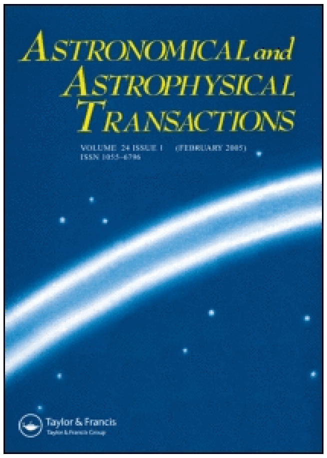

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FRIEDMANN COSMOLOGY IN ALTERNATIVE SPATIAL DIMENSIONS. SOLUTIONS AND TESTS

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Perhaps, spatial dimensionality decreases from 3 to 2 with growth of relative distance between bodies. This seems to be in accordance with observations of rotation curves of galaxies and pecular velocities in clusters of galaxies. This is also suggested by the discrepancy between the age of the Universe and that of globular clusters and galaxies. We obtain cosmological solutions for isotropic and homogeneous universes with spatial dimensionalities between 3 and 2, calculate the ages of the universes for the above dimensionalities and derive the three traditional cosmological tests for determining dimensionality and some other parameters of our Universe.

KEY WORDS Cosmology, spatial dimensionality; cosmological tests.

1 INTRODUCTION

Modern physics is still based on the paradigms of (1) 3-dimensional space and (2) constancy of this dimensionality. This is indeed so, although other values of dimensionalities appear in some recent physical theories like supergravity (West, 1986), strings and superstrings (Green *et al.*, 1988) and the Kaluza–Klein type models with compactified additional dimensions (Duff *et al.*, 1986). However, the appearance (Mandelbrot, 1982) and the rapid development (Fractals in Physics, 1985) of the theory of fractals as objects with non-integer dimensionality, as well as a tendency to describe the physical world as chaotic (Schuster, 1984) stimulate one to revise the above two paradigms.

The notion of dimensionality is always of current interest. The historical and methodological discussion of this notion from antique times up to now by Gorelik (1983) certainly concerns only integer dimensionalities. Connection between dimensionality and fundamental physical laws was also considered, and it was suggested that the number of dimensions in the early Universe could differ from 3 (Gorelik, 1983). Later on, an attempt to understand spacetime in classical and quantum physics was done based on some extension of the principle of relativity and on the geometrical concept of fractals (Nottale, 1989). A more mathematically rigorous approach seems to be that of stochastic metrization of spacetime (Koloskov, 1990), it is also suitable for introducing non-integer number of dimensions: All the geometrical characteristics of Riemannian spaces were reformulated in terms of stochastic metrization and it was shown that any gauge theory of physical interaction may be constructed on the basis of a stochastically metrizied space (Koloskov, 1990). Interesting ideas come from information theory (Harmuth 1989): the finiteness of any information precludes the verification of whether our spacetime represents continuum or not, but in a discrete space packaged in the 3-dimensional continuum space, an arbitrary number of dimensions is possible.

As for the notion of dimensionality, the latter has a relative meaning because if there is only one material point in the world, it cannot know the number of dimensions of a surrounding space – several material points are required for that. The above notion is also relative in the sense that we should match it together with the conception of the absence of the absolute space. Thus, the dimensionality may be only a function of relative distance between two material points: at the point of observation, an observer fixes that any physical phenomenon at the other (observing) point is the same as if there were a space of n dimensions between the two points. If we have determined that there are 3 dimensions on laboratory scales, although a real number may be only very close to 3, it does not follow that the number of dimensions is the same on larger or smaller relative scales.

The first part of the conception that we propose in this paper is that the number of dimensions of physical space varies from the laboratory value 3 to other values (n) when passing to larger relative scales. The change of dimensionality should be very small and possibly undetectable at scales of the Solar system. We should also clarify that since this change occurs when relative distance increases, for all the bodies, laws of microphysics (atomic structure, light emission, spectra, etc.) and those of intermadiate-scale physics (electromagnetic and gravitational interactions) remain the same. Thus they are the same for matter in our locality or for matter in, e.g., other galaxies. We only state that a large scale behavior of cosmic objects may differ from that prescribed by the concept of 3 dimensions.

The question arises of how it is understandable physically? The notion of simple fractals is hardly suitable for this purpose. A fractals is a self-similar object having the same (non-integer) dimensionality at all scales. Perhaps, we should invoke a more general geometrical object with some features of a fractals but with varying dimensionality. It may make sense to give a dynamic definition of dimensionality originating from laws of a spatial variation of forces and laws of light propagation. At any rate, clear image is unnecessary until there is a possibility of calculations. We should admit only that in a space with any (non-integer) number of dimensions, one can always draw a line (for $n \geq 1$) and can construct a 2-dimensional surface (for $n \geq 2$) and so on, that is necessary because, e.g., the physical image of a line is a light ray We should also think of which physical quantities depend on dimensionality and which ones do not. We assume that mass, energy and luminosity are independent quantities, whereas mass and energy densities and visual intensity are dependent quantities in the above sense, because "volume" and "surface area" have the physical dimensionalities (length)^n and (length)$^{n-1}$, respectively.

The most attractive scheme seems to be that the change of dimensionality occurs continuously with a relative distance between bodies. However, it would make a mathematical description too complete. It is simpler to suppose that dimensionality varies spasmodically on some characteristic relative distances $R_0, R_1, ..., R_j$. We give some preliminary classical (in the sense of Newtonian physics) considerations on this subject in Section 2.

As of now, one usually describes gravitational interaction by Einstein equations at scales from the Solar system to cosmological ones. The description at each scale presumes some smoothing at a smaller scales and the definition given of a material point, or, in other words, of a negligible size. However, it may turn out that a choice of a different number of dimensions at each scale would lead to better description of the Universe.

We recall that the ordinary Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\lambda}_{\lambda} = \frac{\kappa^{(n)}}{c^2} T_{\mu\nu} \tag{1.1}$$

has formally the same form in arbitrary (at least, integer) number of dimensions. This is also true concerning the Riemann and Ricci tensors, the Bianchi identities leading to the conservation law

$$T^{\nu}_{\mu;\,\nu} = 0, \tag{1.2}$$

and the hydrodynamic stress-energy tensor

$$T_{\mu\nu} = \varepsilon u_\mu u_\nu - p(g_{\mu\nu} - u_\mu u_\nu). \tag{1.3}$$

The only difference is that the constant $\kappa^{(n)}$, energy density ε and pressure p have other physical dimensionalities. Note that another version of Eq. (1.1),

$$R_{\mu\nu} = \frac{\kappa^{(n)}}{c^2} \left(T_{\mu\nu} - \frac{1}{n-1} g_{\mu\nu} T^{\lambda}_{\lambda} \right) \tag{1.4}$$

shows an explicit dependence on n.

Searching for solutions to (1.1) for arbitrary non-integer n meets with not only technical but rather conceptual difficulties. However, if our physical situation possesses a high degree of symmetry, e.g., like that of homogeneous and isotropic Universe, we may follow a simple way: We obtain cosmological equations for integer n and after that we allow n to acquire arbitrary values when finding solutions to these equations. This is done in Section 3 where we represent solutions for both dust-filled and radiation-filled universes and for all the signs of the Gauss curvature.

Up to this point, we discussed only variable dimensionality, but the next question arising is as how it changes, towards larger or smaller values? The second part of our conception is that the dimensionality changes from 3 at laboratory scales to 2 at a limit of cosmological scales; there are two cosmological indications for this.

The first indication is the discrepancy between the observed dynamics of galaxies themselves and of groups and clusters of galaxies, on the one hand, and the deficiency of luminous matter, on the other hand. This problem is usually formulated as that of hidden mass or dark matter. On needs a dark halo to explain the observed rotation curves of spiral galaxies (Burstein, 1985; Kent, 1986, 1987), high relative velocities in close pairs of galaxies (Turner, 1976) and so on, up to the explanation of high peculiar velocities, e.g., in the Virgo supercluster (Davies *et al.*, 1980). During a long time, people discuss different candidates to the role of dark matter, see, e.g., (Turner, 1989). Note that the standard model for large-scale structure formation where the main mass density is in the form of cold dark matter (Davies *et al.*, 1992a) has a difficulty in matching velocity field of galaxies and the observed structure on very large scales. Actually the most satisfactory model seems to be that of a mixture of cold and hot dark matter, in the framework of which one can overcome the above difficulties (Davies *et al.*, 1992b).

However, any conception of dark matter adopted in order to improve dynamics causes some dissatisfaction by its artificiality, that is why there were attempts to modify Newtonian laws in order to recalculate dynamics in itself. A term inversely proportional to distance (r^{-1}) added to the gravitational force was analyzed (Kuhn and Kruglyak, 1987). An interesting attempt was to modify the first Newton's law at small accelerations (Milgrom, 1983, 1987). However, such non-relativistic considerations can be hardly matched with general relativity. Some other relativistic theories were proposed for example, consequences of the Branch-Dicke theory were considered (Visser, 1988). We think that the required slower law of decay of the gravitational force can be explained if the space dimensionality is less than 3. As mentioned above, this is in agreement with (a somewhat modified) relativity. Moreover, a smoothly decreasing dimensionality at larger scales can explain the fact that dynamics of matter varies when passing to larger scales in such a way that one should introduce larger effective density of dark matter (Davis *et al.*, 1980). By the way, the problem of the origin of the structure of interacting galaxies is better resolved when the law of gravitational force lies somewhere between r^{-2} and r^{-1} (Wright *et al.*, 1992), or in our terms, the dimensionality lies between 3 and 2.

The second indication concerns the discrepancy between the age of galaxies and globular clusters and the age of the Universe coming from the Big Bang theory. This fact follows from dating the age of galaxies by color measurements (Bruzual, 1983), the age of Galaxy from its chemical composition (Lawler *et al.*, 1990) and the age of Galaxy's globular clusters (van Albada *et al.*, 1981). We show in Section 4 that the age of the Universe increases when n decreases. We show in Section 5 that for linearly expanding solution in the case $n = 2$ for the dust-filled Universe with positive curvature the age of the Universe is one and a half of that for $n = 3$ and that there is, for $n = 2$, a closed dust-filled stationary solution without cosmological constant, which formally implies an infinite age. The latter solution can be treated as asymptotical for large times.

