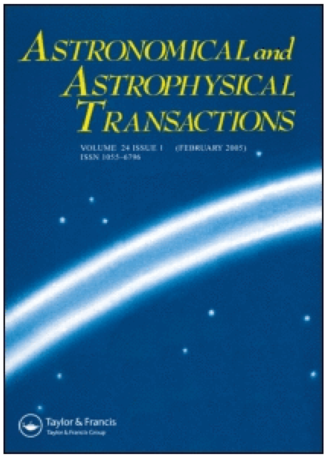

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VACUUM WEYL COSMOLOGIES IN *D* DIMENSIONS

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Vacuum cosmological models are considered in the context of a multidimensional theory of gravity with integrable Weyl geometry. A family of exact solutions with a chain of internal spaces is obtained. Models with one internal space are considered in more detail; nonsingular models are selected.

1. Scalar fields play a significant role in modern cosmology (Staniukovich and Melnikov, 1983), in particular, in various inflationary models. However, there is an inherent problem of the origin of this field. It can be naturally solved in multidimensional models where scalar fields are represented by extra-dimension scale factors (Bronnikov and Melnikov, 1992) and other models with generalized geometries such as the Weyl geometry (Novello *et al.*, 1993). In both cases nonsingular cosmological models have been obtained.

Here we consider a scheme unifying the two approaches, i.e., multidimensional cosmology with an integrable Weyl geometry. Since the most reliable results are obtained on the basis of exact solutions, we try to find them in the simplest cases. Recently some results have been obtained in the same scheme using numerical methods (Konstantinov and Melnikov, 1994).

2. Consider a D-dimensional manifold W_D with an integrable Weyl geometry defined by the metric g_{AB} and the connection

$$\Gamma^A_{BC} = \tilde{\Gamma}^A_{BC} - \frac{1}{2}(\omega_B \delta^A_C + \omega_C \delta^A_B - g_{BC}\omega^A), \tag{1}$$

where $\tilde{\Gamma}^A_{BC}$ are the Christoffel symbols for the metric g_{AB}, ω is a scalar field and $\omega_A = \partial_A \omega$.

Gravitational field is determined by the tensor g_{AB} and the scalar ω, just as in scalar-tensor theories (STT) of gravity. As is the case with STT, the gravitational Lagrangian may in general contain various invariant combinations of g_{AB} and ω. Let us restrict ourselves to Lagrangians (a) linear in the scalar curvature and (b) quadratic in ω_A. Then the general form of the Lagrangian is

$$L = A(\omega)R + B(\omega)\omega^A\omega_A - 2\Lambda(\omega) + L_m, \tag{2}$$

where R is the Weyl scalar curvature corresponding to the connection (1), A, B and Λ are arbitrary functions and L_m is the nongravitational matter Lagrangian.

To simplify the field equations let us make use of the expression of R in terms of the Riemannian curvature $\tilde{R}$ corresponding to the metric g_{AB}:

$$R = \tilde{R} + (D-1)\Box\omega - \frac{1}{4}(D-1)(D-2)\omega^A\omega_A \tag{3}$$

(in $\tilde{R}$ and $\Box$, the Riemannian connection $\tilde{\Gamma}^A_{BC}$ is used) and the conformal mapping well-known in STT (Wagoner, 1970), modified for D dimensions (Bronnikov and Melnikov, 1994):

$$g_{MN} = A^{-2/(D-2)}\overline{g}_{MN}. \tag{4}$$

Consequently, omitting a total divergence, we obtain the following form of the Lagrangian:

$$\overline{L} = A(\omega)\overline{R} + F(\omega)\overline{g}^{AB}\omega_A\omega_B + A^{-D/(D-2)}[-2\Lambda(\omega) + L_m], \tag{5}$$

where $\overline{R}$ is the Riemannian scalar curvature corresponding to the metric $\overline{g}_{AB}$ and $(A_\omega \equiv dA/d\omega)$

$$F(\omega) = \frac{1}{A^2}\left[AB - (D-1)A\left(A_\omega + \frac{D-2}{4}\right) + \frac{D-1}{D-2}A_w^2\right]. \tag{6}$$

3. Let us consider vacuum cosmological models in the theory described: put $L_m = 0$ and postulate the following structure of the space-time W_D:

$$W_D = R \times M_1 \times \ldots \times M_n; \quad \dim M_i = N_i; \tag{7}$$

the subspaces M_i are assumed to be maximally symmetric. The component R corresponds to time τ; besides, we assume $\omega = \omega(\tau)$. Thus, the effective Riemannian metric is written in the form

$$d\overline{s}^2 = \overline{g}_{AB}\, dx^A\, dx^B = e^{2\gamma(\tau)}\, d\tau^2 - \sum_{i=1}^{n} e^{2\beta_i(\tau)}\, ds_i^2, \tag{8}$$

where ds_i^2 are τ-independent metrics of the N_i-dimensional spaces of constant curvature K_i; without loss of generality one can put $K_i = 0, \pm 1$.

