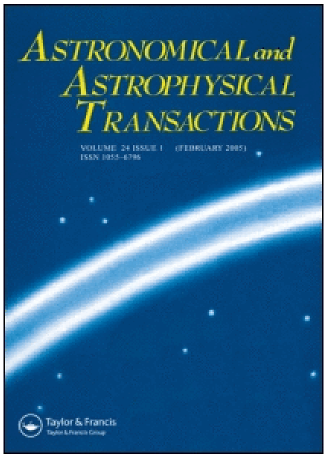

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BOUNDARY LAYERS IN NONLINEAR DYNAMO

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Boundary layers in a nonlinear disk dynamo are considered. The qualitative behavior of the solution and the results of numerical calculations are described.

The steady states of the large-scale magnetic field in a thin-disk galactic dynamo are studied using asymptotic methods. The field is decomposed into a sum of solutions of the degenerate problem plus a boundary layer. The conditions of applicability of this approach are obtained. The solution of the degenerate problem is obtained explicitly, and the boundary layer function is calculated numerically as a solution of a boundary value problem over an infinite interval. This infinity is the main difficulty in the solution of the problem. Therefore, we transform it to an initial value problem depending on a parameter (the shooting method). We searched for the value of the parameter for which the solution fulfills the boundary conditions.

KEY WORDS Boundary layer, nonlinear dynamo, boundary value problem

1 INTRODUCTION

Mean-field dynamo theory describes the appearance of large-scale magnetic fields of galaxies, the Sun, stars and other celestial bodies. Up to now a linear or the so-called kinematic description of the initial stages of the field generation has been well developed. For the subsequent stages, the magnetic field is strong, and, therefore, one has to consider a nonlinear model of the disk dynamo. The use of asymptotic methods is a fruitful way for investigating this. As discussed by Kvasz *et al.* (1992), asymptotic solution of the nonlinear disk dynamo problem is the sum of the slowly changing solution and the boundary layer solution, which is concentrated at the boundary of the disk.

Boundary layers may appear if the intensity of the field generation is sufficiently large. In this case, magnetic field changes strongly in rather small domains of space and large parameters appear in the equations governing the magnetic field. From the mathematical point of view, boundary layers appear due to the necessity to fulfill some boundary conditions which cannot be satisfied only by the slowly changing term of the solution. Equations for boundary layers often cannot be investigated analytically and should be solved only numerically. In this paper we consider these types of problems on the example of the simplest nonlinear dynamo in a thin disk.

The boundary layers have a small characteristic width, which can be hardly observed in detail by modern astronomical equipment. The model approximations used below like vacuum boundary conditions and a simple one-dimensional dynamo model are rather crude. Therefore, the result obtained in the present paper can be hardly directly compared with observational data.

Note, that the main purpose of the present paper is to analyze in detail the asymptotical representation given by Kvasz *et al.* (1992). Indeed, it is supposed in that paper that the solution of the boundary layer equations always exists. However, as we argue in Section 5, this solution exists only under a certain, rather natural condition. Otherwise, one should modify the representation given by Kvasz *et al.* (1992). The technique used can be further applied to other astrophysical and geophysical problems where boundary layers also exist.

2 NONLINEAR DYNAMO IN A THIN DISK

The problem of a steady-state distribution of the large-scale magnetic field in a thin disk can be reduced to the solution of the following differential equation over the interval $z \in [0,1]$, see Vainstein and Ruzmaikin (1972):

$$\frac{d^3}{dz^3}B + D\alpha(z,B)B = 0, \tag{1}$$

where $B = B(z)$ is the azimuthal component of the magnetic field. The so-called dynamo number D is introduced to characterize the intensity of the source of the generation of the large-scale magnetic field (helicity and differential rotation), see e.g. Zeldovich *et al.* (1983). We assume here that the back reaction of the magnetic field on fluid motions can be described in terms of α-quenching. Thus, the helicity becomes a function of B. Following Kvasz *et al.* (1992) we adopt the helicity function as

$$\alpha(z,B) = \begin{cases} \alpha_0(z)[1 - g(z)B^2] & \text{if } \mathrm{g(z)}B^2 < 1, \\ 0 & \text{if } \mathrm{g(z)}\ B^2 \geq 1 \end{cases} \tag{2}$$

where $g(z)$ is a slowly changing function which characterizes the steady state distribution of the magnetic field $B_0(z)$ within most of the disk. Function $\alpha_0(z)$ is the helicity distribution of the kinematic model.