Certainly, we realize that the acceptance of the presented conception wculd lead to a major revision of all ideas about our Universe. To be convinced that the case of the above discrepancies is just the dimensionality, one should bring in correlation all independent observations: photometrical measurements, counts of sources, velocity measurements and so on.

In this paper we present the derivation of the three traditional cosmological tests, namely visual magnitude, angular size and the number of sources versus redshift. We obtain them in Section 4 for arbitrary values of n. In Section 5, we consider a special case $n = 2$, obtain solutions to Einstein equations and rewrite explicity the above tests for this case. Concluding remarks are given in Section 6.

2 CLASSICAL PRELIMINARIES

Following theory of potential, one usually writes the Poisson equation for gravitational attraction of an isolated mass $\mathcal{M}$, embedded in 2- and 3-dimensional spaces, in the forms

$$\Delta^{(2)}\Phi(r) = 2\pi G^{(2)}\mathcal{M}\delta^{(2)}(r),$$

$$\Delta^{(3)}\Phi(r) = 4\pi G^{(2)}\mathcal{M}\delta^{(3)}(r),$$

where $\Delta^{(2)}$ and $\Delta^{(3)}$ are the Laplace operators in the 2- and 3-dimensional space, $\Phi(r)$ is the gravitational potential, $G^{(2)}$ and $G^{(3)}$ are the gravitational constants which we supply with the relevant indices, $\delta^{(2)}$ and $\delta^{(3)}$ are the 2- and 3-dimensional δ-functions. This suggests to generalize the similar Poisson equation in an n-dimensional space as

$$\Delta^{(n)}\Phi(r) = s^{(n)}G^{(n)}\mathcal{M}\delta^{(n)}(r), \tag{2.1}$$

where $\Delta^{(n)}$ is the n-dimensional Laplace operator. The quantity

$$s^{(n)} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

is the $(n-1)$-surface area of a unit n-sphere. We call an n-sphere the set of points of the n-dimensional space which are equidistant from some center. The sense of $G^{(n)}$ and $\delta^{(n)}$ is similar to above. Note that the surface and volume of the n-sphere are

$$S^{(n)} = s^{(n)}r^{n-1}, \quad V^{(n)} = \frac{s^{(n)}}{n}r^n.$$

For the δ-function, we always have

$$\int dV^{(n)}\,\delta^{(n)}(r) = 1.$$

Let us note that the solution to (2.1) depends only on the radial coordinate r. The pure radial part of the Laplace operator in spaces with integer n is

$$\Delta_r^{(n)} = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r},$$

however, we generalize this formula to arbitrary n. The radial-dependent solution to (2.1) decreasing at spatial infinity is $\Phi \propto r^{-(n-2)}$ for $n \neq 2$. Using the generalized Gauss theorem, we can relate the constant in (2.1) to a constant required for the above solution. Thus, for $n \neq 2$, the potential is

$$\Phi^{(n)}(r) = -\frac{1}{n-2}\frac{G^{(n)}\mathcal{M}}{r^{n-2}} + C^{(n)} \tag{2.2}$$

with $C^{(n)}$ an arbitrary constant, whereas the force acting on unit mass is

$$F^{(n)}(r) = -\frac{\partial}{\partial r}\Phi^{(n)} = -\frac{G^{(n)}\mathcal{M}}{r^{n-1}}. \tag{2.3}$$

In the case $n = 3$, (2.3) and (2.4) reduce to their ordinary forms:

$$\Phi^{(3)} = -\frac{G^{(3)}\mathcal{M}}{r}, \quad F^{(3)} = -\frac{G^{(3)}\mathcal{M}}{r^2},$$

where we have imposed $C^{(3)} = 0$. In a special case $n = 2$,

$$\Phi^{(2)} = G^{(2)}\mathcal{M}\ln r + C^{(2)}, \quad F^{(2)} = -\frac{G^{(2)}\mathcal{M}}{r}.$$

Thus, we can see from (2.2) and (2.3) that lesser values of dimensionality (from 3 to $2+\varepsilon$) provide a slower decrease of potentials and forces.

Unfortunately, we have no evident way to describe a smooth change of dimensionality with distance: first of all, we have no law of this change. Thus, we assume that the dimensionality changes abruptly from $n = 3$ to an arbitrary n. From physical arguments, if the leap of dimensionality occurs at a distance R_0 from a material point, at the first step, we should match together not potentials but forces at R_0. Then we obtain the following expression for the gravitational constant ($n \neq 2$):

$$G^{(n)} = G^{(3)}R_0^{n-3}.$$

At the second step, matching $\Phi^{(n)}$ for $n \neq 2$ and $\Phi^{(3)}$ at R_0 gives

$$C^{(n)} = \frac{3-n}{n-2}\frac{G^{(3)}\mathcal{M}}{R_0}.$$

We stress once more that we consider R_0 a relative distance between any two material points. Or, in other words, every material point possesses its own R_0-sphere. If we decided to consider a R_0-sphere which had a rigid location in space, then matching two forces for points located at different distances from the center of the sphere would lead to different "constants", the latter would diverge for points located at the radius R_0 itself.

If the changes of dimensionality occur by leaps several times from 3 to n_1 at R_0, from n_1 to n_2 at R_1 and so on, and from n_j to n_{j+1} at R_j, then we have a chain of relations between the gravitational constants

$$
\begin{aligned}
G^{(n_1)} &= G^{(3)} R_0^{n_1-3}, \\
G^{(n_2)} &= G^{(n_1)} R_1^{n_2-n_1} = G^{(3)} R_0^{n_1-3} R_1^{n_2-n_1}, \\
&\dots \\
G^{(n_{j+1})} &= G^{(n_j)} R_j^{n_{j+1}-n_j} = \dots = G^{(3)} R_0^{n_1-3} R_1^{n_2-n_1} \dots R_j^{n_{j+1}-n_j}.
\end{aligned}
$$

The chain of relations between the constants $C^{(n)}$ is rather cumbersome, but the recurrence relation is

$$C^{(n_j+1)} = C^{(n_j)} + \frac{n_j - n_{j+1}}{(n_{j+1}-2)(n_j-2)} \frac{G^{(n_j)}\mathcal{M}}{R_j^{n_j-2}}.$$

Evidently, the same considerations might be made for the electromagnetic interaction.

The gravitational constant $G^{(n)}$ can be connected with the constant $\kappa^{(n)}$ in the Einstein equations (1.1). Let us write as usual,

$$g_{00} = 1 + \frac{2}{c^2}\Phi^{(n)};$$

then, in the main order in c^{-2},

$$R_0^0 = \frac{1}{c^2}\Delta^{(n)}\Phi^{(n)}.$$

Taking the " $^0_0$ " equation from the set (1.4), and assuming

$$T_0^0 = \mathcal{M}c^2\delta^{(n)}(r), \quad T_i^k = 0$$

for the isolated mass, we obtain

$$\kappa^{(n)} = \frac{(n-1)}{(n-2)} \frac{s^{(n)}G^{(n)}}{c^2}. \tag{2.4}$$

Whence, one immediately sees that in order $\kappa^{(2)}$ be finite for $n = 2$ it is necessary that

$$G^{(n)} \propto (n-2)\,\mathrm{Const}, \tag{2.5}$$

that is $G^{(2)}$ vanishes.

Consider now light propagation. First of all, we consider the situation when an observer is located at a distance r from a light source, outside the R_0-sphere of the source ($r > R_0$). Let, at instance, the observer does not posses his own R_0-sphere, thus he is located in n dimensions (see Figure 1). The postulate that energy is not dissipated when passing from $n = 3$ to an arbitrary n guarantees that the observer can fix the visual intensity

$$E^{(n)} = \frac{L}{s^{(n)}r^{n-1}} \tag{2.6}$$

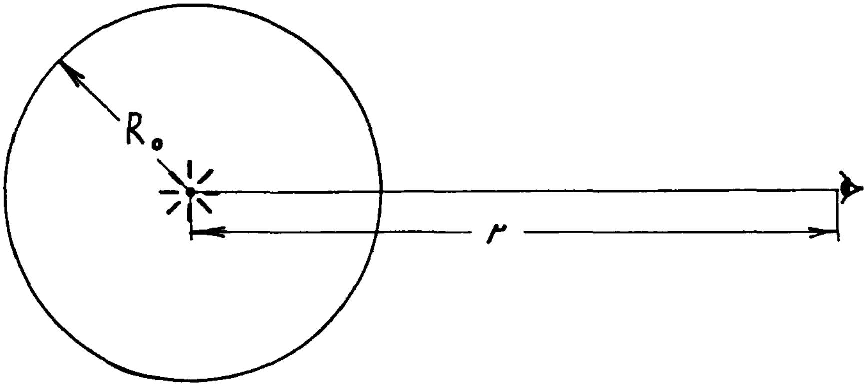

Figure 1 If an observer does not have his own R_0-sphere, he fixes visual light intensity in the n-dimensional space and cannot make the conclusion of whether or not the light source has its R_0-sphere.

in the n-dimensional space. Thus, there are no possibilities to determine the dimensionality near the light source, only the dimensionality near the observer can be determined.

Now, let the observer also possess an R_0-sphere. We construct the segment AB of the radius r going through the observer (the center of the R_0-sphere) up to its intersection with a cone tangent to the R_0-sphere, see Figure 2. Since energy passing through the cone is constant, we write

$$E^{(n)} S^{(n)}_{\text{segm}} = E^{(3)} S^{(3)}_{\text{segm}} = L_{\text{segm}}. \tag{2.7}$$

(We neglect the small parts at the edges of the segment AB lying outside the R_0-sphere). Here, $E^{(3)}$ is the real visual intensity in the 3-dimensional space, $E^{(n)}$ is the visual intensity which could take place in the n-dimensional space, $S^{(n)}_{\text{segm}}$ and $S^{(3)}_{\text{segm}}$ are the surfaces of the n-dimensional and 3-dimensional segments AB:

$$S^{(n)}_{\text{segm}} = \frac{2^{(n-1)/2} s^{(n-1)}}{(n-1)} (rH)^{(n-1)/2} F\left(\frac{n-1}{2}, \frac{3-n}{2}; \frac{n+1}{2}; \frac{H}{2r}\right), \tag{2.8}$$

$$S^{(3)}_{\text{segm}} = 2\pi r H, \tag{2.9}$$

where H is the height of the segment AB:

$$H = r - \sqrt{r^2 - R_0^2}, \tag{2.10}$$

and F is the Gauss hypergeometric function.

