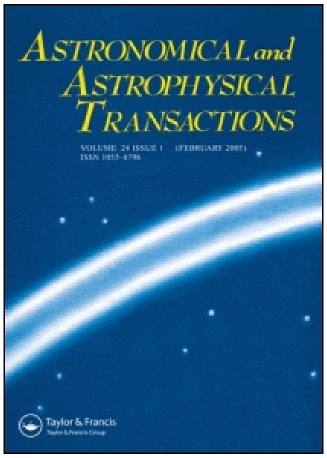

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SYSTEMATIC DIFFERENCES FK5-FK4: OPTIMAL REPRESENTATION

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We argue that spherical functions offer the best way to describe systematic differences in positions and proper motions of the two catalogues, FK4 and FK5. Moreover, it is shown that a covariance function of systematic differences can be easily constructed for the coefficients of the expansion of systematic differences in spherical functions. Since the authors of FK5 used the "Legendre–Fourier" functions to reproduce the systematic differences FK5-FK4, we recalculated them using the spherical functions. Comparison of both approaches is made.

KEY WORDS Fundamental astrometry, catalogue comparison, systematic differences

1 INTRODUCTION

Since the pioneer work of Broche, (1966), many authors have contributed to the representation of systematic differences with sets of orthogonal functions. The original method based on spherical functions was extended by Schwan (1977) in order to include magnitude equations and to use weights, if needed. This method was also actively applied by Jatskiv and Kurjanova (Zverev *et al.*, 1980). A study of the method made in Heidelberg Rechen-Institute resulted in the choice of socalled "Legendre–Hermite–Fourier" functions as a basis of expansion (Bien *et al.*, 1978). In this form the method was applied in FK5 (Fricke *et al.*, 1988). Note that the "Legendre–Hermite–Fourier" functions were first proposed by Valbousquet (1975) for zone catalogues.

The purpose of our paper is to show that the transition from the spherical functions to the "Legendre–Hermite–Fourier" functions was made without taking into account many other properties of the spherical functions, which can become very useful when more general problems than a simple approximation of systematic differences are considered.

2 ANALYTICAL METHODS FOR THE REPRESENTATION OF SYSTEMATIC DIFFERENCES

It is possible to obtain four kinds of individual differences from the comparison of N stars listed in the two catalogues:

$$\left\{ \begin{array}{c} \Delta\alpha_i \cos\delta_i \\ \Delta\mu_i \cos\delta_i \\ \Delta\delta_i \\ \Delta\mu'_i \end{array} \right\}, \quad i = 1, \ldots, N, \tag{1}$$

where i is a number of a star in a catalogue, α and δ are the equatorial coordinates, and μ and μ' are the star proper motions in α- and δ- directions respectively.

Let l be the systematic difference (any of (1)) for the ith star. Now the analytical method represents l as a series expansion in the following form:

$$l(\alpha_i, \delta_i, m_i) = \sum_{j=0}^{g} x_j Y_j(\alpha_i, \delta_i, \overline{m}_i) + \epsilon(\alpha_i, \delta_i, \overline{m}_i), \tag{2}$$

where $Y_j, j = 0, 1 \ldots g$ are the basic functions, x_j is a coefficient of $Y_j, \overline{m}_i$ is the transformed magnitude given by

$$\overline{m}_i = \frac{m_i - \langle m \rangle}{\sigma_m}, \tag{3}$$

where m_i is the apparent magnitude , $\langle m \rangle$ and σ_m^2 are the mean magnitude and dispersion of the magnitudes of stars under consideration.

The first summand in (2) is a systematic part of l, while the second one is a random part of l. Such a division is confirmed by a long-standing experience of astronomical practice. The system of basic functions Y_j must be complete, orthogonal and normalized in the area

$$G = \left\{ \begin{array}{c} 0^h \leq \alpha \leq 24^h \\ -90^\circ \leq \delta \leq +90^\circ \\ m_1 \leq m \leq m_2 \end{array} \right\}.$$

If a chosen system of basic functions is not orthogonal, it can be orthogonalized by means of the Gramm–Schmidt procedure. The algorithm also uses the weight matrix which includes a priori information about observations.

To determine the coefficients of expansion x_j, equation (2) should be solved by the least squares method. Two statistic criteria, the Fisher criterion and the γ-criterion which are discussed in detail by Zverev *et al.*, (1980) are used to find the highest order g of the coefficients x_j.

Let us consider both systems of basic functions.

3 BASIC FUNCTIONS

3.1 The "Legendre-Hermite-Fourier" functions

The authors of FK5 used the "Legendre–Hermite–Fourier" functions for the representation of systematic differences:

$$Y_j(\alpha, \delta, \overline{m}) = R_{pnk} H_p(\overline{m}) L_n(\delta) F_{kl}(\alpha). \tag{4}$$

The Legendre polynomials are defined by the following recurrence formula:

$$L_{n+1}(x) = (2n+1)xL_n(x)(n+1)^{-1} - n(n+1)^{-1}L_{n-1}(x), \quad n = 1, 2 \ldots, \tag{5}$$

where $L_0(x) = 1$ and $L_1(x) = x$. These functions are used to represent the dependence of systematic differences on declination.