The z-axis $(-1 \leq z \leq 1)$ is directed perpendicularly to the disk plane. Further we shall restrict ourselves to considering the quadruple mode, i.e. even solutions of $B(z)$. Therefore, we shall consider only the upper half of the disk $z \geq 0$. It is essential to note here that $\alpha_0(z)$ and $\alpha(z)$ are odd functions, while $g(z)$ and $B_0(z)$ are even ones.

The boundary conditions for Eq. (1) in the case of a disk of the half-thickness $h = 1$ surrounded by a vacuum are (see Zeldovich *et al.*, 1983):

$$B(1) = 0, \tag{3}$$

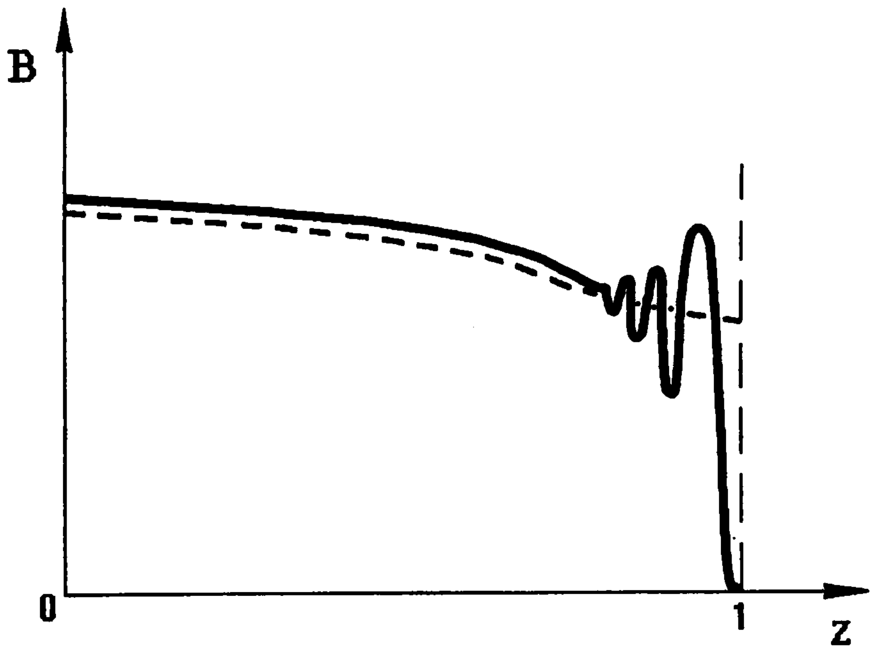

Figure 1 A qualitative behavior of the degenerate (dashed) and asymptotic (solid) solutions of the problem in the interval $z \in [0, 1]$.

$$\frac{d^2}{dz^2}B(1) = 0, \tag{4}$$

To pose the boundary value problem completely we need to add one more boundary condition. Since $B(z)$ is an even function, we have

$$\frac{d}{dz}B(0) = 0. \tag{5}$$

An asymptotic theory for Eq. (1) was developed by Kvasz *et al.* (1992).

