field at the point x and apply second quantization to every local Wheeler–DeWitt equation (2.4) independently. In this manner the operators (2.3) acquire additional dependence of spatial coordinates. The LTQ procedure generalizes relations (2.3) to the following:

$$[C(x,n), C^+(y,m)]_\pm = \delta_{n,m} \delta(x,y). \tag{2.5}$$

Using the operator algebra (2.5), one can construct a set of states with an arbitrary number of domains (with an arbitrary density of points for the physical continuum). In particular, the vacuum state is determined by $C(x,n)|0\rangle = 0$ (for arbitrary $x \in S$) and, therefore, this state corresponds to the absence of all points of physical space and consequently to the absence of all field observables. In other words, this state describes the situation when the physical continuum is absent. The operator $N(x,n) = C^+(x,n)C(x,n)$ has the ordinary meaning of the number of elementary domains $\Delta V(x)$ given in the quantum state U_n and located at the point $x \in S$. Summation over n yields the total number operator for domains having the coordinate x : $N(x) = \sum N(x,n)$ which has also the meaning of an density operator for physical points. The operator $\theta(x) = 1 - N(x)$ may be used then as an indicator of difference in topology of the Universe from that of S.

This theory includes conventional quantum gravity as a particular case. Indeed, let us consider the set of states $\{|A\rangle\}$ which have the form

$$|A\rangle = \sum_{\{n(x)\}} A[n(x)]|n(x)\rangle, |[n(x)]\rangle = \prod_{x\in S} C^{+}(x, n(x))|0\rangle. \tag{2.6}$$

These states describe the case when there is just one elementary domain at each point $x \in S$ and, therefore, the following conditions are fulfilled:

$$\theta(x)|A\rangle = 0, \text{ as } x \in S \tag{2.7}$$

(i.e., the number of point of the physical continuum having the coordinate x coincides with that of S). Obviously, topology of physical space for these states is the same as that of S.

In order to illustrate the nontrivial topology of the Universe, one may construct a handle on S. In our approach, the existence of the handle is indicated by the fact that quantum states of the gravitational field $U_{n(x)}$ are triple-valued functions of spatial coordinates (in some region $K \in S$). Therefore, the states describing the handle can be taken in the form

$$|n; p, q\rangle = \prod_{y\in K} C^{+}(y, p(y))C^{+}(y, q(y)) \prod_{z\in S} C^{+}(z, n(z))|0\rangle.$$

It is obviously that, due to the indistinguishability of domains, one may speak about topology of physical space in a usual sense in the quasi-classical limit only. Indeed, in this limit one can introduce a set of maps such that metric functions become single-valued.

Evidently, one of possible applications of LTQ is a description of effects connected with the "space-time foam" [1], [2]. In particular, it should display itself in the existence of the so-called vacuum fluctuations connected with the creation and annihilation of virtual points of physical space. It should be also noted that the numbers $N(x)$ vary during the evolution [7], [8]. This means that the structure of the foam is not fixed and must be determined dynamically. Furthermore, there is an interesting possibility that the spatial continuum has "hollows" at small distances (i.e. $N(\mathbf{k}) \to 0$ if $\mathbf{k} \to \infty$, where $(N(\mathbf{k}) = (2\pi)^{-2/3} \int N(x) exp(-i\mathbf{k}x) d^3x)$. Thus, in this way, one may attempt to overcome the divergence problem in conventional quantum gravity.

3 ON A MODIFICATION OF THE ORDINARY FIELD THEORY

The foamy structure of the space-time must be reflected in a universal way in the structure of the conventional field theory. As an example, we consider now a free massless scalar field φ

In terms of Fourier expansion for φ

$$\varphi(x, t) = (2\pi)^{-2/3} \int \frac{d^3\mathbf{k}}{\sqrt{2k}} \left\{ A(\mathbf{k}) e^{i\mathbf{k}x - ikt} + A^{+}(\mathbf{k}) e^{i\mathbf{k}x - ikt} \right\} \tag{3.1}$$

(here $k = |\mathbf{k}|$), the field Hamiltonian takes the form of a sum of independent non-interacting oscillators

$$H = \int \frac{k}{2}\left\{A(\mathbf{k})A^+(\mathbf{k}) + A^+(\mathbf{k})A(\mathbf{k})\right\} d^3\mathbf{k}. \tag{3.2}$$

Since, as mentioned in the Section 2, the number of spatial domains $N(\mathbf{k})$ can be a variable quantity so does the number of the field oscillators. This fact can be accounted for in a phenomenological manner by introducing creation and annihilation operators of the field oscillators which obey the same (anti) commutation relations as in (2.5):

$$[C(\mathbf{k},n), C^+(\mathbf{k}',m]_\pm = \delta_{n,m}\delta^3(\mathbf{k}-\mathbf{k}'), \tag{3.3}$$

where dependence of the operators on the quantities $\mathbf{k}$ and n is connected with the classification of the states of an individual oscillator (the spectrum of the oscillator has the form $\epsilon(\mathbf{k},n) = kn + \epsilon_0(\mathbf{k})$, where the quantity $\epsilon_0(\mathbf{k})$ gives the contribution of vacuum fluctuations of the field). In the vacuum state $|0\rangle$(which is determined now by $C(\mathbf{k},n)|0\rangle = 0$) field oscillators (and all field observables) are absent. The operator of total energy of the field can be generalized in a natural way as

$$E = \sum \epsilon(\mathbf{k},n)C^+(\mathbf{k},n)C(\mathbf{k},n). \tag{3.4}$$

The connection with the standard field variables can be determined with the help of operators which increase (decrease) the energy of system on $k([E, A^{(+)}(\mathbf{k})]_- = \pm kA^{(+)}(\mathbf{k}))$

$$A^+(\mathbf{k}) = \sum_{n=0}^{\infty}(n+1)^{1/2}C^+(\mathbf{k},n+1)C(\mathbf{k},n), \tag{3.5}$$

$$A(\mathbf{k}) = \sum_{n=0}^{\infty}(n+1)^{1/2}C^+(\mathbf{k},n)C(\mathbf{k},n+1). \tag{3.6}$$