We suppose that the quantity R_0 exceeds 10 Kpc, then using (2.7) the standard definition for the distance modulus should be replaced by the following one:

$$m - M = 2.5\left(\lg E^{(3)}_{\text{std}} - \lg E^{(3)}\right) = 2.5\left[\lg\left(E^{(3)}_{\text{std}} S^{(3)}_{\text{segm}}\right) - \lg\left(E^{(n)} S^{(n)}_{\text{segm}}\right)\right], \tag{2.11}$$

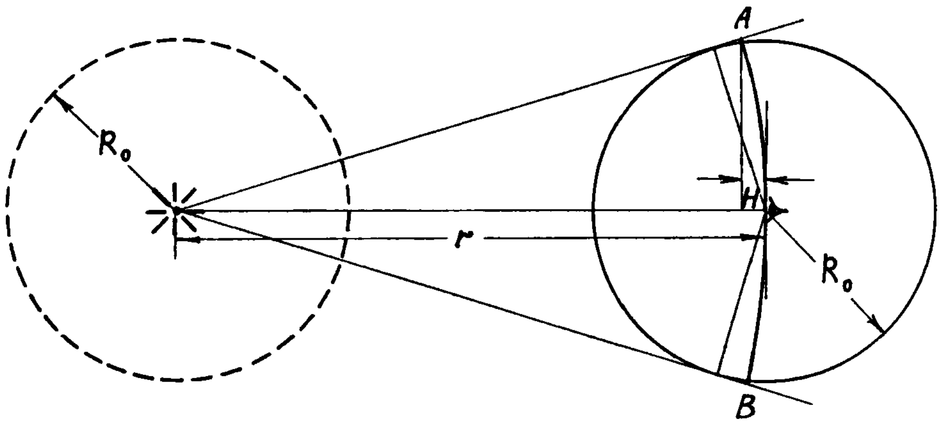

Figure 2 The presence of an R_0-sphere at the observer's location changes visual intensity at the point of observation. The hypotheses is that the luminosity passed through the segment AB in the n-dimensional space and the same segment in the 3-dimensional space does not dissipate.

where m is a visual magnitude, M is an absolute magnitude and

$$E^{(3)}_{\text{std}} = \frac{L}{4\pi 10^2}. \tag{2.12}$$

Combining expressions (2.8)–(2.11) and (2.12), we can write

$$\begin{aligned} m - M &= 2.5(n-1)\lg r - 2.5\lg\beta^{(n)} - 5 - \frac{5(n-3)}{4}\lg r\left(r - \sqrt{r^2 - R_0^2}\right) \\ &- 2.5\lg F\left(\frac{n-1}{2}, \frac{3-n}{2}; \frac{n+1}{2}; \frac{r - \sqrt{r^2 - R_0^2}}{2r}\right), \end{aligned} \tag{2.13}$$

where

$$\beta^{(n)} = \frac{2^{(n+1)/2}}{n-1}\frac{s^{(n-1)}}{s^{(n)}} = \frac{2^{(n+1)/2}\Gamma\left(\frac{n}{2}\right)}{(n-1)\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \tag{2.14}$$

is the number which is equal to unity for $n = 3$: $\beta^{(3)} = 1$.

Under the assumption that R_0 is substantially less than r, $R_0 \ll r$, (2.14) reduces to

$$\begin{aligned} m - M &= 2.5(n-1)\lg r - 2.5\lg\beta^{(n)} - 5 + 2.5(3-n)\lg\frac{R_0}{\sqrt{2}} \\ &- 5\frac{(n-1)(3-n)}{16(n+1)}\lg e\left(\frac{R_0}{r}\right)^2. \end{aligned} \tag{2.15}$$

Obviously, independently of the magnitude of R_0, (2.13) and (2.15) are converted into the standard formula in $n = 3$:

$$m - M = 5\lg r - 5.$$

3 EINSTEIN EQUATION AND SOLUTIONS

Consider the Rienannian space which is the topological product of the time axis and isotropic n-dimensional space. A derivation similar to that in Landau and Lifshitz (1988) shows that the metric interval can be represented as

$$ds^2 = c^2\,dt^2 - a^2(t)\big[dr^2 + \tilde{\varphi}^2(r)\,d\sigma^{(n)2}\big], \tag{3.1}$$

where $d\sigma^{(n)}$ is the angular distance element at the surface of an n-sphere in the Euclidean n-dimensional space. This quantity is not further required in its explicit form. The function $\tilde{\varphi}(r)$ with r the coordinate radius is independent of the number of dimensions,

$$\tilde{\varphi}(r) = \begin{cases} \sin r, & 0 \le r \le \pi, \quad k = +1; \\ r, & r \ge 0, \quad k = 0; \\ \operatorname{sh} r, & r \ge 0, \quad k = -1. \end{cases} \tag{3.2}$$

The parameter $k = +1, 0, -1$ determines, as usual, positive, zero and negative Gauss curvature, respectively.

The Christoffel symbols are formally the same as for the 3-dimensional space (dot denotes differentiation with respect to t):

$$\Gamma^0_{00} = \Gamma^i_{00} = \Gamma^0_{i0} = 0, \;\; \Gamma^k_{0i} = \frac{\dot{a}}{ca}\delta^k_i, \;\; \Gamma^0_{ik} = -\frac{\dot{a}}{ca}g_{ik}$$

(here and below, $i, j, k = 1, 2, ..., n$ and $\alpha, \beta, \gamma = 0, 1, 2, ...n$). The components Γ^i_{jk} correspond to the n-dimensional space of constant curvature. Components of the Ricci tensor are

$$R^0_0 = -n\frac{\ddot{a}}{c^2a}, \;\; R^i_0 = 0, \;\; R^k_i = -\frac{1}{c^2a^2}[a\ddot{a} + (n-1)(\dot{a}^2 + c^2k)]\delta^k_i.$$

Let our spacetime be filled with homogeneous and isotropic matter with energy density $\varepsilon(t)$, pressure $p(t)$ and $(n+1)$-velocity $u^\alpha = (1, 0, \ldots, 0)$. Components of the hydrodynamic stress-energy tensor (1.3) are

$$T^0_0 = \varepsilon, \;\; T^i_0 = 0, \;\; T^k_i = -p\delta^k_i.$$

Note that the trace of this tensor is

$$T = \varepsilon - np. \tag{3.3}$$

The set of the Einstein equations (1.1) now reduces to two equations:

$$\frac{1}{2}n(n-1)\left(\frac{\dot{a}^2}{c^2a^2} + \frac{k}{a^2}\right) = \frac{\kappa^{(n)}}{c^2}\varepsilon \equiv \kappa^{(n)}\rho, \tag{3.4}$$

$$(n-1)\left[\frac{\ddot{a}}{c^2a} + \frac{n-2}{2}\left(\frac{\dot{a}^2}{c^2a^2} + \frac{k}{a^2}\right)\right] = -\frac{\kappa^{(n)}}{c^2}p, \tag{3.5}$$

which are not independent. The constant $\kappa^{(n)}$ is given by (2.4). Conservation law for the stress-energy tensor (1.2) now is

$$\dot{\varepsilon} + n\frac{\dot{a}}{a}(\varepsilon + p) = 0. \tag{3.6}$$

We shall also need an equation with the curvature sign k eliminated by combining Eqs. (3.4) and (3.5):

$$n(n-1)\frac{\ddot{a}}{a} = -\kappa^{(n)}[(n-2)\varepsilon + np]. \tag{3.7}$$

In order to make the set of Eqs. (3.4)–(3.6) fully determined, equation of state $f(\varepsilon, p) = 0$ is required. We consider two interesting and traditional cases, namely, that of dust matter:

$$p = 0, \tag{3.8}$$

and that of radiation-filled Universe. The second case corresponds to the situation when the trace of the stress-energy tensor (3.3) vanishes:

$$p = \frac{\varepsilon}{n}. \tag{3.9}$$

Now, having these equations, we can forget that n is an integer number and assume that it can be an arbitrary number. The analysis below is done separately for all the values of k.

(i) $k = +1$. In this case, the Universe is close and it is easy to calculate its total volume,

$$V^{(n)} = s^{(n)} \int_0^\pi (a \sin r)^{n-1} a \, dr = v^{(n)} a^n, \tag{3.10}$$

where

$$v^{(n)} = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Thus, the volume of the n-dimensional space of the close Universe is equal to the surface area of an $(n+1)$-sphere embedded to the Euclidean $(n+1)$-dimensional space. For $n = 3$, (3.10) gives the well-known result, $V^{(3)} = 2\pi^2 a^3$.

The substitution of the equations of state (3.8) or (3.9) into Eq. (3.6) leads to evolutionary equations for the energy density

$$p = 0: \qquad \varepsilon = C_1 a^{-n}, \tag{3.11}$$

$$p = \frac{\varepsilon}{n}: \qquad \varepsilon = C_2 a^{-(n+1)}, \tag{3.12}$$

where C_1 and C_2 are some constants. We write Eq. (3.4) in a unified form for the above two cases

$$\frac{1}{2}n(n-1)(\dot{a}^2 + c^2) = \kappa^{(n)} C a^{-\alpha}, \tag{3.13}$$

where

$$p = 0: \qquad C = C_1, \quad \alpha = n - 2; \tag{3.14}$$

$$p = \frac{\varepsilon}{n}: \qquad C = C_2, \quad \alpha = n - 1. \tag{3.15}$$

It is clear from (3.13) that the scale factor a passes through a maximum value $a_{\max}$ when $\dot{a} = 0$, so that expansion of the Universe gives way to its contraction,

$$a_{\max} = \left[\frac{2}{n(n-1)} \frac{\kappa^{(n)} C}{c^2}\right]^{1/\alpha}.$$

Eq. (3.13) can now be rewritten in a more convenient form:

$$\dot{a} = c\left[\left(\frac{a_{\max}}{a}\right)^{\alpha} - 1\right]^{1/2}. \tag{3.16}$$