The Hermite polynomials are defined by the following recurrence formula:

$$H_{p+1}(y) = yH_p(y) - pH_{p-1}(y), \quad p = 1, 2 \ldots, \tag{6}$$

where $H_0(y) = 1$ and $H_1(y) = y$. These polinomials describe the dependence of the systematic differences on the transformed apparent magnitude $\overline{m}$. The Fourier terms represent the dependence of the systematic differences on right ascension. They can be written as follows:

$$F_{kl}(\alpha) = \begin{cases} 1 & \text{for k=0,} \\ \cos(lk\alpha) & \text{for } l = +1, \text{ k=1,2,\ldots,} \\ \sin(lk\alpha) & \text{for } l = -1, \text{ k=1,2\ldots.} \end{cases} \tag{7}$$

A product of the functions H_p, L_n, F_{kl} is normalized by the factor R_{pnk} given by

$$R_{pnk} = (2n+1)^{\frac{1}{2}}(p!)^{-\frac{1}{2}} \begin{cases} 1 & \text{for k=0,} \\ \sqrt{2} & \text{for k} \neq 0. \end{cases} \tag{8}$$

The main difference between this system of functions and the spherical functions is that in the former case the dependence of systematic differences in declination is described using ordinary Legendre polynomials alone, whereas in the case of spherical functions this is done using the associated Legendre functions and ordinary Legendre functions as well. The authors of FK5 rejected spherical functions because, in their opinion, these functions yield poor convergence near the poles and, therefore, poorly approximate systematic differences. However, our investigation has shown that the spherical functions are not worse than the "Legendre–Hermite–Fourier" functions near the poles. But this is not the principal reason for the application of spherical functions.

3.2 Spherical functions

The basic functions (Bien *et al.*, 1978) have the following form:

$$Y_j(\alpha, \delta, \overline{m}) = R_s R_p H_p(\overline{m}) K_s(\alpha, \delta), \tag{9}$$

where K_s are the spherical functions. These functions are used to represent the dependence of the systematic differences on right ascension and declination. Hermite polynomials $H_p(\overline{m})$ and transformed apparent magnitude $\overline{m}$ are given by (6) and (3).

K_s can be written as follows:

$$K_s(\alpha, \delta) = \begin{cases} L_{nk}(\delta) & \text{for } k = 0, \text{ l=1}, \\ L_{nk}(\delta)\sin(k\alpha) & \text{for } k \neq 0, \text{ l=0}, \\ L_{nk}(\delta)\cos(k\alpha) & \text{for } k \neq 0, \text{ l=0}, \end{cases} \tag{10}$$

where L_{nk} are the associated Legendre functions. For this form of spherical functions, there is a relation between the indices n, k, l and s:

$$s = n^2 + 2k + l - 1,$$

and therefore,

$$n = [\sqrt{s}],$$

$$k = \left[\frac{s - n^2 + 1}{2}\right],$$

$$l = s - n^2 - 2k + 1.$$

Square brackets, [], denote the integer part of the corresponding quantity. The associated Legendre functions are defined by the following recurrence formula:

$$L_n^{k+2}(x) - 2(k+1)\frac{x}{\sqrt{1-x^2}}L_n^{k+1}(x) + (n-k)(n+k+1)L_n^k(x) = 0, \tag{11}$$

where $n = 0, 1 \ldots, \infty$, $k = 0, 1 \ldots, n$, $L_1^1(x) = \sqrt{1-x^2}$, $L_2^1(x) = 3x\sqrt{1-x^2}$ and $L_2^2(x) = 3(1-x^2)$. A product of the functions K_s and H_p is normalized by the factors R_s and R_p given by:

$$R_p = \frac{1}{\sqrt{p!}},$$

$$R_s = \sqrt{2n+1}\begin{cases} \sqrt{\frac{2(n-k)!}{(n+k)!}} & \text{for k>0}, \\ 1 & \text{for k=0}. \end{cases} \tag{12}$$

4 THE COLLOCATION METHOD

Let us consider some important consequences of the application of the spherical functions to the determination of the systematic differences.

The approximation algorithm given above is a particular case of accidental field filtration based on the least squares collocation method. Mathematically, collocation is a least squares adjustment in Hilbert space. As written above, an analytical method of representation of systematic differences based on the "Legendre–Hermite–Fourier" functions was applied by the authors of FK5. In this case these functions are considered as elements of a Hilbert space without any supplementary conditions. However, the whole problem may be considered from another point of view if one allows the existence of the reproducing kernel in the Hilbert space (Gubanov *et al.*, 1993).

Let us consider some elements of the Hilbert space theory. Let H be a Hilbert space of functions $f(P)$, where point P belongs to some area B in the R^n space. Let us introduce a function $K(P,Q)$ that satisfied the following conditions:

$$K(P,Q) \in H, \quad \forall Q,$$
$$f(Q) = (f(P), K(P,Q))_p, \quad \forall f \in H, \tag{13}$$

where (,) denotes a scalar product. The function $K(P,Q)$ that satisfies these conditions is called a reproducing kernel. Any Hilbert space has no more than one reproducing kernel, which is defined uniquely by (13). Owing to the separability of the space H, the reproducing kernel can be expanded in terms of a complete orthonormalized system of functions $\varphi_i(P)$ (Moritz, 1980, p. 156):

$$K(P,Q) = \sum_{i=1}^{\infty} \varphi_i(P)\varphi_i(Q). \tag{14}$$

It is known (Moritz, 1980, p. 76) that estimations of parameters obtained with the least squares collocation method have a minimum of dispersion if the covariance function and the reproducing kernel are identical.