In asymptotic study of the equation for $|D| \gg 1$ the third-derivative term can be neglected in the main part of the disk in comparison with the nonlinear term. Thus, we obtain the so-called degenerate solution for the magnetic field:

$$B(z) = [g(z)]^{-1/2}. \tag{6}$$

This is shown in Figure 1 as a dashed line. The greater is $|D|$, the better this solution satisfies Eq. (1). However, the degenerate solution (6) does not satisfy the boundary conditions (3) and (4). Therefore, a boundary layer appears in the neighborhood of $z = 1$. Here the value of the third-derivative term is comparable with the nonlinear term and the former cannot be neglected, i.e., the magnetic field changes abruptly within the layer. The boundary conditions (5) at $z = 0$ is satisfied since $g(z)$ is an even function and $\frac{d}{dz}g(0) = 0$. Thus, we seek the solution of Eqs. (1), (3), (4) and (5) in the form:

$$B = [g(z)]^{-1/2} + \Phi_0(x) + |D|^{-\eta}\Phi_1(x) + \dots, \tag{7}$$

where $x = (z-1)|D|^{\kappa}$ is the so-called fast variable, and η and κ are constants. The characteristic thickness of the boundary layers is $|D|^{-\kappa}$. Here, the first term is the degenerate solution, and the other terms correspond to a boundary layer at the disk's surface. The latter are power series in $|D|$. We shall search for constants η and κ and functions $\Phi_0, \Phi_1, \dots$ assuming that they depend only on the fast variable x but not on D.

Let us derive boundary conditions for the functions $\Phi_0, \Phi_1, \dots$. Substituting (7) into (3) and taking into account that the large parameter $|D|$ should be eliminated in equations which determine Φ_0 and that $z = 1$ corresponds to $x = 0$, we obtain

$$\Phi_0(0) = -[g(1)]^{-1/2}. \tag{8}$$

Then, taking $dx = |D|^{\kappa}\,dz$, we obtain from Eq. (4)

$$\frac{d^2}{dz^2}[g(z)]^{-1/2}|_{z=1} + |D|^{2\kappa}\Phi_0''(0) + |D|^{2\kappa-\eta}\Phi_1''(0) + \dots = 0. \tag{9}$$

Here prime denotes derivative with respect to x. Consecutive terms in Eq. (9) have the orders $O(1)$, $O(|D|^{\kappa})$, $O(|D|^{\kappa-\eta})$, etc., respectively. Since none of the remaining terms can be equal to the second one for $\kappa > 0$ and $\eta > 0$, we have

$$\Phi_0''(0) = 0. \tag{10}$$

The following is necessary to balance the two remaining terms in Eq. (9):

$$\eta = 2\kappa$$

and

$$\Phi_1''(0) = -\frac{d^2}{dz^2}[g(z)]^{-1/2}|_{z=1}.$$

The third boundary condition (5) should be rewritten as

$$\Phi_0(-\infty) = 0, \tag{11}$$

because $z = 0$ corresponds to $x \to \infty$ for $|D| \to \infty$. Let us substitute (7) into (1). We require the function $\Phi_0'''(x)$ to enter the equation of the lowest order in $|D|$. It is necessary to fulfill all the boundary conditions (8), (10) and (11). Equation for $\Phi_0(x)$ and a constant κ, which determines the thickness of the boundary layer, depends on $\alpha_0(1)$. It is possible to consider two different cases: $\alpha_0(1) = 0$ and $\alpha_0(1) \neq 0$. Further we shall assume that $\alpha_0(1) \neq 0$. The other case will be also briefly considered below.

3 A NUMERICAL STUDY OF THE BOUNDARY LAYER

When $\alpha_0(1) \neq 0$, since $\Phi_0'''(x)$ should appear in the equation of the lowest power in $|D|$, we find that $\kappa = \frac{1}{3}$ and $\eta = \frac{2}{3}$, and the following equation for Φ_0 emerges:

$$u\Phi_0'''(x) + \alpha_0(1)[u\Phi_0(x))^3 + 3(u\Phi_0(x))^2 + 2(u\Phi_0(x))] = 0, \tag{12}$$

where $u = [g(z)]^{1/2}|_{z=1}$. Let us introduce a new function

$$\Psi(x) = -[g(1)]^{1/2}\Phi_0(x) = -u\Phi_0(x)$$

and a new variable

$$t = -[\alpha_0(1)]^{1/3}x.$$

Then we obtain from Eqs. (12), (8), (10) and (11) the following boundary value problem for an ordinary nonlinear differential equation:

