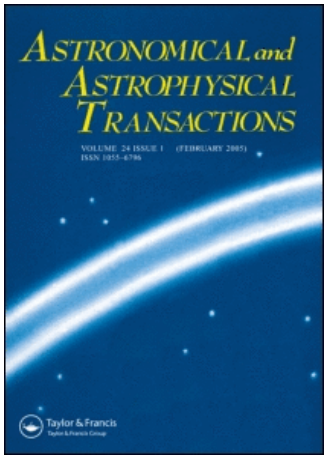

Astronomical & Astrophysical Transactions
The Journal of the Eurasian Astronomical Society
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713453505

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Online Publication Date: 01 June 1996
To cite this Article: Kirillov, A. A. and Melnikov, V. N. (1996) 'On properties of metrics inhomogeneities in the vicinity of a singularity in Kaluza-Klein cosmological models', Astronomical & Astrophysical Transactions, 10:2, 101 - 109
To link to this article: DOI: 10.1080/10556799608203014
URL: http://dx.doi.org/10.1080/10556799608203014

Astronomical and Astrophysical Transactions, 1996,
Vol. 10, pp. 101–109
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ON PROPERTIES OF METRICS INHOMOGENEITIES IN THE VICINITY OF A SINGULARITY IN KALUZA-KLEIN COSMOLOGICAL MODELS

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We discuss the dynamics of inhomogeneities of metric in a general solution to D-dimensional Einstein equation with matter sources satisfying the inequality $\epsilon \geq p$ in the vicinity of a cosmological singularity. It is shown that a local behavior of a part of metric functions near the singularity is described by a billiard on a space of a constant negative curvature. If $D \leq 10$, the billiard has a finite volume and, consequently, it is a mixing one. It is shown for this case that statistical properties of inhomogeneities of metric admit a complete description. An invariant measure describing statistics of inhomogeneities is obtained and the role of a minimally-coupled scalar field in the dynamics of the inhomogeneities is also considered.

1 INTRODUCTION

Qualitative features of a general solution to D-dimensional Einstein equations in the vicinity of the cosmological singularity are known to depend critically on the number of spacetime dimension [1]. In the case $D < 11$ an oscillatory regime occurs obtained first in Refs. [2]. In the opposite case $D \geq 11$ the oscillatory regime turns out to be unstable and the last stage of cosmological collapse is described by the stable generalized Kasner metric. In the case $D = 4$, the existence of the oscillatory stage in the evolution of metric was shown to result in the fractioning of the coordinate scale of inhomogeneities of metric and finally in the formation of spatial chaos in metric functions [3]. The aim of this paper is to generalize the approach suggested in Ref. [3] and to investigate dynamics and properties of inhomogeneities of metric near the singularity in Kaluza–Klein cosmology.

It is well known that matter with equation of state satisfying the inequality $\epsilon < p$ does not change the behavior of metric near the singularity [4] and the only kind of matter effecting the dynamics of metric is a scalar field [2]. The scalar field results in the instability of the oscillatory regime as that of the dimensions

exceeding $D = 10$. Therefore, it seems to be sufficient to consider Kaluza–Klein cosmologies with $D \leq 10$ filled with a scalar field.

Thus, in this paper we consider D-dimensional Einstein equations with the matter source given by a minimally-coupled scalar field. Using generalized Kasner variables, we divide the dynamical functions connected with physical degrees of freedom in two parts. One part has a simple behavior while the other is described by a billiard on an appropriate Lobachevsky space. In $D < 11$, the billiard has a finite volume and shows stochastic properties. This stochasticity results in inhomogeneity of dynamical functions and leads to the formation of spatial chaos. The presence of a scalar field results in the fact that lengths of trajectories on the billiard take finite values. This destroys the chaotic properties which, however, are restored in the limit when the ADM energy density for the scalar field turns out to be small as compared with that of gravitational variables.