Accordingly to Moritz, (1980), the covariance function on a sphere is determined by

$$\begin{aligned} Q(\psi) &= \frac{1}{8\pi^2}\int_{\Sigma} f(P)f(Q)\,d\Sigma \\ &= \frac{1}{8\pi^2}\int_{\alpha}\int_{\delta}\int_{\varphi} f(\alpha',\delta')\cos\delta\, d\alpha d\delta d\varphi. \end{aligned} \tag{15}$$

The covariance function depends only on an angular distance ψ between the points $P(\alpha,\delta)$ and $Q(\alpha',\delta')$. Therefore, $Q(\psi)$ can be introduced in terms of the Legendre polynomials

$$Q(\psi) = \sigma^2 \sum_{n=0}^{\infty} \lambda_n L_n(\cos\psi), \tag{16}$$

where σ^2 is the systematic difference dispersion.

Applying the well-known addition theorem for the Legendre polynomials,

$$L_n(\cos\psi) = \sum_{k=0}^{n} h_k L_n^k(\sin\delta) L_n^k(\sin\delta') \cos k(\alpha - \alpha'), \tag{17}$$

where

$$h_k = \frac{2}{\beta_k}\frac{(n-k)!}{(n+k)!} \begin{cases} \beta_k = 2 & \text{for k=0,} \\ \beta_k = 1 & \text{for k>0,} \end{cases}$$

one can write (15) as follows:

$$\begin{aligned} Q(\psi) &= \sigma^2 \sum_{n=0}^{\infty} \lambda_n \sum_{k=0}^{n} h_k L_n^k(\sin\delta) L_n^k(\sin\delta') \cos k(\alpha - \alpha') \\ &= \sigma^2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \overline{\lambda}_{nk} L_n^k(\sin\delta) L_n^k(\sin\delta')(\cos k\alpha \cos k\alpha' + \sin k\alpha \sin k\alpha'). \end{aligned} \tag{18}$$

Applying the spherical functions,

$$\begin{aligned} \varphi_{nk}(P) &= \sigma^2 \overline{\lambda}_{nk} L_n^k(\sin\delta) \begin{cases} \cos k\alpha \\ \sin k\alpha \end{cases}, \\ \varphi_{nk}(Q) &= \sigma^2 \overline{\lambda}_{nk} L_n^k(\sin\delta') \begin{cases} \cos k\alpha' \\ \sin k\alpha' \end{cases}, \end{aligned}$$

where $\overline{\lambda}_{nk} = \lambda_n h_k$, one can obtain

$$Q(\psi) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \varphi_{nk}(P) \varphi_{nk}(Q), \tag{19}$$

which is nothing else but Eq. (14). Thus the spherical functions provide identity of the covariance function and the reproducing kernel and consequently are the best functions to ensure the optimal representation of the systematic differences (in general case).

Now, we are going to show that the "Legendre–Fourier" functions do not satisfy the "identity condition". For this purpose we expand an associated function into a series in terms of the "Legendre–Fourier" functions

$$L_n^k(\sin\delta) = \sum_{i=0}^{\infty} A_{ink} L_i(\sin\delta),$$

where expansion coefficients A_{ink} are given by

$$A_{ink} = \frac{(L_n^k(\sin\,\delta), L_i(\sin\,\delta))}{||L_i(\sin\,\delta)||^2}.$$

Therefore, Eq. (17) can be written in the form:

$$\begin{aligned} L_n(\cos\,\psi) &= \sum_{k=0}^{n} h_k A_{ink} L_i(\sin\,\delta) \sum_{j=0}^{\infty} A'_{jnk} L_i(\sin\,\delta') \cos\,k(\alpha-\alpha') \\ &= \sum_{k=0}^{n} h_k \left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{ink}A'_{ink} L_i(\sin\,\delta) L_j(\sin\,\delta') \right] \cos\,k(\alpha-\alpha') \end{aligned}$$

and Eq. (16) transforms to

$$\begin{aligned} Q(\psi) &= \sigma^2 \sum_{n=0}^{\infty}\sum_{k=0}^{n} \overline{\lambda}_{nk} \left[\sum_{i=0}^{\infty} A_{ink}A'_{ink} L_i(\sin\,\delta) L_i(\sin\,\delta') \right. \\ &+ \left. \sum_{i,j=0\ i\neq j}^{\infty} A_{ink}A'_{jnk} L_i(\sin\,\delta) L_j(\sin\,\delta') \right] \cos\,k(\alpha-\alpha'). \end{aligned} \tag{20}$$

Applying the "Legendre-Fourier" functions

$$\begin{aligned} \theta_{ink}(P) &= \sigma^2 \overline{\lambda}_{nk} A_{ink} L_i(\sin\,\delta) \begin{cases} \cos\,k\alpha \\ \sin\,k\alpha \end{cases}, \\ \theta_{ink}(Q) &= \sigma^2 \overline{\lambda}_{nk} A'_{ink} L_i(\sin\,\delta') \begin{cases} \cos\,k\alpha' \\ \sin\,k\alpha' \end{cases}, \end{aligned}$$

we obtain

$$Q(\psi) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{\infty} \theta_{ink}(P)\theta_{ink}(Q) + B(\psi), \tag{21}$$

where

$$B(\psi) = \sigma^2 \sum_{n=0}^{\infty}\sum_{k=0}^{n} \overline{\lambda}_{nk} \sum_{i,j=0\ i\neq j}^{\infty} A_{ink}A'_{jnk} L_i(\sin\,\delta) L_j(\sin\,\delta') \cos\,k(\alpha-\alpha').$$

Ordering summation of the components in the first term of Eq. (21) by single index i, one can write

$$Q(\psi) = \sum_{l=1}^{\infty} \theta_l(P)\theta_l(Q) + B(\psi), \tag{22}$$

which is not identical with Eq. (14). For the systematic differences FK5-FK4, we have found that $B(\psi) \not\equiv 0$ and $B(0)/Q(0) = 5\%$. This means that in general case there is no reproducing kernel which would be produced by the "Legendre–Fourier" functions.