Using the freedom to choose the time coordinate τ, let us introduce the harmonic time by putting

$$\gamma = \sum_{i=1}^{n} N_i\beta_i. \tag{9}$$

Then the Ricci tensor for $\overline{g}_{AB}$ has the following nonzero components:

$$
\begin{aligned}
\overline{R}^{\tau}_{\tau} &= e^{-2\gamma}\left(\ddot{\gamma} - \dot{\gamma}^2 + \sum_{i=1}^{n} N_i \dot{\beta}_i^2\right), \\
\overline{R}^{m_i}_{n_i} &= \delta^{m_i}_{n_i}\left[e^{-2\gamma}\ddot{\beta}_i + (N_i - 1) K_i e^{-2\beta_i}\right], \qquad (10)
\end{aligned}
$$

where the indices m_i and n_i belong to the subspace M_i.

4. The field equations take an especially simple form under the additional constraint $\Lambda \equiv 0$:

$$\overline{R}_{MN} + F(\omega)\omega_M\omega_N = 0, \qquad (11)$$

$$2\overline{\nabla}_M[F(\omega)\omega^M] - F_\omega\omega^M\omega_M = 0. \qquad (12)$$

They can be integrated completely under the above assumptions if (i) all the subspaces M_i are Ricci-flat and (ii) if one of M_i (for instance, M_1) is a space of nonzero constant curvature (K_1). Indeed, putting $K_i = 0$ ($i > 1$), we obtain:

$$
\begin{aligned}
(F\dot{\omega}^2) = 0 \quad &\Rightarrow \quad F\dot{\omega}^2 = \text{const}; \qquad (13) \\
\ddot{\beta}_i = 0 \quad &\Rightarrow \quad \beta_i = \beta_{i0} + h_i\tau, \quad i > 1; \qquad (14) \\
\ddot{\gamma} - \ddot{\beta}_1 \quad &= \quad -K_1 d^2 e^{2\gamma - 2\beta_1} \qquad (15)
\end{aligned}
$$

where $d + 1 = N_1 = \dim M_1$. Equation (15) leads to different results for different K_1: for $K_1 = 0$ (case (i)) Eq. (14) may be regarded to include $i = 1$; for $K_1 \neq 0$ (case (ii)) we get:

$$e^{\beta_1 - \gamma} = \frac{d}{k}\cosh k\tau, \quad k > 0 \quad (K_1 = +1), \qquad (16)$$

$$e^{\beta_1 - \gamma} = d \cdot s(k,\tau) \equiv \begin{cases} (d/k)\sinh k\tau, & k > 0, \\ d \cdot \tau, & k = 0; \\ (d/k)\sin k\tau, & k < 0, \end{cases} \quad (K_1 = -1) \qquad (17)$$

where $k =$ const and another integration constant is eliminated by a particular choice of the origin of τ. Finally, a combination of components of (11) representing the time component of the Einstein equations (the initial data equation) leads to the following relation among the integration constant:

$$\left(\sum_{i=1}^{n} N_i h_i\right)^2 - \sum_{i=1}^{n} N_i h_i^2 = S, \qquad K_1 = 0; \qquad (18)$$

$$\frac{d+1}{d}k^2 \operatorname{sign} k = \frac{1}{d}\left(\sum_{i=2}^{n} N_i h_i\right)^2 + \sum_{i=2}^{n} N_i h_i^2 = S, \qquad K_1 \neq 0. \qquad (19)$$

Thus the set of equations (11), (12) has been integrated by quadratures.

Insofar as the original functions $A(\omega)$ and $B(\omega)$ and hence $F(\omega)$ are arbitrary, it is difficult to describe the physical properties of the models in a general form. Therefore here we restirct ourselves to some simple special cases.