Solutions to (3.16) give a as implicit functions of time

$$p = 0: \qquad \frac{2}{n} a_{\max} \left(\frac{a}{a_{\max}}\right)^{n/2} F\left(\gamma_1, \frac{1}{2}; \gamma_1 + 1; \left(\frac{a}{a_{\max}}\right)^{n-2}\right) = ct, \tag{3.17}$$

$$p = \frac{\varepsilon}{n}: \qquad \frac{2}{n+1} a_{\max} \left(\frac{a}{a_{\max}}\right)^{(n+1)/2} F\left(\gamma_2, \frac{1}{2}; \gamma_2 + 1; \left(\frac{a}{a_{\max}}\right)^{n-1}\right) = ct, \tag{3.18}$$

where F is, as before, Gauss' hypergeometric function,

$$\gamma_1 = \frac{n}{2(n-2)}, \tag{3.19}$$

$$\gamma_2 = \frac{n+1}{2(n-1)}. \tag{3.20}$$

It is worth noting that in the case of dust matter, (3.8), the constant $a^{(n)}_{\max}$ can be related to the total mass $\mathcal{M}$ of the Universe. Indeed, Eq. (3.11) can be rewritten in terms of the total volume (3.10) and the mass $\mathcal{M}$:

$$\varepsilon v^{(n)} a^n = \mathcal{M} c^2 \left(= C_1 v^{(n)}\right).$$

Then

$$a^{(n)}_{\max} = \left[\frac{\Gamma\left(\frac{n+1}{2}\right)}{n(n-1)\pi^{(n+1)/2}} \kappa^{(n)} \mathcal{M}\right]^{1/(n-2)} = \left[\frac{2\Gamma\left(\frac{n+1}{2}\right)}{n(n-2)\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)} \frac{G^{(n)} \mathcal{M}}{c^2}\right]^{1/(n-2)}$$

Table 1. The values of $t_{\max}^{(n)}$, i.e., the time intervals required to reach the maximum values of the scale factor ($a_{\max}^{(n)}$), in units $a_{\max}^{(n)}/c = 1$ and for various values of n between 3 and 2. These values are in accordance with solutions presented at Figures 3 and 4 which correspond to the closed Universe

n	*Dust* ($p = 0$)	*Radiation* ($p = \varepsilon/n$)
3.0	1.571	1
2.8	1.797	1.075
2.6	2.125	1.164
2.4	2.667	1.271
2.2	3.866	1.403
2.0	∞	1.571

With the use of (3.17) and (3.18), it is easy to calculate time interval $t_{\max}^{(n)}$ after which $a_{\max}^{(n)}$ is reached:

$$p = 0: \qquad t_{\max}^{(n)} = \frac{2\sqrt{\pi}}{n} \frac{\Gamma(\gamma_1 + 1)}{\Gamma\left(\gamma_1 + \frac{1}{2}\right)} \frac{a_{\max}^{(n)}}{c},$$

$$p = \frac{\varepsilon}{n}: \qquad t_{\max}^{(n)} = \frac{2\sqrt{\pi}}{n+1} \frac{\Gamma(\gamma_2 + 1)}{\Gamma\left(\gamma_2 + \frac{1}{2}\right)} \frac{a_{\max}^{(n)}}{c},$$

see numerical values of $t_{\max}^{(n)}$ in Table 1.

(ii) $k = 0$. In the case of flat n-dimensional space, we obtain explicit solutions:

$$p = 0: \qquad a = a_0 \left(\frac{t}{t_0}\right)^{2/n}, \tag{3.21}$$

$$p = \frac{\varepsilon}{n}: \qquad a = a_0 \left(\frac{t}{t_0}\right)^{2/(n+1)}, \tag{3.22}$$

henceforth, index "0" is used for the modern values of corresponding quantities.

(iii) $k = -1$. After substituting (3.11) and (3.12), Eq. (3.4) acquires the form

$$\frac{1}{2} n(n-1)(\dot{a}^2 - c^2) = \kappa^{(n)} C a^{-\alpha} \tag{3.23}$$

in terms of the notations (3.14) and (3.15). In this case, there exists no $a_{\max}$, and expansion of the Universe continues forever. However, Eq. (3.23) can be written in a more compact form similar to (3.16):

$$\dot{a} = c \left[\left(\frac{A}{a}\right)^{\alpha} + 1\right]^{1/2},$$

where the constant A is connected in some way with the constant C. The solutions are

$$p = 0: \qquad \frac{2}{n} A \left(\frac{a}{A}\right)^{n/2} F\left(\gamma_1, \frac{1}{2}; \gamma_1 + 1; -\left(\frac{a}{A}\right)^{n-2}\right) = ct, \tag{3.24}$$

$$p = \frac{\varepsilon}{n}: \qquad \frac{2}{n+1} A \left(\frac{a}{A}\right)^{(n+1)/2} F\left(\gamma_2, \frac{1}{2}; \gamma_2 + 1; -\left(\frac{a}{A}\right)^{n-1}\right) = ct, \quad (3.25)$$

with the same expressions (3.19) and (3.20) for γ_1 and γ_2.

Note that for arbitrary n the hypergeometric function cannot be expressed via elementary functions, although, due to its special form, it can be reduced to the generalized Legendre functions with non-integer indices (Handbook of Mathematical Functions, 1964). It is also interesting to note the formal mathematical fact that, for every k, the solution with $p = \varepsilon/n$ in the n-dimensional space corresponds to the solution with $p = 0$ in the $(n+1)$-dimensional space.

The physical property of the solutions (3.17), (3.18) and (3.24), (3.25) is that for small t

$$p = 0: \qquad a \propto t^{2/n},$$
$$p = \frac{\varepsilon}{n}: \qquad a \propto t^{2/(n+1)}.$$

These asymptotic forms correspond to solutions (3.21) and (3.22) in the case of flat space. Thus, the fact that curvature does not affect the origin of expansion applies for all n.

Clearly, solutions (3.17) and (3.24) are not fit directly for $n = 2$ (see the case $n = 2$ in Section 5 and captions to Figures 3 and 7).

In the case $n = 3$, we can derive from the solutions (3.17), (3.18), (3.21), (3.22), (3.24) and (3.25) the well-known ones:

(i) $k = +1$.

$$\begin{aligned} p = 0: &\qquad a_{\max}\left[\arcsin\left(\frac{a}{a_{\max}}\right)^{1/2} - \left(\frac{a}{a_{\max}}\right)^{1/2}\left(1 - \frac{a}{a_{\max}}\right)^{1/2}\right] = ct, \\ p = \frac{\varepsilon}{3}: &\qquad a = c\left(2\frac{a_{\max}}{c}t - t^2\right)^{1/2}, \end{aligned} \quad (3.26)$$

with

$$a_{\max}^{(3)} = \frac{4GM}{3\pi c^2}$$

in the case $p = 0$ (cf. Landau and Lifshitz, 1988). The times of reaching $a_{\max}^{(3)}$ are:

$$\begin{aligned} p = 0: &\qquad t_{\max}^{(3)} = \frac{\pi}{2}\frac{a_{\max}^{(3)}}{c}, \\ p = \frac{\varepsilon}{3}: &\qquad t_{\max}^{(3)} = \frac{a_{\max}^{(3)}}{c}. \end{aligned} \quad (3.27)$$

(ii) $k = 0$.

$$\begin{aligned} p = 0: &\qquad a = a_0\left(\frac{t}{t_0}\right)^{2/3}, \\ p = \frac{\varepsilon}{3}: &\qquad a = a_0\left(\frac{t}{t_0}\right)^{1/2}. \end{aligned} \quad (3.28)$$

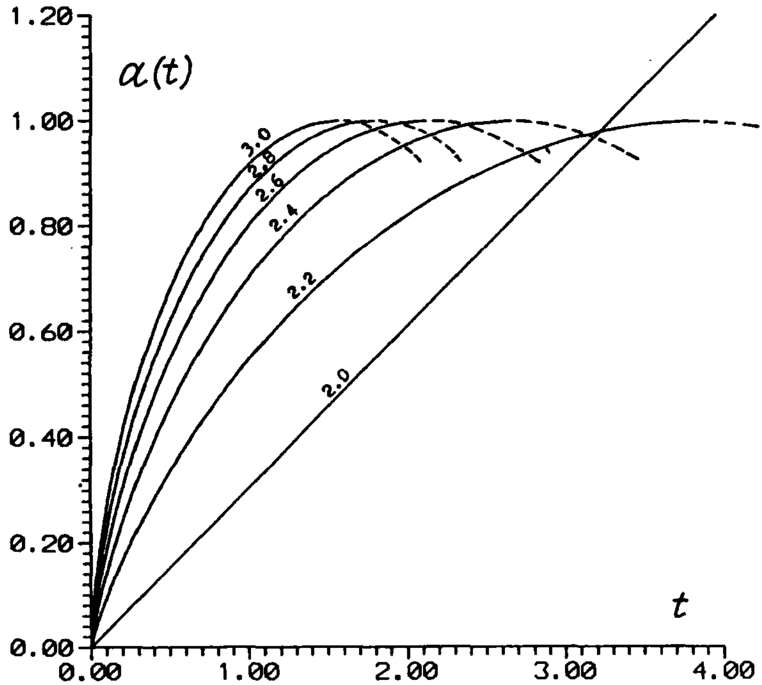

Figure 3 The family of solutions (3.17) for the dust-filled Universe ($p = 0$) with positive curvature ($k = +1$) is presented for $n = 3.0$, 2.8, 2.6, 2.4, 2.2 and 2.0; units are such that $c = 1$ and $a^{(n)}_{\max} = 1$. Solutions (3.17) determine in fact the only branches which increase up to $a = a^{(n)}_{\max} = 1$. Each full solution should be obtained by adding a decreasing branch which is symmetric to the above branch under the reflection with respect to the vertical line $t = t^{(n)}_{\max}$ where $t^{(n)}_{\max}$ is the time for reaching $a^{(n)}_{\max}$. Decreasing branches are shown dashed. The straight line for $n = 2$ is drawn with arbitrary inclination. The real inclination is connected with fundamental constants and the mass of the Universe (see Section 5). There is no finite limit of the solution (3.17) for n tending to 2 because $t^{(n)}_{\max}$ tends to infinity, thus such a straight line would have zero inclination.

(iii) $k = -1$.

$$\begin{aligned} p = 0: &\quad A\left[\left(\frac{a}{A}\right)^{1/2}\left(1+\frac{a}{A}\right)^{1/2} - \operatorname{arsinh}\left(\frac{a}{A}\right)^{1/2}\right] = ct, \\ p = \frac{\varepsilon}{3}: &\quad a = c\left[\left(\frac{a_0^2}{c^2 t_0} - t_0\right)t + t^2\right]^{1/2}. \end{aligned} \tag{3.29}$$

All the above solutions are presented in Figures 3–8 for the two equations of state (3.8) and (3.9) and for all the values of k when n acquires various values from 3 to 2.