2 GENERALIZED KASNER VARIABLES

We consider the theory in canonical formulation. Basic variables are the Riemann metric components $g_{\alpha\beta}$ with signature $(+,-,\ldots,-)$ and a scalar field ϕ specified on the n-manifold S, and its conjugate momenta $\Pi^{\alpha\beta} = \sqrt{g}(K^{\alpha\beta} - g^{\alpha\beta}K)$ and Π_ϕ, where $\alpha = 1,\ldots,n$ and $K^{\alpha\beta}$ is the extrinsic curvature of S. For the sake of simplicity we shall consider S to be compact, i.e., $\partial S = 0$. The action has the following form in Planck's units:

$$I = \int_S (\Pi^{ij}\frac{\partial g_{ij}}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - NH^0 - N_\alpha H^\alpha) d^n x\, dt, \tag{2.1}$$

where

$$H^0 = \frac{1}{\sqrt{g}}\{\Pi^\alpha_\beta \Pi^\beta_\alpha - \frac{1}{n-1}(\Pi^\alpha_\alpha)^2 + \frac{1}{2}\Pi^2_\phi + g(W(\phi) - R)\}, \tag{2.2}$$

$$H^\alpha = -2\Pi^{\alpha\beta}_{|\beta} + g^{\alpha\beta}\partial_\beta\phi\Pi_\phi, \tag{2.3}$$

$$W(\phi) = \frac{1}{2}\{g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\}. \tag{2.4}$$

We consider the following representation for metric components and their conjugate momenta [3]

$$g_{\alpha\beta} = \sum_a \exp\{q^a\} l^a_\alpha l^a_\beta, \tag{2.5}$$

$$\Pi^\alpha_\beta = \sum_a p_a L^\alpha_a l^a_\beta, \tag{2.6}$$

where $L^\alpha_a l^b_\alpha = \delta^b_a$ $(a, b = 0,\ldots,(n-1))$, and the vectors l^a_α contain only $n(n-1)$ arbitrary functions of spatial coordinates. Further parametrization may be taken in the following form:

$$l^a_\alpha = U^a_b S^b_\alpha,\ U^a_b \in SO(n),\ S^a_\alpha = \delta^a_\alpha + R^a_\alpha, \tag{2.7}$$

where R^a_α denotes a triangular matrix ($R^a_\alpha = 0$ as $a < \alpha$). Substituting (2.5)–(2.7) into (2.1) one gets the following expression for the action functional:

$$I = \int_S (p_a \frac{\partial q^a}{\partial t} + T^\alpha_a \frac{\partial R^a_\alpha}{\partial t} + \Pi_\phi \frac{\partial \phi}{dt} - NH^0 - N_\alpha H^\alpha) d^n x\, dt, \tag{2.8}$$

where $T^\alpha_a = 2\sum_b p_b L^\alpha_b U^b_a$ and the Hamiltonian constraint takes the form

$$H^0 = \frac{1}{\sqrt{g}}\{\sum p_a^2 - \frac{1}{n-1}(\sum p_a)^2 + \frac{1}{2}\Pi_\phi^2 + V\},\ V = g(W - R). \tag{2.9}$$

In the case of $n = 3$ the functions R^a_α are connected purely with transformations of a coordinate system and may be removed by solving momentum constraints $H^\alpha = 0$ [3]. In the multidimensional case the functions R^a_α contain $\frac{n(n-3)}{2}$ dynamical functions as well.

3 AN ASYMPTOTIC MODEL IN THE VICINITY OF A COSMOLOGICAL SINGULARITY

In order to investigate inhomogeneities in the vicinity of a singularity, it is more convenient to use an asymptotic expression for the potential [3]. For this purpose we represent the potential in the following form:

$$V = \sum_{A=1}^{k} \lambda_A g^{\sigma_A}, \tag{3.1}$$

where λ_A is a set of functions of all dynamical variables and their derivatives and σ_a are linear functions of the anisotropy parameters $Q_a = \frac{q^a}{\sum q}(\sigma_A = \sigma_A(Q))$. Assuming the finiteness of the functions λ_A and considering the limit $g \to 0$ we find that the potential V may be modeled by potential walls

$$g^{\sigma_A} \to \theta_\infty[\sigma_A(Q)] = \begin{cases} +\infty, & \sigma_A < 0, \\ 0, & \sigma_A > 0. \end{cases} \tag{3.2}$$