$$\frac{d^3}{dt^3}\Psi = \Psi^3 - 3\Psi^2 + 2\Psi, \tag{13}$$

$$\Psi(0) = 1, \tag{14}$$

$$\frac{d^2}{dt^2}\Psi(0) = 0, \tag{15}$$

$$\Psi(\infty) = 0. \tag{16}$$

This problem cannot be solved analytically, so we have to solve it numerically. We describe here qualitatively the behavior of the solution observed in our numerical investigation. The main difficulty here is a numerical implementation of the boundary condition (16) posed at infinity. Using the shooting method (see, e.g., Tikhonov *et al.*, 1985; Ortega and Poole, 1981), we replace it by

$$\frac{d}{dt}\Psi(0) = p, \tag{17}$$

and we solve further the initial value problem (13), (14), (15) and (17) for several values p. Thus, the problem is reduced to a search for that value of p which allows the condition (16) to be satisfied. As we shall show now, an unsuccessful choice of the parameter causes the integral curve of the solution to tend not to zero but to infinity for $t \to \infty$.

We consider this phenomenon in detail. Let us represent Eq. (13) as

$$\frac{d^3}{dt^3}\Psi = \Psi(\Psi - 1)(\Psi - 2), \tag{18}$$

and consider in the phase space of (Ψ, t) (Figure 2) the interval in which the right-hand side of Eq. (18) is sign-constant. The right-hand side is positive in the intervals A ($\Psi > 2$) and C ($0 < \Psi < 1$), and negative in the intervals B ($1 < \Psi < 2$) and D ($\Psi < 0$). Hence, the second derivative of Ψ increases in the intervals A and C, and decreases in B and D.

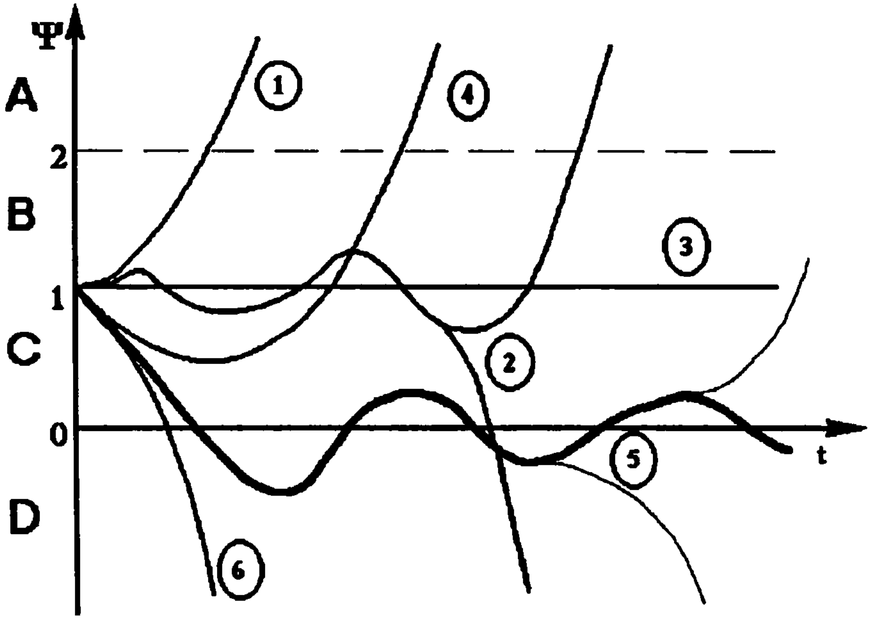

Figure 2 A qualitative behavior of the solution of the boundary layer problem. The dependence of the boundary layer solution on the fast variable t is shown. *1*, $p \geq 4 \times 10^{-4}$; *2*, $0 < p \leq 4 \times 10^{-4}$; *3*, $p = 0$; *4*, $p_0 < p < 0$; *5*, $p = p_0$; *6*, $p < p_0$.