For the derivation of the cosmological tests, it is convenient to introduce the parameter $\Omega^{(n)}(t)$:

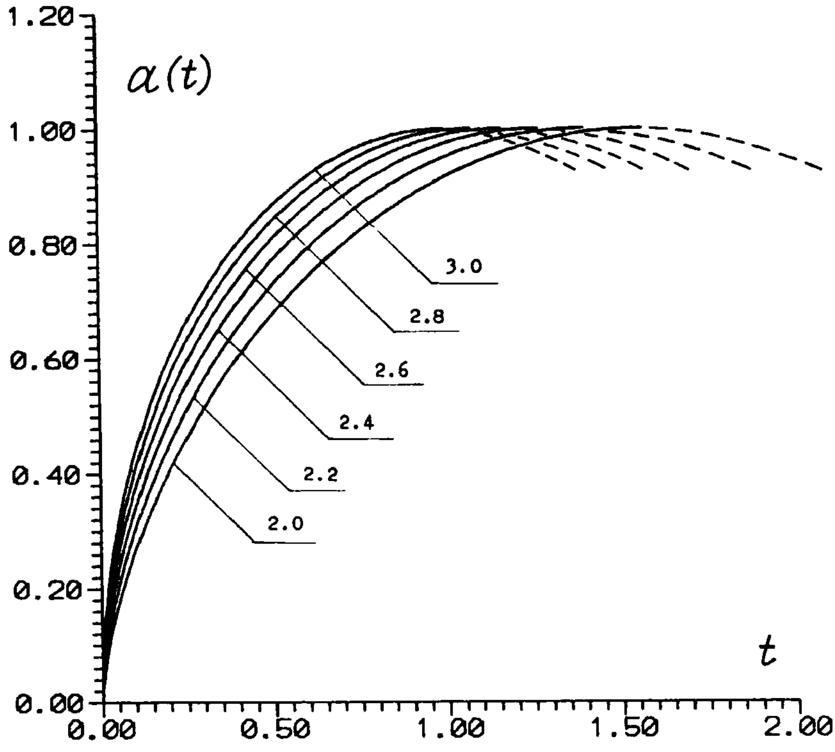

Figure 4 The family of solutions (3.18) for the radiation-filled Universe ($p = \varepsilon/n$) with positive curvature ($k = +1$) is presented for $n = 3.0$, 2.8, 2.6, 2.4, 2.2 and 2.0; units are such that $c = 1$ and $a_{\max}^{(n)} = 1$. As in the case $p = 0$ (see Figure 3) each full solution can be obtained by adding a decreasing branch shown dashed. Solution for $n = 2$ coincides here with the dust solution for $k = +1$ and $n = 3$.

$$\Omega^{(n)}(t) = \frac{\rho^{(n)}(t)}{\rho_c^{(n)}(t)} \tag{3.30}$$

where the (time-dependent) critical density is determined from Eq. (3.4) when $k = 0$

$$\rho_c^{(n)}(t) = \frac{n(n-1)}{2} \frac{H^2}{c^2\kappa^{(n)}} = \frac{n(n-2)}{2} \frac{H^2}{s^{(n)}G^{(n)}}, \tag{3.31}$$

$H(t) = \dot{a}/a$ is the Hubble parameter. As usual, $0 < \Omega^{(n)} < 1$ for $k = -1$, $\Omega^{(n)} = 1$ for $k = 0$, and $\Omega^{(n)} > 1$ for $k = +1$. As a rule, index "(n)" at Ω will be omitted below. The deceleration parameter is defined as always

$$q^{(n)} = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{aH^2}. \tag{3.32}$$

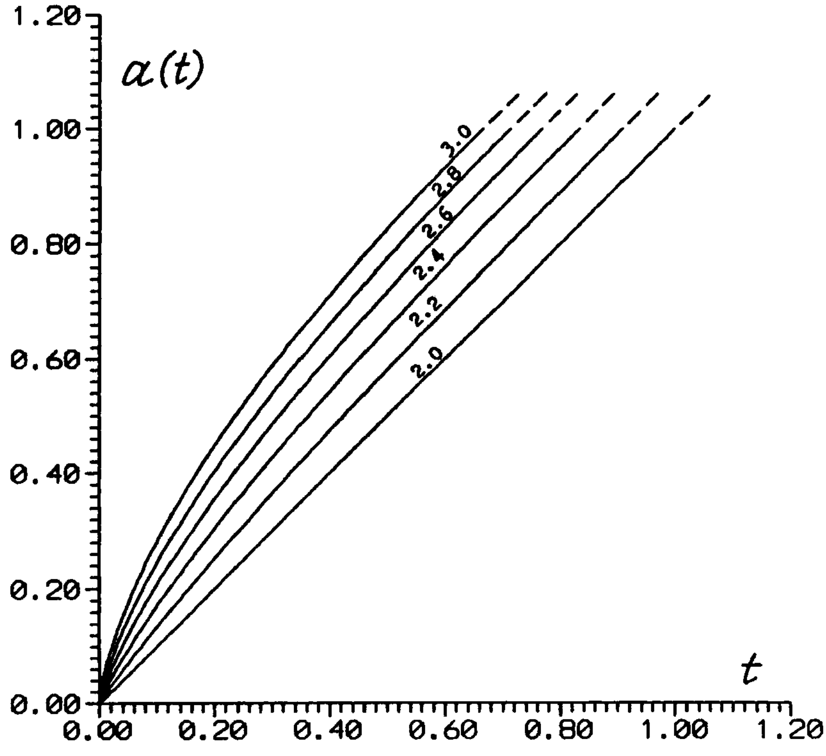

Figure 5 The family of solutions (3.21) for a dust-filled Universe ($p = 0$) with zero curvature ($k = 0$) is presented for $n = 3.0, 2.8, 2.6, 2.4, 2.2$ and 2.0; we put $t_0 = 1$ and $a_0 = 1$. The solutions are calculated up to $a = 1$ and continued by dashes. A solution for $n = 2$ is a straight line with the inclinations equal to unity.

Using (3.30), (3.31) and (3.32), we obtain from (3.7):

$$p = 0: \qquad q^{(n)}(t) = \frac{n-2}{2}\Omega(t), \tag{3.33}$$

$$p = \frac{\varepsilon}{n}: \qquad q^{(n)}(t) = \frac{n-1}{2}\Omega(t).$$

4 THE AGE OF THE UNIVERSE AND COSMOLOGICAL TESTS

In order to obtain the age and cosmological tests, we should make some preliminary work and express all necessary quantities via the redshift z. We now deal only with the case $p = 0$.

Clearly, the relation determining z (Zeldovich and Novikov, 1975; Weinberg, 1972),

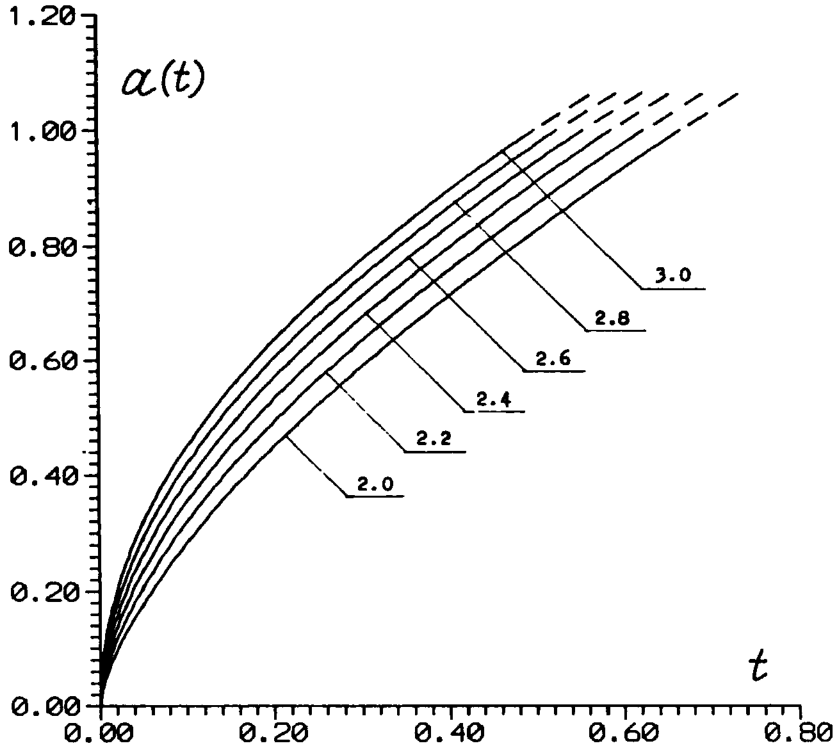

Figure 6 The family of solutions (3.22) for a radiation-filled Universe ($p = \varepsilon/n$) with zero curvature ($k = 0$) is presented for $n = 3.0$, 2.8, 2.6, 2.4, 2.2 and 2.0; we put $t_0 = 1$ and $a_0 = 1$. Solutions are calculated up to $a = 1$ and continued by dashes. A solution for $n = 2$ coincides here with a dust solution for $k = 0$ and $n = 3$.