5 SYSTEMATIC DIFFERENCES FK5-FK4 IN TERMS OF THE SPHERICAL FUNCTIONS

Now we can see that at least theoretically the "Legendre–Fourier" representation of the differences FK5-FK4 from Fricke *et al.*, (1988) is unsufficient in the problems where the optical representation is required. For this reason we expanded the differences in spherical functions combined with the Hermit polynomials. Since the stars in the catalogues FK5 and FK4 are almost uniformly distributed on the sky, we applied the algorithm which calculates the coefficients of the expansion directly using the least squares method without the Gram–Schmidt orthogonalization. In addition, we modified the γ-criterion as described by Zverev *et al.*, (1980) by introducing

$$\gamma = \frac{Q_\xi^\psi(1)}{Q_\xi^\psi(0)},$$

where $Q_\xi^\psi(\tau), \tau = 0, 1, 2, \ldots$ is the covariance function of residuals ξ. $Q_\xi^\psi(\tau)$ was using Eq. (15). The following formula was used for the determination of the highest order of the expansion q:

$$\gamma < -\frac{q+1}{N} + \frac{\beta_\mu}{\sqrt{N}},$$

where β_μ is the μ-percent point of the normal distribution. We chose $\mu = 0.01$. When using Fisher's statistical criterion for the determination of significant expansion terms, the significance level α was chosen as 0.01.

Elements of the weight matrix were calculated for each ith star as

$$p_i = \frac{\sigma_0^2}{\sigma_{\mathrm{FK4}}^2 + \sigma_{\mathrm{FK5}}^2} i,$$

where σ_{FK4} and σ_{FK5} are the coordinate and proper motion errors taken from FK4 and FK5 catalogues and σ_0 is the unit weight error.

6 NUMERICAL RESULTS

Expansion coefficients of systematic differences obtained in terms of spherical functions are presented in Tables 1–4.

Table 1. Expansion coefficients for the systematic difference $\Delta\alpha \cos \delta$

j	n	k	l	p	x	rms
1	0	0	1	0	−.86	.15
2	1	0	1	0	1.62	.17
3	2	0	1	0	−1.82	.15
4	2	1	1	0	−.48	.14
5	2	2	1	0	−.35	.13
6	3	0	1	0	1.60	.14
7	3	1	0	0	−.55	.14
8	5	0	1	0	−.25	.13
9	5	1	0	0	−.28	.13
10	6	2	1	0	−.31	.13
11	6	3	1	0	.34	.13
12	7	0	1	0	.21	.13
13	8	0	1	0	.84	.14
14	9	0	1	0	−1.22	.13
15	10	3	1	0	.31	.13
16	10	4	1	0	.29	.15
17	11	4	1	0	−0.29	.14
18	11	9	0	0	.28	.13
19	12	0	1	0	−.32	.12
20	14	0	1	0	.38	.12
21	15	1	1	0	−.34	.12
22	16	1	0	0	−.29	.12
23	18	0	1	0	−.46	.12
24	0	0	1	1	.33	.15
25	2	0	1	1	.12	.13
26	0	0	1	2	−.26	.11
27	2	0	1	2	.18	.10

Table 2. Expansion coefficients for the systematic difference $\Delta\delta$

j	n	k	l	p	x	rms
1	0	0	1	0	−.39	.18
2	1	0	1	0	−1.07	.20
3	1	1	0	0	.67	.16
4	1	1	1	0	−.82	.17
5	2	0	1	0	.41	.16
6	2	1	1	0	−.43	.19
7	2	2	1	0	−.36	.16
8	4	0	1	0	−.93	.16
9	5	3	0	0	−.38	.17
10	6	6	1	0	−.38	.15
11	7	0	1	0	.27	.15
12	7	6	1	0	.29	.16
13	8	1	1	0	−.43	.16
14	8	2	1	0	−.22	.17
15	9	3	0	0	−.33	.17

Table 2. (Continued)

j	*n*	*k*	*l*	*p*	*x*	*rms*
16	10	8	0	0	.30	.16
17	11	0	1	0	.95	.16
18	12	0	1	0	–.58	.15
19	14	0	1	0	.86	.14
20	14	3	0	0	.46	.16
21	14	11	1	0	.42	.16
22	15	1	1	0	.29	.15
23	15	2	0	0	–.28	.16
24	17	0	1	0	–.43	.14
25	18	1	1	0	–.39	.15
26	18	5	1	0	.41	.16
27	18	10	1	0	.31	.17

Table 3. Expansion coefficients for the systematic difference $\Delta\mu \cos\delta$

j	*n*	*k*	*l*	*p*	*x*	*rms*
1	0	0	1	0	–2.99	.38
2	1	0	1	0	5.44	.43
3	1	1	0	0	3.30	.28
4	2	0	1	0	–7.14	.36
5	2	2	1	0	–1.59	.31
6	3	0	1	0	9.47	.33
7	3	1	1	0	–1.48	.31
8	3	2	1	0	1.32	.36
9	3	3	1	0	–1.16	.27
10	4	0	1	0	–3.10	.37
11	4	1	1	0	–1.79	.30
12	4	2	1	0	–.99	.34
13	5	0	1	0	–1.59	.37
14	5	1	0	0	–1.09	.28
15	6	0	1	0	–3.62	.32
16	7	1	1	0	–1.08	.27
17	7	3	0	0	1.08	.29
18	7	3	1	0	1.00	.29
19	8	0	1	0	4.78	.32
20	9	0	1	0	–3.96	.34
21	10	0	1	0	.95	.32
22	10	3	1	0	.87	.28
23	12	0	1	0	–1.06	.26
24	14	0	1	0	1.34	.29
25	15	0	1	0	–1.48	.27
26	0	0	1	1	1.60	.37
27	1	0	1	1	–1.49	.40
28	2	0	1	1	1.36	.29
29	0	0	1	2	–1.52	.23
30	1	0	1	2	1.00	.24