Thus, we assume $A \equiv 1$ while $B(\omega)$ remains arbitrary, so that the metric $\overline{g}_{AB}$ and g_{AB} coincide.

5. As the first step consider 4-dimensional cosmologies: put $n = 1, d = 2$ and $\beta_1 \equiv \beta(\tau)$. The condition that τ is a harmonic coordinate takes the form $\gamma = 3\beta$ and for the scale factor we get:

$$e^{2\beta} = a^2(\tau) = \begin{cases} 1/2s(k,\tau), & K_1 = 1, \\ e^{k\tau}, & K_1 = 0, \\ 1/2\cosh k\tau, & K_1 = -1, \end{cases} \tag{20}$$

while the physical time is determined by the integral $l = \pm \int e^{\gamma(\tau)}\, d\tau$. The constant k is connected with the "scalar charge" S according to (18), (19) where one should substitute $h_i = 0 (i > 1)$ and $h_1 = k/2$:

$$2S = \begin{cases} 3k^2 \operatorname{sign} k, & K_1 = \pm 1, \\ 3k^2, & K_1 = 0. \end{cases} \tag{21}$$

It is easy to obtain that in the case of a spherical world ($K_1 = 1$) the values $\tau = \pm\infty$ correspond to finite times t_1 and t_2 at which $a = 0$ (the initial and final singularities). For a flat world ($K_1 = 0$) at $k \neq 0$ and a hyperbolic one ($K_1 = -1$) at $k > 0$ an initial or final singularity is observed at infinite τ. In the special case $K_1 = -1$, $k = 0$ we obtain the Milne vacuum model which is known to describe a domain in flat space-time (in this case $S = 0$, so that the scalar field is trivial).

Finally in the case where $K_1 = -1$ and $k < 0$ we see that the limits $\tau \to 0,\ \pi/|k|$ correspond to $t \to \pm\infty$; the scale factor $a(t)$ decreases in an asymptotically linear manner in the remote past ($t \to -\infty$), reaches a minimum at $\tau = \pi/2|k|$ and grows in an asymptotically linear manner at $t \to \infty$. The model is time-symmetric with respect to the moment of maximum contraction.

From (21), a necessary condition for the existence of nonsingular solutions is the restriction $F < 0$ on the function (6), or, in terms of the initial function $B(\omega) : B < 3/2$.

These results confirm those of Novello *et al.* (1993).

6. Consider now the metric $\overline{g}_{AB}$ for $n = 2$; let $a(t) \equiv e^{\beta_1(\tau)}$ be the scale factor of the ordinary physical space ($N_1 = 3$), while $b(t) \equiv e^{\beta_2(\tau)}$ is that of the internal space ($N_2 = N$).

6.1. In the case $K_1 = 0$ (spatially flat models) we obtain:

$$d\overline{s}^2 = e^{2(3h_1 + Nh_2)\tau}\, d\tau^2 - e^{2h_1\tau}\, ds_1^2 - e^{2h_2\tau}\, ds_2^2, \tag{22}$$

where, without loss of generality, the scales in M_1 and M_2 are chosen so that $\beta_{10} = \beta_{20} = 0$. Herewith,

$$6(h_1 + Nh_2/2)^2 = N(N + 1/2) + S. \tag{23}$$

In the special case $3h_1 + Nh_2 = 0$ the time coordinate τ is synchronous, or physical. The metric (22) is nonsingular at finite τ and described an exponential expansion (inflation) of one of the space (e.g., the physical one, M_1) and a simultaneous exponential contraction of the other, M_2, since h_1 and h_2 have different signs. However, from (23) and (13)

$$S = F\dot{\omega}^2 = -h_1^2(2N + 1)/N < 0. \tag{24}$$

So a necessary condition for the existence of the special solution (22) is

$$B(\omega) < (D - 1)(D - 2)/4, \tag{25}$$

which is more general than $B < 3/2$ from Section 4.

In the more general case $3h_1 + Nh_2 = H \neq 0$ a transition to the physical time $dt = e^{H\tau} d\tau$ leads to the metric

$$d\bar{s}^2 = dt^2 - t^{2h_1/H} ds_1^2 - t^{2h_2/H} ds_2^2, \tag{26}$$

which is singular at $t = 0$ if at least one of the constants h_1 or h_2 is nonzero. At $h_1 = h_2 = 0$ the metric is static and (24) implies that either $\dot{\omega} = 0$ (the solution is trivial), or $F \equiv 0$, a special choice of B such that $\omega(\tau)$ has no dynamics.