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e}$$

is independent of spatial dimensionality. Here λ denotes the wavelength of emitted radiation, index "0" corresponds to the moment of observation and index "e" corresponds to that of emission ($\lambda_e \equiv \lambda$ is the proper wavelength). Since $\lambda(t) \propto a(t)$ independently of n, we can write as usually

$$a(t_e) \equiv a_e = a_0(1+z)^{-1}. \tag{4.1}$$

The quantity a_0 can be expressed (Zeldovich and Novikov, 1975) via the modern values of the Hubble parameter, H_0, and the density parameter, Ω_0. Taking Eq. (3.4) at the moment t_0 and expressing its right-hand via Ω_0 (see (3.30) and (3.31)), we obtain

$$a_0 = cH_0^{-1}[k(\Omega_0 - 1)]^{-1/2} = cH_0^{-1}(|\Omega_0 - 1|)^{-1/2}. \tag{4.2}$$

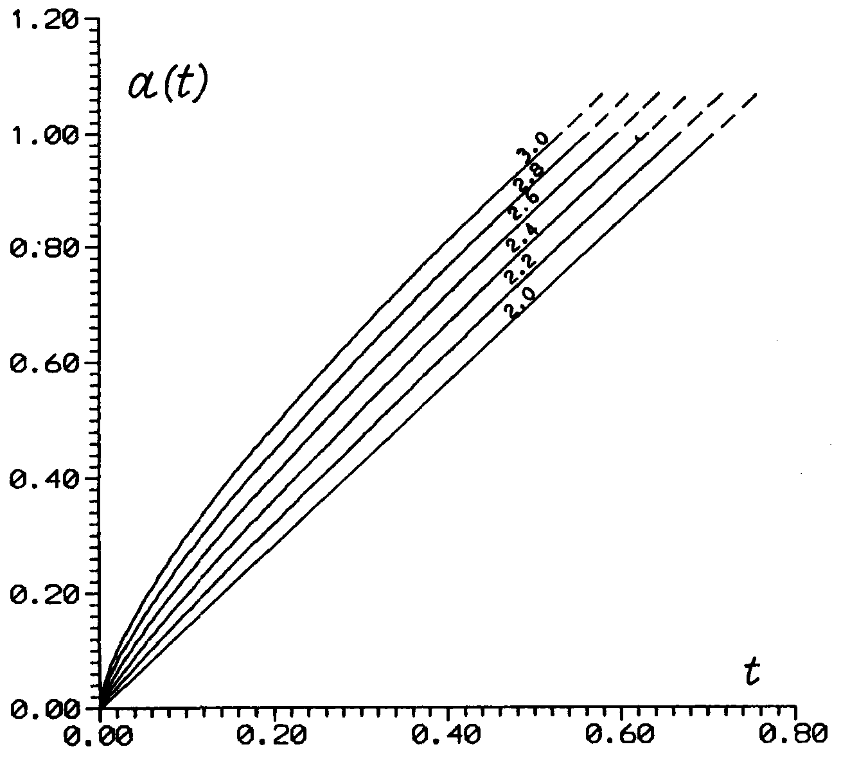

Figure 7 The family of solutions (3.24) for a dust-filled Universe ($p = 0$) with negative curvature ($k = -1$) is presented for $n = 3.0, 2.8, 2.6, 2.4, 2.2$ and 2.0; we put $c = 1$ and $A = 1$. Solutions are calculated up to $a = 1$ and continued by dashes. There exists a finite limit of the solution (3.24) for n tending to 2. This is a straight line with the inclination $1/\sqrt{2}$ and this solution is shown on the graph. The real inclination depends on arbitrary constants, see Section 5.

In the case $\Omega_0 = 1$, a_0 remains an arbitrary parameter and it does not enter final formulae.

Using (4.1), Eq. (3.11) for energy density can be rewritten in terms of matter density:

$$\rho = \rho_{co}\Omega_0(1+z)^n, \tag{4.3}$$

where ρ_{co} is the modern value of the critical density, it is expressed via H_0 similarly to (3.31).

Now we obtain equation for the dimensionless Hubble parameter $h = H/H_0$. To do this, Eqs. (3.7) and (3.6) should be represented in the form

$$\frac{dH}{dt} = -\left(\frac{n-2}{2}H_0^2\frac{\rho}{\rho_{co}} + H^2\right), \tag{4.4}$$

$$\frac{d\rho}{dt} = -nH\rho. \tag{4.5}$$

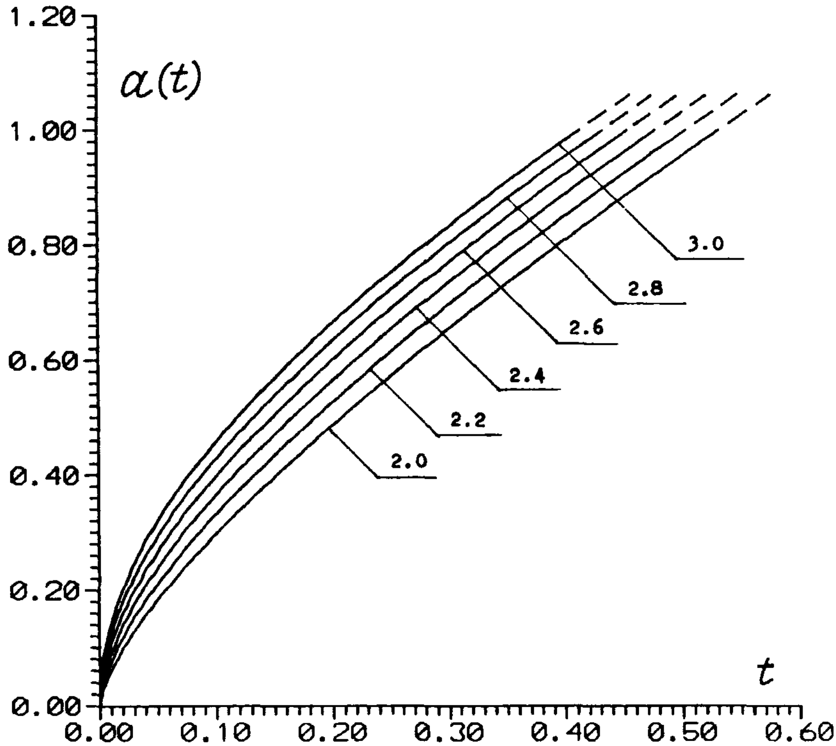

Figure 8 The family of solutions (3.25) for a radiation-filled Universe ($p = \varepsilon/n$) with negative curvature ($k = -1$) is presented for n = 3.0, 2.8, 2.6, 2.4, 2.2 and 2.0; we put $c = 1$ and $A = 1$. Solutions are calculated up to $a = 1$ and continued by dashes. A solution for $n = 2$ coincides here with a dust solution for $k = -1$ and $n = 3$.

We need $d\rho$ in the form following from (4.3):

$$d\rho = n\rho_{\mathrm{co}}\Omega_0(1+z)^{n-1}\,dz. \tag{4.6}$$

Eliminating dt from the set of equations (4.4) and (4.5) and substituting (4.6) there, we obtain

$$\frac{dh}{dz} = \frac{n-2}{2}\frac{\Omega_0}{h}(1+z)^{n-1} + \frac{h}{1+z}. \tag{4.7}$$

The solution to (4.7) with the initial condition $h = 1$ at $z = 0$ is

$$h = (1+z)\left[1 - \Omega_0 + \Omega_0(1+z)^{n-2}\right]^{1/2}. \tag{4.8}$$

After substituting (4.8) and (4.2) into (4.5), the differential dt can be expressed via dz:

$$dt = -\frac{dz}{H_0(1+z)h(z)} = -\frac{dz}{H_0(1+z)^2[1-\Omega_0+\Omega_0(1+z)^{n-2}]^{1/2}}. \tag{4.9}$$

Table 2. The values of the dimensionless age $t^{(n)}$ for various values of n and $\Omega_0^{(n)}$, see (4.10)

n	$\Omega_0^{(n)}$					
	0	0.1	0.3	1	3	10
3.0	1	0.898	0.809	0.667	0.513	0.351
2.8	1	0.925	0.848	0.714	0.560	0.390
2.6	1	0.950	0.888	0.769	0.619	0.442
2.4	1	0.971	0.929	0.833	0.696	0.515
2.2	1	0.988	0.967	0.909	0.806	0.638
2.0	1	1	1	1	1	1

Equation (4.9) is very convenient for the calculation of the age of the Universe. It can be obtained by performing integration of (4.9) over z from $z = \infty$ up to $z = 0$:

$$T^{(n)} \equiv H_0^{-1} t^{(n)} = H_0^{-1} \int\limits_0^\infty \frac{dz}{(1+z)^2[1-\Omega_0+\Omega_0(1+z)^{n-2}]^{1/2}}. \tag{4.10}$$

This integral can be expressed via elementary functions for the cases $n = 3$ and $n = 2$ with arbitrary Ω_0 or for the case $\Omega_0 = 0$ and $\Omega_0 = 1$ with arbitrary n. For $n = 3$ the resulting expression is in agreement with that given by Zeldovich and Novikov (1975). For the case $n = 2$, see Section 5. We give a table of the values of $t^{(n)}$ for various n and Ω_0 (see Table 2); $t^{(n)} = 1$ for $\Omega_0 = 0$ and $t^{(n)} = 2/n$ for $\Omega_0 = 1$.

The metric radius at the moment of light emission, r_e, should be also expressed via z. For the light ray propagating from a light source to us, using (4.9),

$$dr_e = -\frac{c\,dt}{a_e} = \begin{cases} (|1-\Omega_0|)^{1/2} \dfrac{dz}{(1+z)[1-\Omega_0+\Omega_0(1+z)^{n-2}]^{1/2}}, & \Omega_0 \neq 1; \\ \dfrac{c}{a_0 H_0} \dfrac{dz}{(1+z)^{n/2}}, & \Omega_0 = 1. \end{cases} \tag{4.11}$$

The metric radius is given by the integral of (4.11) over z from $z = 0$ up to a current z:

$$r_e(z) = \begin{cases} \dfrac{1}{n-2} \left\{ \ln \left| \dfrac{\xi(z,\Omega_0) - (1-\Omega_0)^{1/2}}{\xi(z,\Omega_0) + (1-\Omega_0)^{1/2}} \right| - \ln \left| \dfrac{1-(1-\Omega_0)^{1/2}}{1+(1-\Omega_0)^{1/2}} \right| \right\}, & 0 < \Omega_0 < 1; \\ \dfrac{2}{n-2} \dfrac{c}{a_0 H_0} \left[1 - (1+z)^{-(n-2)/2} \right], & \Omega_0 = 1; \\ \dfrac{2}{n-2} \left[\arctan \dfrac{\xi(z,\Omega_0)}{(\Omega_0-1)^{1/2}} - \arctan \dfrac{1}{(\Omega_0-1)^{1/2}} \right], & \Omega_0 > 1; \end{cases} \tag{4.12}$$

where we denote

$$\xi(z,\Omega_0) = \left[\Omega_0(1+z)^{(n-2)} + 1 - \Omega_0\right]^{1/2}.$$

Equations (4.12) are valid for $n \neq 2$ only (see Section 5 for $n = 2$).