Thus, putting $N^\alpha = 0$, we can remove the passive dynamical function T^α_a, R^a_α from the action (2.8) and get the reduced dynamical system

$$I = \int_S \left\{ (p_a \frac{\partial q^a}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - \lambda \left\{ \sum p^2 - \frac{1}{n-1}(\sum p)^2 + \frac{1}{2}\Pi_\phi^2 + U(Q) \right\} \right\} d^n x\, dt, \tag{3.3}$$

where λ is expressed via the lapse function as $\lambda = \frac{N}{\sqrt{g}}$. In harmonic variables, the action (3.3) takes the form formally coinciding with the action for a relativistic particle

$$I = \int_S \{P_r \frac{\partial z^r}{\partial t} - \lambda'(P_i^2 + U - P_0^2)\} d^n x\, dt, \tag{3.4}$$

where $r = 0, \ldots, n$, $i = 1, \ldots, n$, $q^a = A_j^a z^j + z^0$ $(j = 1, \ldots, n-1)$, $z^n = \sqrt{\frac{2}{n(n-1)}}\phi$, $\lambda' = \frac{\lambda}{n(n-1)}$ and the constant matrix Λ_j^a obeys the following conditions:

$$\sum_a A_j^a = 0, \ \sum_a A_j^a A_k^a = n(n-1)\delta_{jk}, \tag{3.5}$$

and can be expressed in the following form:

$$\Lambda_j^a = \sqrt{\frac{n(n-1)}{j(j+1)}}(\theta_j^a - j\delta_j^a), \ \theta_j^a = \begin{cases} 1, & j > a, \\ 0, & j \le a. \end{cases}$$

Since the timelike variable z^0 varies during the evolution as $z^0 \sim \ln g$, position of the potential walls turn out to be moving. It is more convenient to fix positions of the walls. This may be done by using the so-called variables of the Misner–Chitre type [3,5] ($\vec{y} = y^j$),

$$z^0 = -e^{-\tau}\frac{1+y^2}{1-y^2}, \ \vec{z} = -2e^{-\tau}\frac{\vec{y}}{1-y^2}, \ y = |\vec{y}| < 1. \tag{3.6}$$

Using these variables one can find the following expressions for the anisotropy parameters:

$$Q_a(y) = \frac{1}{n}\left\{1 + \frac{2A_j^a y^j}{1+y^2}\right\}, \tag{3.7}$$

which are now independent of the time-like variable τ. In the vacuum case, expressions (3.7) give, under the restriction $|y| = 1$, a parametrization of the standard Kasner exponents [1], [2]. From (3.7) one can find the range of the anisotropy functions $-\frac{n-2}{n} \le Q_a \le 1$.

Choosing as a time variable the quantity τ(i.e., in the gauge $N = \frac{n(n-1)}{2}\sqrt{g}$ $\exp(-2\tau)/P^0$) we put the action (3.4) into the ADM form

$$I = \int_S \{\vec{P}\frac{\partial}{\partial\tau}\vec{y} + P^n \frac{\partial}{\partial\tau} z^n - P^0(P, y)\} d^n x\, d\tau, \tag{3.8}$$

where the quantity

$$P^0(P, y) = (\epsilon^2(\vec{y}, \vec{P}) + V[y] + (P^n)^2 e^{-2\tau})^{1/2}, \tag{3.9}$$