Table 4. Expansion coefficients for the systematic difference $\Delta\mu'$

j	*n*	*k*	*l*	*p*	*x*	*rms*
1	1	1	1	0	−2.15	.38
2	2	0	1	0	−3.19	.33
3	2	1	1	0	−1.35	.41
4	2	2	1	0	−2.19	.34
5	4	3	0	0	.60	.41
6	4	4	0	0	.76	.33
7	5	0	1	0	−3.33	.35
8	5	1	1	0	−1.06	.35
9	5	3	0	0	−1.31	.44
10	6	0	1	0	1.31	.41
11	6	3	0	0	1.00	.40
12	6	6	1	0	−.83	.32
13	7	0	1	0	1.91	.40
14	8	0	1	0	−.76	.37
15	8	1	1	0	−1.00	.34
16	8	2	1	0	−1.09	.36
17	9	3	0	0	−.95	.40
18	10	0	1	0	.68	.39
19	10	3	0	0	.48	.39
20	11	0	1	0	3.12	.40
21	12	0	1	0	−1.82	.38
22	13	0	1	0	−.37	.38
23	14	0	1	0	2.36	.35
24	14	3	0	0	.99	.35
25	17	0	1	0	-1.33	.32

Table 5. Systematic difference $\Delta\alpha \cos\delta$ (units 0ˢ001)

	0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*
80	1.	0.	−1.	−2.	−2.	−2.	−1.	0.	0.	0.	−2.	−3.	−3.	−3.	−2.	0.	0.	0.	−2.	−3.	−3.	−3.	−1.	0.
70	−1.	−2.	−2.	−3.	−3.	−2.	−2.	−1.	−1.	0.	0.	0.	0.	1.	1.	1.	1.	0.	0.	−1.	−2.	−2.	−2.	−2.
60	0.	0.	0.	0.	0.	0.	0.	−1.	−1.	−1.	−1.	−1.	−1.	−1.	−2.	−2.	−2.	−2.	−2.	−2.	−2.	−2.	−1.	−1.
50	1.	1.	1.	1.	0.	0.	0.	1.	1.	1.	1.	1.	1.	1.	2.	3.	3.	2.	2.	1.	1.	1.	1.	1.
40	−1.	−1.	0.	−1.	−1.	0.	−1.	0.	−1.	−2.	−1.	−2.	−2.	−1.	−1.	0.	0.	0.	0.	−1.	−1.	−1.	−2.	−1.
30	0.	0.	1.	0.	0.	1.	0.	1.	−1.	−2.	0.	0.	1.	2.	0.	1.	2.	2.	4.	3.	2.	2.	0.	1.
20	−1.	−1.	0.	0.	0.	0.	−1.	0.	0.	0.	0.	−1.	0.	1.	0.	1.	1.	0.	1.	0.	0.	0.	−1.	0.
10	1.	2.	1.	1.	0.	−1.	0.	0.	1.	1.	0.	0.	0.	0.	1.	1.	1.	0.	−1.	0.	0.	0.	1.	1.
0	−1.	0.	−1.	0.	−1.	−1.	1.	0.	0.	0.	0.	2.	2.	1.	1.	−1.	−1.	−1.	−2.	−1.	−1.	−2.	−1.	−2.
−10	0.	1.	1.	2.	2.	2.	2.	0.	0.	0.	0.	2.	3.	2.	1.	−1.	−2.	−1.	0.	1.	1.	0.	0.	−1.
−20	−3.	−2.	0.	0.	1.	0.	−1.	1.	1.	1.	1.	0.	1.	2.	1.	2.	1.	−1.	0.	0.	1.	1.	−2.	−2.
−30	−2.	−2.	0.	−1.	−1.	0.	0.	1.	1.	0.	0.	−1.	−1.	0.	0.	1.	0.	0.	1.	0.	0.	−1.	−3.	−2.
−40	2.	2.	1.	0.	0.	2.	2.	3.	2.	1.	2.	2.	4.	4.	3.	3.	3.	3.	5.	4.	3.	2.	0.	2.
−50	0.	0.	1.	0.	0.	0.	1.	2.	2.	2.	2.	1.	2.	2.	2.	1.	0.	0.	0.	0.	−1.	−1.	−1.	−1.
−60	3.	3.	4.	3.	2.	0.	1.	3.	5.	5.	3.	1.	0.	2.	4.	5.	3.	1.	0.	1.	3.	3.	3.	2.
−70	−3.	−2.	−1.	−1.	−2.	−2.	−1.	1.	3.	2.	0.	−2.	−2.	−2.	−1.	−1.	−2.	−4.	−5.	−5.	−4.	−4.	−4.	−4.
−80	0.	1.	1.	1.	0.	0.	0.	1.	0.	−1.	−2.	−2.	−2.	−2.	−1.	−2.	−2.	−1.	0.	1.	1.	1.	0.	−1.