Distance along the n-dimensional sphere is determined by the functions (3.2):

$$\tilde{\varphi}(z) = \begin{cases} \frac{1}{2}\Omega^{-2/(n-2)}(1+z)^{-1}\Big\{\left[\xi(z,\Omega_0) - (1-\Omega_0)^{1/2}\right]^{2/(n-2)} \\ \times \left[1 + (1-\Omega_0)^{1/2}\right]^{2/(n-2)} - \left[\xi(z,\Omega_0) + (1-\Omega_0)^{1/2}\right]^{2/(n-2)} \\ \times \left[1 - (1-\Omega_0)^{1/2}\right]^{2/(n-2)}\Big\}, \quad 0 < \Omega_0 < 1; & (4.13) \\ \dfrac{2}{n-2}\dfrac{c}{a_0 H_0}\left[1 - (1+z)^{-(n-2)/2}\right], \quad \Omega_0 = 1; & (4.14) \\ \sin\left\{\dfrac{2}{n-2}\left[\arctan\dfrac{\xi(z,\Omega_0)}{(\Omega_0-1)^{1/2}} - \arctan\dfrac{1}{(\Omega_0-1)^{1/2}}\right]\right\}, \quad \Omega_0 > 1. & (4.15) \end{cases}$$

Expanding (4.13) and (4.15) in series in z, the results are the same in the first and second approximations

$$\tilde{\varphi}(z) = (|1-\Omega_0|)^{1/2} z\left[1 - \left(\frac{1}{2} + \frac{n-2}{4}\Omega_0\right)z\right], \quad \Omega_0 \neq 1, \tag{4.16}$$

while (4.14) leads to

$$\tilde{\varphi}(z) = \frac{c}{a_0 H_0} z\left(1 - \frac{n}{4}z\right), \quad \Omega_0 = 1. \tag{4.17}$$

We need also the quantity $a_0\tilde{\varphi}$ which is the same for all the values of Ω_0:

$$a_0\tilde{\varphi} = \frac{c}{H_0} z\left[1 - \left(\frac{1}{2} + \frac{n-2}{4}\Omega_0\right)z\right]. \tag{4.18}$$

Now we are in the position to derive the cosmological tests.

(i) *Visual magnitude - redshift.* Following Zeldovich and Novikov (1975), we should write expression for bolometric visual intensity in our curved n-dimensional space in two equivalent forms:

$$E^{(n)}_{\text{bol}} = \frac{L}{s^{(n)}(a_0\tilde{\varphi})^{n-1}(1+z)^2} = \frac{L}{s^{(n)}\tilde{R}^{n-1}(1+z)^{n+1}}. \tag{4.19}$$

where $\tilde{R}$ is the distance determined from the angular size of a source (see below). It is evidently independent of n:

$$\tilde{R} = a(t_e)\tilde{\varphi}(r_e). \tag{4.20}$$

This formula admits two interpretations. The first equality states that the luminosity L is distributed over the surface area $s^{(n)}(a_0\tilde{\varphi})^{n-1}$ at the moment of observation; one factor $(1+z)$ describes decreasing the frequency of an individual quantum, and the other factor $(1+z)$ describes decreasing the frequency of the appearance of subsequent quanta. The second interpretation is suggested by the second equality in (4.19). The quantity $s^{(n)}\tilde{R}^{n-1}$ describes the surface area on which the radiation would flow unless the scale factor increases, the factor $(1+z)^{n+1}$ describes decreasing the brightness which is proportional to energy decreasing. The latter statement is valid because the law (3.6) is the most general conservation law, it is also suitable for quanta propagating in the Universe filled by dust: $\varepsilon_{\text{quanta}} \propto (1+z)^{n+1}$. By the way, due to the Boltzmann law, it follows that

$$T \propto (1+z)^{\frac{n+1}{4}}$$

for the brightness temperature.

The substitution of (4.19) into (2.11) gives

$$m_{\text{bol}} - M_{\text{bol}} = 2.5\left\{\lg[s^{(n)}(a_0\tilde{\varphi})^{n-1}(1+z)^2] - \lg 4\pi + \lg S^{(3)}_{\text{segm}} - \lg S^{(n)}_{\text{segm}}\right\} - 5.$$

Evidently, the distance r in the expressions (2.8) and (2.9) for the surface area of the segments should be replaced by the quantity $a_0\tilde{\varphi}$. We suppose that R_0 is much less than $a_0\tilde{\varphi}$: $R_0 \ll a_0\tilde{\varphi}$. Then, in accordance with (2.15),

$$\begin{aligned} m_{\text{bol}} - M_{\text{bol}} &= 5\lg(1+z) - 5 + 2.5\left[(3-n)\lg\frac{R_0}{\sqrt{2}} + (n-1)\lg(a_0\tilde{\varphi})\right. \\ &- \left.\lg\beta^{(n)} - \frac{(n-1)(3-n)}{8(n+1)}\lg e\left(\frac{R_0}{a_0\tilde{\varphi}}\right)^2\right]. \end{aligned}$$

For small z under the assumption that $R_0H_0/cz \ll 1$, using (4.18)

$$\begin{aligned} m_{\text{bol}} - M_{\text{bol}} &= 5z\left[\frac{5-n}{4} - \frac{(n-1)(n-2)}{8}\Omega_0\right]\lg e - 5 \\ &+ 2.5\left[(n-1)\lg\frac{cz}{H_0} + (3-n)\lg\frac{R_0}{\sqrt{2}} - \lg\beta^{(n)}\right] \\ &- \frac{5}{16}\frac{(n-1)(3-n)}{n+1}\lg e\left(\frac{R_0H_0}{cz}\right)^2. \end{aligned} \tag{4.21}$$

In the case $n=3$, (4.21) can be reduced to that given by Zeldovich and Novikov (1975):

$$m_{\text{bol}} = 5\lg z + 1.086(1 - q_0^{(3)})z + \text{Const}.$$

(ii) *Angular size – redshift.* Following Zeldovich and Novikov (1975), the quantity $\tilde{R}$ in (4.19) is just determined as the size l of an object divided by its angular size θ:

$$\tilde{R} = \frac{l}{\theta}.$$

Exact formulae for $\tilde{R}$ can be obtained by the substitution of (4.1), (4.2), (4.13), (4.14) and (4.15) into (4.21). Considering the first and second approximations in z

and using (4.1), (4.2), (4.16) and (4.17), we obtain for all the values of Ω_0:

$$\tilde{R} = \frac{c}{H_0}\left[z - \left(\frac{3}{2} + \frac{n-2}{4}\Omega_0^{(n)}\right) z^2\right]. \tag{4.22}$$

For $n = 3$, (4.22) reduces to the known expression

$$\tilde{R} = \frac{c}{H_0}\left[z - \left(\frac{3}{2} + \frac{1}{4}\Omega_0^{(3)}\right) z^2\right].$$

Recalling (3.32), (4.22) may be equivalently rewritten in the form

$$\tilde{R} = \frac{c}{H_0} z\left[1 - \left(\frac{3}{2} + \frac{1}{2}q_0^{(n)}\right) z\right] \tag{4.23}$$

without explicit dependence on the dimensionality of space.

(iii) *Number of sources – redshift.* The differential of the number of sources at a moment t_e is equal to the product of the density of sources $\mathcal{N}$, area of the n-dimensional sphere at the distance r_e and the differential of the radial distance:

$$dN = \mathcal{N}(t_e) s^{(n)} [a(t_e)\tilde{\varphi}(r_e)]^{n-1} a(t_e)\, dr_e. \tag{4.24}$$

Let us assume that the density of sources is proportional to a total matter density, which is natural for $p = 0$:

$$\mathcal{N} = \mathcal{N}_0(1+z)^n. \tag{4.25}$$

Then, substituting (4.25) into (4.24) and using (4.1), (4.2), (4.16), and (4.17) for remaining quantities, the resulting expression is suitable for all the Ω_0:

$$dN = \mathcal{N}_0 s^{(n)} \left(\frac{c}{H_0}\right)^n z^{n-1}\left[1 - \frac{n+1}{2}\left(1 + \frac{n-2}{2}\Omega_0^{(n)}\right) z\right] dz,$$

or, in terms of the deceleration parameter,

$$dN = \mathcal{N}_0 s^{(n)} \left(\frac{c}{H_0}\right)^n z^{n-1}\left[1 - \frac{n+1}{2}(1 + q_0^{(n)}) z\right] dz.$$

5 THE CASE OF TWO-DIMENSIONAL SPACE

We have been convinced that the case $n = 2$ is quite special because many of expressions obtained are not suitable for it: they contain the factor $(n-2)$ in undesired combinations. That is why this case is treated separately.

The metric interval (3.1) now becomes

$$ds^2 = c^2\, dt^2 - a^2(t)(dr^2 + \tilde{\varphi}^2(r)\, d\varphi^2)$$

with $0 \le \varphi \le 2\pi$.

For $n = 2$, the set of equations (3.4)–(3.6) acquires the form

$$\frac{\dot{a}^2}{c^2a^2} + \frac{k}{a^2} = \frac{\kappa^{(2)}}{c^2}\varepsilon, \tag{5.1}$$

$$\frac{\ddot{a}}{a} = -\kappa^{(2)}p,$$

$$\dot{\varepsilon} + 2\frac{\dot{a}}{a}(\varepsilon + p) = 0. \tag{5.2}$$

Certainly, we assume that $\kappa^{(2)}$ is finite, or, in other words, (2.5) is fulfilled.

As in Section 3, we analyze separately the cases $k = +1, 0, -1$ and the two equations of state, (3.8) and (3.9).

(i) $k = +1$. In this case, $\tilde{\varphi}(r) = \sin r$ so that the coordinate r is equivalent to the angle of latitude on the 2-dimensional sphere embedded in the flat 3-dimensional space $(0 \le r \le \pi)$.

It is seen at glance that there exists a stationary solution for dust matter ($p = 0$). Let $\dot{a} = 0$, then it follows from (5.2) that $\dot{\varepsilon} = 0$ and Eq. (5.1) gives the connection between the scale factor and the energy density:

$$a \equiv a_0 = \frac{c}{(\kappa^{(2)}\varepsilon)^{1/2}}. \tag{5.3}$$

It is a remarkable fact that this situation is impossible for $n \neq 2$: there are no stationary solution without the cosmological constant. Some preliminary investigations assure us that small fluctuations on the background of this solution do not grow in time. Thus, this solution seems to be stable.