plays the role of the ADM Hamiltonian density and

$$\epsilon^2 = \frac{1}{4}(1-y^2)^2 \vec{P}^2. \tag{3.10}$$

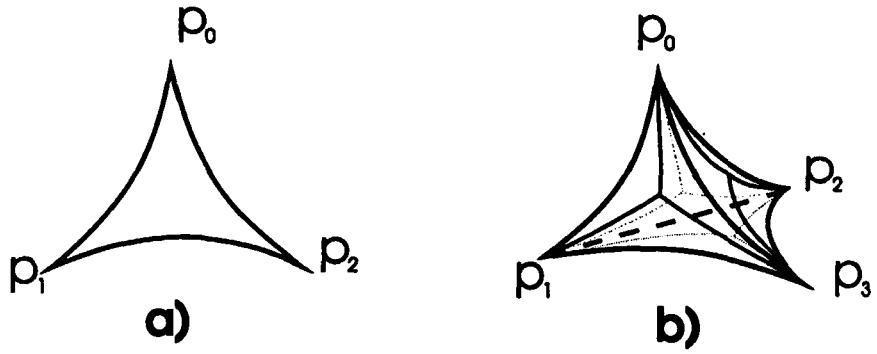

Figure 1

The part of the configuration space connected with the variables $\vec{y}$ is a realization of the $(n-1)$-dimensional Lobachevsky space and the potential V cuts a part of it. Thus, the action (3.9) describes locally (at a particular point of S) a billiard in the Lobachevsky space. Positions of the walls that form the boundary of the billiard are determined, due to (3.1) by the inequalities (see also [1], [2])

$$\sigma_{abc} = 1 + Q_a - Q_b - Q_c \geq 0, \; a \neq b \neq c, \tag{3.11}$$

and the total number of the walls is $\frac{n!}{2!(n-3)!}$. Using the matrix (3.5), one can find that walls are formed by spheres determined by the equations

$$\begin{aligned} \sigma_{abc} &= \frac{n-1}{n(1+y^2)}\{(\vec{y}+\vec{B}_{abc})^2 + 1 - B^2_{abc}\}, \\ \vec{B}_{abc} &= \frac{1}{n-1}(\vec{A}^a - \vec{A}^b - \vec{A}^c), \\ B^2 &= \frac{3n-1}{n-1}. \end{aligned} \tag{3.12}$$

In a general case n points of the billiard having the coordinates $\vec{P}_a = \frac{1}{n-1}\vec{A}^a$ lie on the absolute (at infinity of the Lobachevsky space). The trajectories which end with these points correspond to the set of Kasner exponents $(0, \ldots 0, 1)$. It was shown in Refs [1] that if $n \geq 10$, there appear open accessible domains on the absolute in addition to the points P_a and the volume of the billiard turns out to be infinite. If on contrary $n < 10$ the volume of the billiard is finite and the billiard turns out to be a mixing one. In order to illustrate the billiard we give two simplest examples in Figure 1. The case $n = 3$ in Figure 1a coincides with the well-known "mixmaster" model and in Figure 1b we illustrate the case $n = 4$.

4 DYNAMICS OF INHOMOGENEITIES

The system (3.8) has the form of a direct product of "homogeneous" local systems. Each local system in (3.8) has two variables ϵ and P^n as integrals of motion. A

solution of this local system for the remaining functions represents a geodesic flow on a manifold with negative curvature. It is well known that the geodesic flow on a manifold with negative curvature is characterized by exponential instability [6]. This means that, during the motion along a geodesic, the normal deviations grow no slower than the exponential of the traversed path ($\xi \simeq \xi_0 e^s$, where the traversed path is determined by the expression

$$s = \int_{\tau_0}^{\tau} dl = \int_{\tau_0}^{\tau} \frac{2|\frac{\partial y}{\partial \tau}|}{(1-y^2)} d\tau = \frac{1}{2} \ln \left| \frac{P^0 - \epsilon}{P^0 + \epsilon} \right|_{\tau_0}^{\tau} \tag{4.1}$$

This instability leads to the stochastic nature of the corresponding geodesic flow. The system possesses the mixing property [7] and an invariant measure induced by the Liouviulle one

$$d\mu(y, P) = \text{const}\, \delta(E - \epsilon)\, d^{n-1}y\, d^{n-1}P, \tag{4.2}$$

where E is a constant. Integrating this expression over ϵ we find

$$d\mu(y, m) = \text{const}\, \frac{d^{n-1}y d^{n-2}m}{(1-y^2)^n}, \tag{4.3}$$

where $\vec{m} = \frac{\vec{P}}{\epsilon}$ and $|m| = 1$.