6.2. For a spherical world ($K_1 = 1$) the metric is

$$d\bar{s}^2 = \frac{e^{-Nh\tau}}{2\cosh k\tau}\left[\frac{d\tau^2}{4\cosh^2 k\tau} - ds_1^2\right] - e^{2h\tau} ds_2^2, \tag{27}$$

where ds_1^2 is the line element on a unit sphere. A consideration similar to that in Section 5.1 leads to the following conclusions:

(a) The model behavior is classified by the values of the constant $h = h_2$ as compared with $k > 0$. The physical time $l = \pm \int e^{\gamma(\tau)} d\tau$ varies either within a finite segment $[t_1, t_2]$ (if $|Nh| < 3k$), or within a semi-infinite range (if $|Nh| \geq 3k$).

(b) At any finite boundary of the range of t at least one of the scale factors $a(t)$ or $b(t)$ vanishes, i.e., a singularity takes place.

(c) At $t \to \pm\infty$ either $a \to 0, b \to \infty$, or conversely, $a \to \infty, b \to 0$.

The value of $S = F\dot{\omega}^2$ is determined at $K_1 = \pm 1$ from

$$3k^2 \operatorname{sign} k = N(N + 2)h^2 + 2S. \tag{28}$$

6.3. For hyperbolic models ($K_1 = -1$) the metric has the form

$$d\bar{s}^2 = \frac{e^{-Nh\tau}}{2s(k,\tau)}\left[\frac{d\tau^2}{4s^2(k,\tau)} - ds_1^2\right] - e^{2h\tau}\, ds_2^2, \tag{29}$$

which is the same as (27), but with the function $\cosh k\tau$ replaced by $s(k,\tau)$ defined in (17). Without loss of generality, let us assume $\tau > 0$.

The model behavior can be briefly described as follows:

(a) At $k > 0, Nh \le -3k$ or $k = 0, h < 0$ the physical time $t = \pm\int e^{\gamma(\tau)}\, d\tau$ ranges from $-\infty$ to $+\infty$. The factor $b(t) = e^{h(\tau)}$ varies from a finite value at $\tau = 0 (t = -\infty)$ to zero at $\tau \to \infty (t \to \infty)$. The factor $a(t)$ describes a power-law contraction from infinity (at $t \to -\infty$) to a regular minimum and an infinite (in general, power-law) expansion at $t \to \infty$. There is no singularity at finite t.

(b) At $k \ge 0, Nh > 3k$ the model is singular at finite t corresponding to $\tau \to \infty$. In the special case $h = k = 0$ we come again to the Milne model (see Section 4) supplemented with the space M_2 having a constant scale factor.

(c) At $k < 0$ the time t ranges again from $-\infty$ to $+\infty$. The factor $a(t)$ behaves as it did in item (a), however, its variation at $t \to \pm\infty$ is linear (but in general with unequal slopes at the two asymptotics). The factor $b(t)$ changes monotonically between two finite boundary values.

Unlike the 4-dimensional models (Section 4), the nonsingular multidimensional ones with $h \neq 0$ exhibit a time-asymmetric behavior of $a(t)$.

It is seen in a straightforward way that in all the nonsingular models the requirement (25) is imposed on $B(\omega)$, which, as it could be formulated in general relativity, implies a negative scalar field energy density.

Some properties of the above models have been discovered in numerical calculations for a number of special cases with $D = 5$ and $D = 6$ (Konstantinov and Melnikov, 1994).

We conclude that some of the multidimensional Weyl cosmologics are nonsingular: there are special flat-space models with eternally increasing or decreasing scale factor (such models are absent in the 4-dimensional approach) and there are more general hyperbolic models with a cosmological bounce generalizing the 4-dimensional ones (Novello *et al.*, 1993). However, it should be taken into account that we have considered only one conformal gauge (although, in a certain sense, the most natural one), while, in the others the picture of singularities may change. The choice of a conformal gauge, connected with the choice of a system of measurements, is a separate problem (Staniukovich and Melnikov, 1993), especially when a generalized geometry is used; its solution depends on the specific form of interaction between matter and geometry, which subject is beyond the scope of vacuum cosmologics.

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