Table 6. Systematic difference $\Delta\delta$ (units $0\overset{''}{.}01$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
80	−1.	0.	1.	1.	1.	0.	0.	0.	1.	1.	2.	2.	3.	3.	2.	2.	3.	2.	1.	0.	−1.	−2.	−2.	−2.
70	−1.	0.	1.	1.	−1.	−2.	0.	1.	1.	0.	0.	2.	3.	2.	0.	0.	1.	2.	0.	−1.	−1.	1.	1.	0.
60	−2.	0.	−1.	−1.	−2.	−3.	−3.	−3.	0.	2.	3.	2.	0.	−1.	−1.	−1.	−1.	0.	2.	0.	−1.	−4.	−5.	−3.
50	−2.	0.	0.	1.	0.	0.	1.	0.	1.	0.	0.	2.	1.	2.	1.	1.	2.	1.	2.	0.	0.	0.	−1.	0.
40	−1.	−1.	−1.	−1.	−2.	0.	−1.	−1.	0.	−1.	−1.	−2.	−1.	−3.	0.	−1.	−1.	0.	−1.	0.	0.	1.	0.	1.
30	−1.	2.	2.	2.	−1.	−1.	−2.	−3.	−1.	−1.	0.	0.	0.	−3.	0.	−1.	0.	2.	1.	0.	0.	0.	−2.	1.
20	1.	1.	1.	0.	0.	0.	0.	−1.	1.	0.	0.	1.	0.	0.	1.	0.	0.	0.	1.	0.	2.	2.	1.	3.
10	0.	−3.	−1.	−2.	−1.	−1.	−2.	0.	−1.	0.	−1.	0.	0.	1.	−1.	1.	0.	0.	−1.	0.	−1.	−2.	−1.	−3.
0	2.	1.	−2.	0.	3.	1.	2.	1.	1.	−1.	−2.	−1.	0.	2.	0.	1.	3.	−1.	0.	1.	2.	2.	2.	2.
−10	1.	0.	−3.	0.	1.	1.	−1.	1.	1.	−1.	−1.	−3.	2.	−1.	0.	−1.	1.	−1.	−3.	0.	1.	1.	−1.	1.
−20	1.	1.	1.	1.	2.	0.	−2.	−1.	2.	1.	−1.	2.	1.	−1.	−2.	−1.	1.	1.	0.	−1.	1.	−2.	−3.	0.
−30	3.	−1.	1.	0.	3.	4.	3.	4.	5.	4.	1.	5.	3.	4.	2.	3.	2.	2.	0.	1.	4.	1.	1.	1.
−40	−2.	−5.	−2.	−2.	0.	0.	−1.	0.	0.	−1.	−3.	−1.	−2.	−1.	−2.	−1.	−1.	0.	−3.	−3.	−2.	−4.	−3.	−4.
−50	1.	1.	0.	1.	3.	2.	3.	1.	2.	0.	−2.	−1.	−3.	−1.	−2.	−1.	0.	−1.	1.	1.	3.	2.	1.	2.
−60	−2.	−2.	−2.	−1.	0.	0.	2.	1.	0.	−2.	−3.	−1.	−1.	1.	0.	0.	0.	−1.	−1.	−2.	−1.	1.	1.	0.
−70	6.	5.	3.	3.	6.	7.	5.	4.	4.	4.	4.	3.	3.	4.	6.	6.	4.	3.	5.	6.	6.	4.	4.	6.
−80	−11.	−14.	−15.	−13.	−10.	−6.	−4.	−3.	−4.	−5.	−6.	−6.	−4.	−2.	−2.	−3.	−5.	−7.	−8.	−8.	−7.	−6.	−7.	−8.

Table 7. Systematic difference $\Delta\mu\cos\delta$ (units $0\overset{s}{.}001$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
80	−5.	−5.	−7.	−8.	−8.	−8.	−6.	−3.	0.	1.	0.	−4.	−8.	−9.	−5.	1.	7.	7.	3.	−3.	−9.	−10.	−9.	−6.
70	−3.	−3.	−3.	−2.	−2.	−2.	−4.	−5.	−4.	−4.	−3.	−5.	−7.	−8.	−6.	−3.	1.	2.	1.	−2.	−4.	−4.	−4.	−3.
60	−2.	−2.	−2.	−1.	0.	0.	−2.	−4.	−4.	−4.	−3.	−2.	−2.	−2.	−2.	0.	1.	1.	0.	−1.	−1.	−1.	−1.	−1.
50	−1.	−2.	−3.	−4.	−4.	−4.	−4.	−3.	−2.	−2.	−2.	−2.	−2.	−1.	1.	3.	4.	3.	1.	−2.	−3.	−3.	−1.	0.
40	1.	1.	−1.	−4.	−6.	−7.	−7.	−4.	−2.	−1.	−1.	−1.	−2.	−1.	1.	4.	6.	6.	4.	0.	−3.	−4.	−3.	−1.
30	−1.	1.	1.	−2.	−5.	−7.	−8.	−6.	−3.	−1.	0.	−1.	−2.	−2.	0.	2.	4.	6.	5.	2.	−1.	−4.	−5.	−3.
20	−1.	1.	1.	−1.	−3.	−5.	−4.	−2.	0.	1.	1.	0.	0.	0.	2.	3.	5.	5.	4.	2.	−1.	−4.	−5.	−3.
10	−3.	−2.	−2.	−2.	−2.	−1.	1.	3.	3.	2.	1.	1.	2.	4.	5.	5.	4.	3.	1.	−1.	−3.	−5.	−5.	−4.
0	−7.	−6.	−4.	−2.	−1.	1.	2.	1.	0.	−1.	−1.	0.	3.	4.	4.	2.	−1.	−3.	−4.	−4.	−4.	−5.	−6.	−7.
−10	−4.	−2.	0.	1.	1.	1.	0.	−1.	−1.	−2.	−2.	0.	1.	2.	1.	−2.	−4.	−4.	−4.	−3.	−3.	−3.	−4.	−5.
−20	5.	6.	5.	2.	−1.	−3.	−2.	0.	1.	1.	0.	−2.	−3.	−3.	−2.	−1.	0.	0.	−1.	−3.	−4.	−3.	−1.	2.
−30	8.	8.	6.	1.	−3.	−5.	−3.	0.	2.	4.	3.	0.	−2.	−2.	−1.	1.	2.	2.	0.	−2.	−4.	−3.	0.	4.
−40	0.	0.	0.	0.	−1.	−2.	−2.	−2.	0.	2.	5.	7.	7.	6.	4.	1.	0.	−1.	−2.	−2.	−2.	−1.	−1.	−1.
−50	−7.	−7.	−6.	−2.	0.	1.	0.	−1.	−1.	2.	6.	11.	12.	11.	7.	3.	−1.	−3.	−4.	−2.	−1.	0.	−1.	−4.
−60	−3.	−5.	−4.	−2.	2.	4.	4.	4.	5.	6.	9.	11.	12.	11.	9.	7.	4.	2.	0.	0.	1.	2.	2.	0.
−70	−2.	−4.	−4.	−3.	0.	1.	2.	2.	3.	5.	6.	6.	5.	4.	4.	5.	5.	4.	3.	1.	0.	1.	1.	0.
−80	−8.	−10.	−9.	−7.	−5.	−5.	−5.	−5.	−4.	−2.	0.	1.	−1.	−3.	−3.	−1.	1.	3.	2.	1.	−1.	−2.	−3.	−6.