Since the inhomogeneous system (3.8) is the direct product of "homogeneous" systems, one can simply describe its behavior as in ref [3]. In particular, the scale of the inhomogeneity decreases as

$$\lambda_i \sim \left(\frac{\partial y}{\partial x}\right)^{-1} \sim \lambda_i^0 \exp(-s), \tag{4.4}$$

and after a sufficiently large time ($s(\tau) \to \infty$) the dynamical functions $\vec{y}(x)$ and $\vec{P}(x)$ become random functions of spatial coordinates. Their statistics is described by the invariant distribution (4.3) and asymptotic expressions for averages and correlating functions have the form

$$\langle y(\vec{x})\rangle = \langle P(\vec{x})\rangle = 0,\ \langle y_k(x), y_l(x')\rangle = \langle y_k, y_l\rangle \delta(x, x'), \tag{4.5}$$

for $|x - x'| \gg \lambda_i^0 \exp(-s)$.

Here it is necessary to point out the role of the scalar field in dynamics and statistical properties of inhomogeneities. As can be easily seen from (4.1) that in the absence of a scalar field (i.e., $P^n = 0$) the transversed path coincides with the duration of motion (we have $s = \Delta\tau = \tau - \tau_0$ instead of (4.1)). Thus, the effect of scalar fields is displayed in a modification of the dependence of the transversed path on time variable and, therefore, affects the growth rate of the inhomogeneities. This modification does not change qualitatively the evolution of the universe in the case of cosmological expansion. But in the case of a contracting universe the situation changes drastically. Indeed, in the limit $\tau \to -\infty$ we find from (4.1) that the

transversed path s takes a limited value s_0 and therefore growth of inhomogeneities turns out to be finite. One of consequences of such a behaviour is the fact that the functions $\vec{y}$ and $\vec{P}$ take constant values at the singularity. In other words, a cosmological collapse ends with a stable Kasner-like regime (2.6) in the presence of scalar fields. This can be obtained in the other way. Indeed, in the limit $\tau \to -\infty$ the scalar field gives the leading contribution to the ADM Hamiltonian (3.9) and P^0 does not depend on gravitational variables at all.

The finiteness of the transversed path $s(\tau)$ leads, generally speaking, to the destruction of the mixing properties [7], since it is necessary to satisfy the condition $s_0 \to \infty$ for the establishment of the invariant measure. Evidently, this condition requires the smallness of the energy density for the scalar field as compared with the ADM energy of gravitational field (the last term in (3.9) in comparison with the first ones). Indeed, in this case s_0 is determined by $s_0 = -\ln \frac{P^n e^{\tau_0}}{2\epsilon}$, which follows from (4.1), and as $P^n \to 0$ one obtains $s_0 \to \infty$ (i.e., s can have arbitrary large values).

Thus, in the case of cosmological contraction one may speak of mixing and, therefore, of the establishment of the invariant statistical distribution just only for those spatial domains which have a sufficiently small energy density of the scalar field.

5 ESTIMATES AND CONCLUDING REMARKS

In this manner the large-scale structure of the space in the vicinity of the singularity acquires a quasi-isotropic nature. The distribution of inhomogeneities is determined by the set of functions of spatial coordinates $\epsilon(x), \Pi_\phi(x), R^a_\alpha$ and T^α_a which conserve during the evolution a primordial degree of inhomogeneity of the space. The scale of inhomogeneity of other functions grows as $\lambda \approx \lambda_0 e^{-s(\tau)}$. In this section we give some estimates clarifying the behaviour of the inhomogeneities. For simplicity we consider the case when the scalar field is absent.