It can be seen from (3.4)–(3.6) that the operators A and A^+ satisfy the commutation relations

$$[A(\mathbf{k}), A^+(\mathbf{k}')]_- = N(\mathbf{k})\delta^3(\mathbf{k}-\mathbf{k}'), \tag{3.7}$$

where $N(\mathbf{k}) = \sum_{n=0}^{\infty} C^+(\mathbf{k},n)C(\mathbf{k},n)$ is the complete number of spatial domains related to the wave number $\mathbf{k}$. If one restricts oneself to the states (2.6) with $N(\mathbf{k}) = 1$, the operator $A^+(\mathbf{k})$ and $A(\mathbf{k})$ certainly coincide with the standard creation and annihilation operators of scalar particles.

As mentioned in Section 2, the quantities $N(\mathbf{k},n) = C^+(\mathbf{k},n)C(\mathbf{k},n)$ must be determined by dynamics. However, they can be estimated from simple considerations. It is clear that in the absence of gravitational interaction the quantities $N(\mathbf{k},n)$ remain constant. Then, for instance, under the assumption of bounded density $N < \infty$ of oscillators satisfying the Fermi statistics it is easy to find that the occupation numbers corresponding to the ground state are

$$N(\mathbf{k},n) = \theta(\mu - \epsilon(\mathbf{k},n)), \tag{3.8}$$

where $\theta(x) = \{0 \text{ for } x < 0 \text{ and } 1 \text{ for } x > 0\}$, and μ is determined via the total number of oscillators $N = \sum N(\mathbf{k}, n)$. Using (3.8), one can found the number of oscillators corresponding to a wave vector $\mathbf{k}$ as

$$N(\mathbf{k}) = \sum_{n=0}^{\infty} \theta(\mu - \epsilon(\mathbf{k}, n)) = [1 + (\mu - \epsilon_0(\mathbf{k}))/k], \tag{3.9}$$

where $[x]$ denotes the entire part of x. In particular, one can see from (3.9) that $N(\mathbf{k}) = 0$ for $\mu < \epsilon_0(\mathbf{k})$.

4 CONCLUDING REMARKS

For the excited states formed by the action of the operators $A^+(\mathbf{k})$ on the ground state (3.8), the operator $N(\mathbf{k})$ is the usual function (3.9). Let us consider excitations of the field (scalar particles) described by the thermal equilibrium state corresponding to temperature T (one could expect that the spatial domains created near the singularity have a thermal spectrum [8]). Then the correlation function for the potentials of the field (3.1) takes form

$$\langle \varphi(x)\varphi(x + \mathbf{r}) \rangle = (2\pi^2)^{-1} \int \Phi^2(k) \frac{\sin kr}{kr} \frac{dk}{k}, \tag{4.1}$$

where $\Phi^2(k) = k^2 N(k) \frac{1}{2} \mathrm{cotanh}(\frac{k}{2T})$. In the wave number range $k \ll (T, \mu)$ the spectrum of the field fluctuations is scale-independent: $\Phi^2(k) \approx TkN(k) = T\mu$.

We also note that he ground state determined by the occupation numbers (3.8) has a bounded energy density of the field which can be considered as a "dark matter". In addition, we note that the above property of the spectrum to be scale-invariant at large scales for the thermal equilibrium state, actually, does not depend on the statistics of the oscillators (i.e., upon the sign $\pm$ in (3.3) and (2.5)).

References

1. Wheeler, J. A. (1964) In *Relativity Groups and Topology*, B. S. DeWitt and C. M. DeWitt (eds.), Gordon and Breach, New York.
2. Hawking, S. W. (1978) *Nuclear Phys.* **B114**, 349.
3. Hawking, S. W. (1988) *Phys. Rev.* **D37**, 904.
4. Gidings, S. and Strominger, A. (1988) *Nucl. Phys.* **B307**, 854.
5. Lavrelashvili, G. V., Rubakov, V. A., and Tinyakov, P. G. (1988) *Nucl. Phys.* **B299**, 757.
6. Coleman, S. (1988) *Nucl. Phys.* **B310**, 643.
7. Rubakov, V. A. (1988) *Phys. Lett.* **B214**, 503.
8. Kirillov, A. A. (1992) *Sov. Phys. JETP Lett.* **55**, 561.
9. Grishchuk, L. P. and Zeldovich, Ya. B. (1982) *Quantum Structure of Space and Time*, Cambridge Univ. Press, p. 387;
Zeldovich, Ya. B. and Starobinsky, A. A. (1984) *Pis'ma Astron. Zh.* **10**, 323 [*Sov. Astron. Lett.* **10**, 135, 135 (1984)].
10. Kirillov, A. A. (1993) *Sov. Phys. JETP* **76**, 355.
11. Kirillov, A. A. (1994) *Int. Jour. Mod. Phys.* **D3**, 431 (see also Kirillov, A. A. (1993) *Ph. D. Thesis*, Center for Surface and Vacuum Research, Moscow (in Russian)).
12. Misner, C. W. (1969) *Phys. Rev.* **186**, 1319.