Tables 5–8 show the above differences for the whole celestial sphere. Systematic differences $\Delta\alpha\cos\delta$ obtained for the both function systems for declinations $+60°$, $0°$, $-60°$ are shown in Figures 1–3. The figures indicate that systematic differences are different, especially in the Southern hemisphere. Thus, the choice of the basis can strongly affect the systematic difference estimation.

Table 8. Systematic difference $\Delta\mu'$ (units $0.''01$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
80	−6.	−4.	−3.	−3.	−4.	−3.	0.	3.	5.	5.	3.	2.	2.	2.	4.	5.	8.	11.	14.	13.	9.	3.	−3.	−6.
70	1.	−1.	−3.	−4.	−4.	−3.	−1.	0.	0.	−1.	−1.	0.	2.	4.	5.	5.	5.	5.	7.	8.	9.	8.	5.	3.
60	−5.	−5.	−6.	−7.	−9.	−11.	−10.	−8.	−5.	−3.	−2.	−2.	−3.	−3.	−3.	−3.	−1.	1.	3.	3.	1.	−2.	−4.	−5.
50	1.	−2.	−2.	−1.	1.	3.	5.	5.	4.	3.	2.	2.	3.	4.	6.	7.	7.	7.	7.	8.	8.	7.	6.	3.
40	−1.	−4.	−4.	−2.	1.	3.	4.	3.	2.	0.	−2.	−2.	−1.	−1.	0.	1.	2.	2.	1.	2.	3.	4.	3.	1.
30	2.	2.	1.	2.	2.	0.	−1.	0.	2.	4.	4.	4.	2.	−2.	−4.	−2.	1.	1.	1.	1.	1.	0.	−1.	1.
20	5.	3.	1.	2.	4.	2.	0.	2.	5.	6.	6.	8.	8.	5.	2.	3.	4.	2.	−1.	0.	2.	2.	2.	4.
10	−1.	−3.	−5.	−2.	0.	−3.	−5.	−2.	1.	1.	1.	3.	3.	−1.	−4.	−1.	2.	0.	−2.	−2.	−1.	−2.	−4.	−2.
0	4.	−2.	−5.	−2.	4.	4.	3.	3.	2.	−2.	−4.	−2.	2.	2.	1.	4.	5.	2.	−1.	1.	5.	6.	5.	5.
−10	4.	−2.	−6.	−3.	2.	2.	0.	−1.	−2.	−6.	−8.	−5.	−1.	0.	−1.	2.	3.	−1.	−4.	−1.	4.	6.	5.	6.
−20	5.	4.	2.	3.	2.	−1.	−4.	−2.	2.	4.	5.	6.	6.	2.	0.	3.	6.	6.	5.	5.	5.	2.	1.	3.
−30	1.	−1.	−3.	−1.	1.	2.	1.	2.	4.	5.	6.	8.	9.	9.	8.	9.	9.	7.	4.	4.	4.	3.	2.	2.
−40	−6.	−8.	−8.	−7.	−4.	−3.	−3.	−2.	−1.	−1.	−1.	1.	2.	3.	3.	4.	4.	3.	1.	0.	−1.	−2.	−3.	−5.
−50	−1.	−3.	−4.	−5.	−5.	−5.	−5.	−5.	−5.	−4.	−3.	−1.	1.	2.	2.	2.	1.	0.	−1.	−1.	0.	0.	0.	0.
−60	8.	5.	1.	−2.	−3.	−4.	−3.	−4.	−4.	−4.	−2.	2.	6.	9.	9.	5.	0.	−4.	−6.	−4.	1.	6.	9.	10.
−70	17.	14.	10.	7.	5.	5.	6.	7.	7.	7.	9.	13.	19.	22.	21.	15.	7.	1.	−1.	1.	7.	13.	17.	18.
−80	−12.	−21.	−26.	−26.	−20.	−12.	−7.	−9.	−17.	−24.	−26.	−20.	−9.	0.	2.	−6.	−20.	−32.	−36.	−31.	−20.	−9.	−4.	−5.