To find an estimate for the growth of the inhomogeneity in the synchronous time $t(dt = Nd\tau)$ we put $y = 0$. Then for variation of the variable τ one may find the following estimate $\sqrt{g} \sim \exp(-\frac{n}{2}e^{-\tau}) \sim P^0 t$ (here the point $t = 0$ corresponds to the singularity). According to (4.4), the dependence of the coordinate scale of the inhomogeneity upon the time t takes the form

$$\lambda \approx \lambda_0 \ln(1/g_0)/ln(1/g)$$

in the case of contracting universe ($g \to 0$) and

$$\lambda \approx \lambda_0 \ln(1/g)/ln(1/g_0)$$

in the case of expanding universe.

A rapid generation of smaller and smaller scales leads to the formation for spatial chaos in metric functions and so the large-scale structure acquires a quasi-isotropic

nature. The rates of expansion (Hubble constants) for different directions turn out to be equal after averaging over spatial domains having the size $\approx \lambda_0$. Indeed, using (3.7) one can find the expressions for the averages, $\langle Q_a \rangle = 1/n$.

In spite of the isotropic nature of the spatial distribution of the field, a strong local anisotropy displays itself in the anomalous dependence of spatial lengths on time variable for vectors and curves. Indeed, the moment of the scale function $\langle g^{MQ_a} \rangle$ (where $M > 0$) decreases in the asymptotic $g \to 0$ as the Laplace integral $\int\limits_{Q_{min}}^{1} g^{MQ_a} \rho(Q_a)\, dQ_a$, where $\rho(Q_a)$ is the distribution which follows from (4.3). The main contribution in this integral is given by the point $Q = Q_{min} - \frac{(n-1)^2-(n+1)}{n(n+1)}$ and in the case $n > 3$ and in the limit $(Q - Q_{min}) \to 0$ one can find $\rho(Q) \approx C(Q - Q_{min})^{n-1}$, where C is a constant and we obtain the estimate

$$\langle g^{MQ_a} \rangle \approx \text{const} \frac{g^{MQ_{min}}}{[M \ln(1/g)]^{n-1}}. \tag{5.1}$$

Thus, for $n > 3$ the lengths even increase when one approaches the singularity. The case $n = 3$ must be considered separately. For this case we have $Q_{min} = 0$ and the explicit form of the distribution function $\rho(Q_a)$ that follows from (4.3) is

$$\rho(Q) = \frac{2}{\pi}[Q(1-Q)]^{-1/2}(1+3Q)^{-1}. \tag{5.2}$$

As $Q \ll 1$, one has $\rho(Q_a) \approx \frac{2}{\pi}(Q_a)^{-1/2}$ and, thus, in the limit $g \to 0$ we obtain the estimate [3]

$$\langle g^{MQ_a} \rangle \approx [M \ln(1/g)]^{-1/2}. \tag{5.3}$$

Thus, in the case $n = 3$, average scales decrease in the asymptotic $g \to 0$ but with a logarithmical behaviour.

In conclusion, we briefly formulate the main results. The general inhomogeneous solution of D-dimensional Einstein equations ($D = n + 1$) with any matter sources satisfying the inequality $\epsilon \geq p$ near the cosmological singularity is constructed. It is shown that near the singularity a local behaviour of metric functions (at a particular point of the coordinate space) is described by a billiard in the $(n-1)$-dimensional Lobachevsky space. In the case $D < 11$ the billiard has a finite volume and, consequently, is a mixing one. The rate of growth of metric inhomogeneities is obtained. Statistical properties of inhomogeneities are described by the invariant measure. It is shown that a minimally-coupled scalar field leads, in general, to the destruction of stochastic properties of inhomogeneous models as that of additional dimensions exceeding $D = 10$.

Acknowledgments

This work was supported in part by the Russian Ministry of Science.

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