Although solution (5.3) is distinct, it is not the only solution for dust matter. For $p = 0$, the general solution to (5.2) is

$$p = 0: \quad \varepsilon = C_1 a^{-2}. \tag{5.4}$$

Recalling that, from (3.10), the volume of the 2-dimensional closed Universe is $4\pi a^2$, so that (5.4) can be written as follows

$$\varepsilon a^2 = \frac{\mathcal{M}c^2}{4\pi}. \tag{5.5}$$

The substitution of (5.5) into (5.1) gives the general solution:

$$a = c\left(\frac{\kappa^{(2)}\mathcal{M}}{4\pi} - 1\right)^{1/2} t + \text{Const.} \tag{5.6}$$

For the initial condition $a = 0$ at $t = 0$,

$$a = c\left(\frac{\kappa^{(2)}\mathcal{M}}{4\pi} - 1\right)^{1/2} t.$$

The solution (5.4) can be derived from (5.6) if we impose

$$\frac{\kappa^{(2)}\mathcal{M}}{4\pi} = 1,$$

which is equivalent to (5.4). In this case

$$\mathrm{Const} = \frac{c}{(\kappa^{(2)}\varepsilon)^{1/2}}.$$

In summary, the 2-dimensional closed Universe filled with dust matter expandes forever with a linear law of expansion or it is stationary.

In the case of a radiation filled-Universe, as before,

$$p = \frac{\varepsilon}{2}: \quad \varepsilon = C_2 a^{-3}, \tag{5.7}$$

and Eq. (5.1) has the equivalent form

$$\dot{a} = c\left(\frac{a_{\max}}{a} - 1\right)^{1/2}.$$

The solution

$$p = \frac{\varepsilon}{2}: \quad a_{\max}\left[\arcsin\left(\frac{a}{a_{\max}}\right)^{1/2} - \left(\frac{a}{a_{\max}}\right)^{1/2}\left(1 - \frac{a}{a_{\max}}\right)^{1/2}\right] = ct$$

coincides with the solution (3.26) in the case $n = 3$ for dust matter as mentioned above. The time for reaching $a_{\max}^{(2)}$ is also the same as in (3.27).

(ii) $k = 0$. On substituting Eq. (5.4) into (5.1), we obtain

$$p = 0: \quad a = \mathrm{Const}_1\, t + \mathrm{Const}_2,$$

that is the linear law of expansion with a non-vanishing constant coefficient at t (it might vanish only together with energy density). In terms of a_0 and t_0,

$$a = a_0 \frac{t}{t_0}. \tag{5.8}$$

Further, from (5.1) and (5.6),

$$p = \frac{\varepsilon}{2}: \quad a = a_t \left(\frac{t}{t_0}\right)^{2/3} \tag{5.9}$$

(cf. (3.21)–(3.22)). The solution (5.9) coincides with (3.28).

(iii) $k = -1$. For dust matter, the solution is the same (5.8). For pure radiation,

$$p = \frac{\varepsilon}{2}: \quad a^{1/2}(A + a)^{1/2} + \frac{A}{2}\ln\frac{(A + a)^{1/2} - a^{1/2}}{(A + a)^{1/2} + a^{1/2}} = ct \tag{5.10}$$

(cf. (3.29) which is merely another form of (5.10)).

Expressions of Section 3 for the critical density and $\Omega^{(2)}$ and $q^{(2)}$ remain valid after imposing $n = 2$ in (3.31) and (3.33) and under the assumption (2.5):

$$\rho_c^{(2)} = (c^2\kappa^{(2)})^{-1}H^2(t),$$

$$\begin{aligned} p = 0: &\qquad q_1^{(2)} = 0, \\ p = \frac{\varepsilon}{2}: &\qquad q_2^{(2)} = \frac{1}{2}\Omega^{(2)}(t). \end{aligned} \tag{5.11}$$

Thus, for all the values of k, the Universe with dust matter in $n = 2$ is described by the linear law of expansion. This can be also seen from (5.11).

The age of the Universe is given by (4.10) after putting $n = 2$

$$T^{(2)} = \int\limits_0^\infty \frac{dz}{H_0(1+z)^2} = \frac{1}{H_0}.$$

It is clear that the age of the 2-dimensional Universe is independent of Ω_0. Its value is one and a half of the age of the flat 3-dimensional dust-filled Universe (see Table 1).

Now we return to the above cosmological tests which have been also derived in the case $n = 2$. For $n = 2$, Eq. (4.7) is independent of Ω_0:

$$\frac{dh}{dz} = \frac{h}{1+z}. \tag{5.12}$$

Solution to (5.12) is

$$h = (1+z).$$

For all the values of Ω_0 (see (4.11) and (4.1) which are valid in arbitrary n for the differential dr_e)

$$r_e = \frac{c}{a_0 H_0}\ln(1+z).$$

(Recall that a_0 is expressed by (4.2) in the case $\Omega_0 \neq 1$ and is arbitrary in the case $\Omega_0 = 1$). The functions (3.2) now are:

$$\tilde{\varphi}(z) = \begin{cases} \mathrm{sh}[(1-\Omega_0)^{1/2}\ln(1+z)], & 0 < \Omega_0 < 1; \qquad (5.13) \\ \dfrac{c}{a_0 H_0}\ln(1+z), & \Omega_0 = 1; \qquad (5.14) \\ \sin[(\Omega_0 - 1)^{1/2}\ln(1+z)], & \Omega_0 > 1. \qquad (5.15) \end{cases}$$

The z-expansions of the functions (5.13)–(5.15) have the forms

$$\tilde{\varphi}(z) = \begin{cases} (|1-\Omega_0|)^{1/2} z\left(1 - \dfrac{z}{2}\right), & \Omega_0 \neq 1; \qquad (5.16) \\ \dfrac{c}{a_0 H_0} z\left(1 - \dfrac{z}{2}\right), & \Omega_0 = 1. \qquad (5.17) \end{cases}$$

In any case,

$$a_0\tilde{\varphi} = \frac{c}{H_0}z\left(1 - \frac{z}{2}\right);$$

as before, the two first orders in z of the expressions (5.13)–(5.15) give the same result independently of Ω_0.

(i) *Visual magnitude - redshift.* Imposing $n = 2$ in (4.21) gives the immediate result

$$m_{\mathrm{bol}} - M_{\mathrm{bol}} = \frac{15}{4}\lg e\, z - 5 + 2.5\left(\lg R_0 - \lg\frac{4}{\pi} + \lg\frac{cz}{H_0}\right) - \frac{5}{48}\frac{R_0 H_0}{cz}. \quad (5.18)$$

However, this equality is hardly justified physically because following our scheme the dimensionality 2 is reached far away as a limit case. Perhaps, one should improve the formula (5.18) by taking into account several characteristic radii $R_0, \ldots, R_j$ with several suitable dimensionalities.

(ii) *Angular size - redshift.* In accordance with (4.20), it follows from (5.13)–(5.15) that

$$\tilde{R} = \begin{cases} \dfrac{c}{2H_0(1-\Omega_0)(1+z)^{1+(1-\Omega_0)^{1/2}}}\left[(1+z)^{2(1-\Omega_0)^{1/2}} - 1\right], & 0 < \Omega_0 < 1; \\ \dfrac{c}{H_0}\dfrac{\ln(1+z)}{1+z}, & \Omega_0 = 1; \\ \dfrac{c}{H_0(\Omega_0 - 1)(1+z)}\sin\left[(\Omega_0 - 1)^{1/2}\ln(1+z)\right], & \Omega_0 > 1. \end{cases}$$

In the same orders as before,

$$\tilde{R} = \frac{c}{H_0}z\left(1 - \frac{3}{2}z\right),$$

which is in surprising agreement with (4.22) and (4.23). It is interesting that in the given test the case $\Omega^{(n)} = 0$ is indistinguishable from the case $n = 2$ (see (4.22)).

(iii) *Number of sources - redshift.* The differential of the number of sources (4.24) now is

$$dN = 2\pi\mathcal{N}a^2(t_e)\tilde{\varphi}(r_e)\, dr_e, \quad (5.19)$$

with

$$\mathcal{N} = \mathcal{N}_0(1+z). \quad (5.20)$$

Substituting (5.10), (5.16) or (5.17), (4.1) and (4.2) into (5.19), we obtain for all the values of Ω_0

$$dN = 2\pi\mathcal{N}_0\left(\frac{c}{H_0}\right)^2 z\left(1 - \frac{3}{2}z\right) dz.$$

6 CONCLUDING REMARKS

(1) From the viewpoint of observations, we should note that we hardly make easier the tasks of observers because the derived cosmological tests involve one or more additional parameters. The main test, namely that of visual magnitude seems to be too complicated because, apart from the number of dimensions, it involves explicitly the characteristic radius R_0 (or radii $R_j, j = 0, 1, \ldots$) which should be adjusted. The test of angular size contains a product of two parameters, $(n-2)$ and $\Omega^{(n)}$, and thus, taken alone, it does not allow to distinguish between them, although it could give information on the change of the deceleration parameter in time. The most powerful test would be that of the number of sources but this test is most vulnerable to selection and evolution effects. In principle, this test could indicate dimensionality decreasing if the number of sources were described by a non-polynomial law.

(2) All of the solutions to the Einstein equations have been obtained for constant (although non-integer) values of n. Recall that n is the dimensionality of space-like hypersurfaces which are orthogonal to time-coordinate lines in the given case. However, following our conception, the spatial dimensionality can change in the process of expansion: if it decreases with the growth of relative distance between bodies, then the mean spatial dimensionality of the Universe should decrease in time. If the fall of dimensionality is sufficiently slow, then the solutions obtained are valid locally.

(3) In the context of the above note light propagating along null geodesics from a point of emission to a point of observation, would intersect different space-like hypersurfaces with different values of n. That is why it would be correct to improve our tests in order to take into account the change of n because they have been also derived under the assumption of constant values of n.

(4) The most interesting case is certainly that of $n = 2$. We consider this case as a limiting one: First, some of the solutions and tests which are suitable for the general case are undefined for $n = 2$ and required a limiting procedure to come to $n = 2$. Second, the remarkable fact is that the case $n = 2$ is the only case where a stationary solution exists for a closed dust-filled Universe without a cosmological constant. We think that just this solution provides the stationary asymptotic regime for our Universe which seems to be more attractive for us to imagine as closed.

(5) We cannot immediately take into account the change of dimensionality because of the following two reasons. First, we do not know a law of this change. Second, even if one knows this law, this account is very difficult to realize technically. Probably, this law could follow from a future theory extending general relativity. We suggest that in such a theory the dimensionality would be found from equations just as the metric is. Note that effects of curvature can be distinguished from those of dimensionality. One can imagine a flat space with changing dimensionality but with zero curvature.

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