Depending on the values of p, we obtain six qualitatively different cases of behavior of the solution (see Figure 2):

1. The solution increases and tends to $+\infty$.

2. The solution oscillates near the point $\Psi = 1$, then grows to $+\infty$ or $-\infty$.

3. The solution is identical unity.

4. The solution decreases, becomes negative, then increases and tends to $+\infty$.

5. The solution decays with oscillations near the t-axis ($\Psi = 0$).

6. The solution tends to $-\infty$ after possible oscillations near point $\Psi = 0$.

To study the first three cases, let us introduce a new function in Eq. (18): $\Psi(t) = 1 + \xi(t)$. Thus, we obtain the following Cauchy problem for ξ:

$$\frac{d^3}{dt^3}\xi = \xi^3 - \xi, \tag{19}$$

$$\xi(0) = 0, \tag{20}$$

$$\frac{d^2}{dt^2}\xi(0) = 0, \tag{21}$$

$$\frac{d}{dt}\xi(0) = p. \tag{22}$$

The term ξ^3 on the right-hand side of Eq. (19) can be neglected for small $\xi(|\xi| \gg 1)$

as compared with the remaining term. Thus, we consider an approximate equation

$$\frac{d^3}{dt^3}\xi + \xi = 0.$$

This equation has three linearly independent solutions e^{-t}, $e^{t/2}\sin\frac{\sqrt{3}}{2}t$, and $e^{t/2}\cos\frac{\sqrt{3}}{2}t$. The general solution is their linear combination

$$\xi(t) = C_1 e^{-t} + e^{t/2}(C_2 \sin\frac{t\sqrt{3}}{2} + C_3 \cos\frac{t\sqrt{3}}{2}, \tag{23}$$

where C_1, C_2, and C_3 are constants. The first term decays with t. The second and third terms cause the growth of the solution and lead it out of the applicability domain of the approximation $|\xi \ll 1$.

Case 3 ($p = 0$) corresponds to $C_1 = C_2 = C_3 = 0$ and $\xi(t)$ is identically equal to zero (dashed line in Figure 2). If $p \neq 0$, substituting Eq. (23) into initial conditions (20), (21), and (22) yields a system of linear equations for C_1, C_2 and C_3. It has a unique solution $C_1 = -\frac{1}{2}p$, $C_2 = \frac{1}{2\sqrt{3}}p$ and $C_3 = \frac{1}{2}p$. This explains why the solution grows for any value of $p > 0$ and leaves the frames of the approximation $|\xi| \ll 1$. Only for very small values of p the solution has a few oscillations in the domain $|\xi| \ll 1$ and continues either upwards plus-infinity or downwards to minus-infinity, depending on the number of the oscillations. The smaller is p, the more oscillations the solution will have. Hence, the behavior of the solution in the cases 1–3 can be completely explained by considering linearized Eq. (13) near the point $\Psi = 1$. They never fulfill condition (16) and are, thus, of no interest.

Let us consider cases 4, 5 and 6 in detail. Since p is negative, Ψ first decreases although the right-hand side of Eq. (18) is positive. For not too small absolute values of p it can reach values $\Psi \simeq 0$. Using approximation $|\Psi| \ll 0$, the terms Ψ^3 and $-3\Psi^2$ can be neglected in Eq. (13) and we obtain the following linear differential equation:

$$\frac{d^3}{dt^3}\Psi = 2\Psi.$$

This equation also has three linearly independent solutions, of which the linear combination is the general solution of the equation:

$$\Psi(t) = C_1 e^{\lambda_1 t} + e^{\lambda_2 t}(C_2 \sin\omega t + C_3 \cos\omega t), \tag{24}$$

where $\lambda_1 = 2^{1/3}$, $\lambda_2 = -2^{-2/3}$, $\omega = 2^{-2/3}3^{1/2}$, and C_1, C_2, and C_3 are constants.