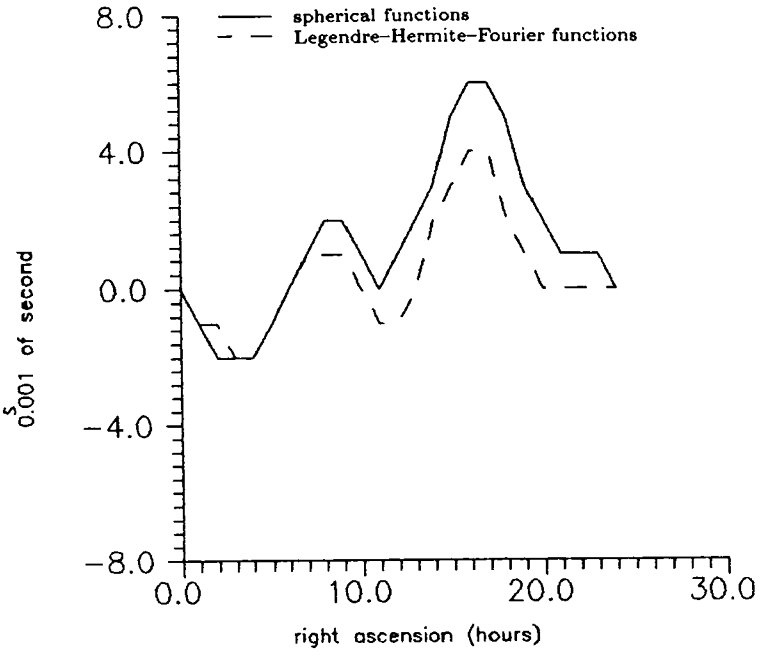

Figure 1 FK5-FK4 systematic difference $\Delta\alpha \cos\delta$ for $\delta = 60°$.

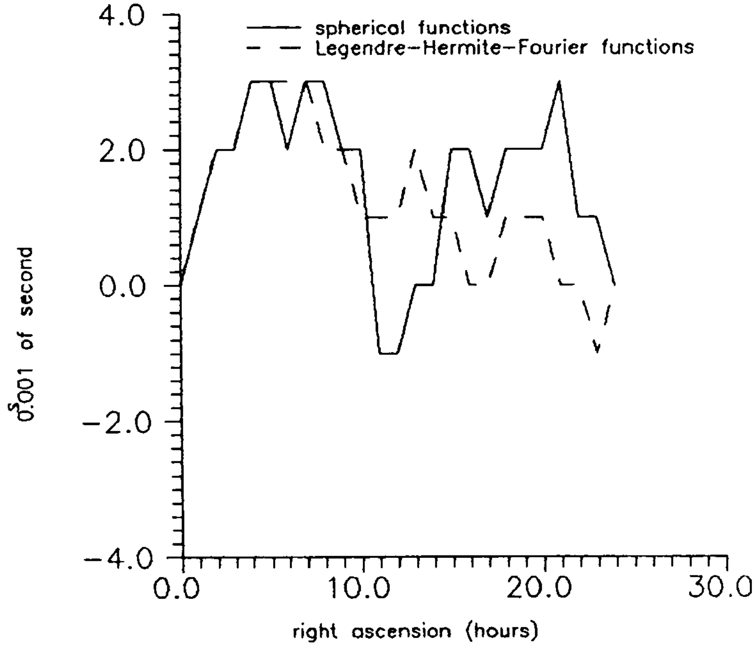

Figure 2 FK5-FK4 systematic difference $\Delta\alpha \cos\delta$ for $\delta = 0°$.

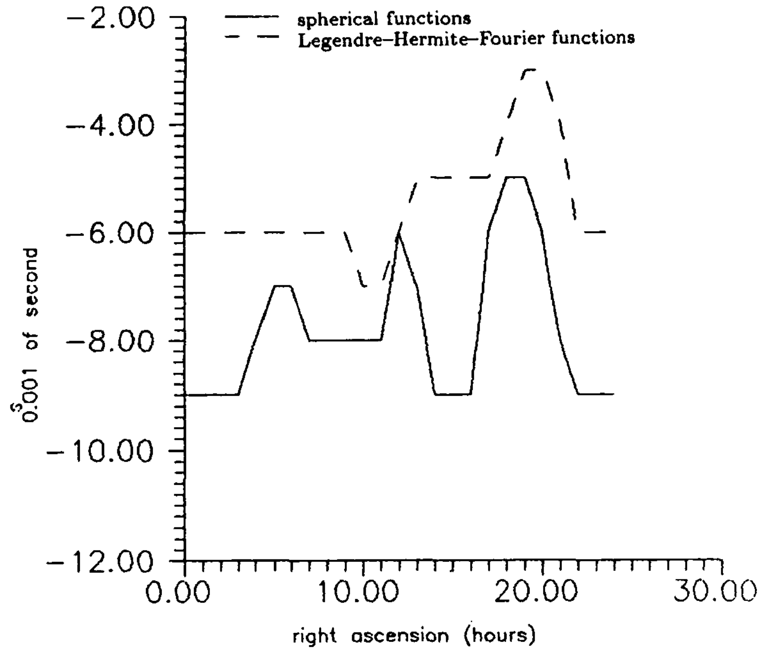

Figure 3 FK5-FK4 systematic difference $\Delta\alpha \cos\delta$ for $\delta = -60°$.

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