The values of the constants C_1, C_2 and C_3 depend on how the solution enters the domain of approximation $|\Psi| \ll 1$, i.e. on the parameter p. If we take a critical value of p which provides $C_1 = 0$ then we obtain only decaying oscillations near $\Psi = 0$. This corresponds to case 5 and is the unique case when condition (16) is fulfilled. However, this value cannot be reached exactly in any real numerical calculations, and with a growth of t the exponentially growing term $C_1 e^{\lambda_1 t}$ leads the solution out from the domain of approximation $|\Psi| \ll 1$ and then towards infinity in the upper or lower half-plane.

4 NUMERICAL RESULTS

The critical value of p yielding $C_1 = 0$ can be estimated by varying p in the numerical solution of the problem (13), (14), (15) and (17). For strongly negative values of p (e.g., $p < -0.7$) the behavior of the solution corresponds to case 6, and for greater values of p (e.g., $p > -0.6$), to case 4. Hence, we conclude that the desired critical value of p belongs to the interval $[-0.7, -0.6]$.

Seeking the critical value of p we try to hold the solution within the limits $|\Psi| \ll 1$. Any real numerical solution leaves these limits towards infinity in the upper or lower half-plane. Let us introduce a function which is either positive or negative depending on the direction towards which the solution grows after it leaves the domain $|\Psi| \ll 1$. To determine the critical value of p we can use the bisection method (see, e.g., Forsythe *et al.*, 1977) applying it in the above interval $p \in [-0.7, -0.6]$. Let us consider solutions of the boundary layer problem for values of p at the ends of the interval. One of them should grow in the upper and the other in the lower half-plane, i.e., our function has different signs at these points. Then we solve the problem for p at the middle position within the interval. Depending on the direction towards which the solution grows, or on which sign our function has, we bisect the interval in such a way that our function has different signs at different ends of the new interval. Since the critical value of p belongs to this interval, we can interpret the length of the interval to be accuracy of the determination of p. Hence, we can bisect the interval until we reach the required accuracy.

We have to reach here a very high accuracy of the determination of p due to the exponential terms in Eq. (24). If a low precision of, say, five digits is used, only one oscillation of the solution can be recovered, and it is not possible to study the oscillations. For a higher accuracy, the interval of the oscillations near the point $\Psi = 0$ becomes wider, the numerical solution approximates Ψ better and can be quantitatively compared, in this interval, with the approximate solution (24) for $C_1 = 0$. Nevertheless, this numerical solution ultimately tends to infinity.

For a 16-digit accuracy, we obtain six oscillations. The first five of them are rather stable and do not change significantly if the accuracy increases. The qualitative behavior of the solution over this interval is found with a sufficient accuracy. The zeros and extreme of the oscillations of the solution are shown in Table 1.

The solution is shown in Figure 2 as a thick line. We obtain the critical value $p_0 = -0.685\,139\,987\,740\,998 \pm 1 \times 10^{-15}$. It is easy to see, that the numerical half-period of oscillations corresponds to $T/2 = \pi/\omega = 2.879227$ and the decay rate corresponds to $d = -2\pi\lambda_2/\omega = 3.627599$, which are derived explicitly (see Eq. (24)) from the analysis of the linearized problem. This confirms that our qualitative and numerical analyses are correct.

Thus, we can calculate the boundary layer solution of the nonlinear dynamo problem as accurately and for as long interval as necessary. The first few oscillations of the solution can be calculated numerically, for the subsequent oscillations the solution can be obtained as an approximate solution (24) for $C_1 = 0$.

Table 1. Numerical results: N is the ordinal number of the oscillation, Z_n is the zero of the solution, A_n is the maximum of the oscillation amplitude, $T/2$ is the half-period of the oscillation and d is the decay rate. The latter has been calculated as $d = 2\ln\frac{A_{n-1}}{A_n}$

N	Z_n	A_n	$T/2$	d
1	1.73663	1.30666×10^{-1}	–	–
2	4.48213	2.27303×10^{-2}	2.7455	3.497889
3	7.38638	3.66336×10^{-3}	2.9043	3.650639
4	10.26188	5.98227×10^{-4}	2.8755	3.624329
5	13.14188	9.74825×10^{-5}	2.8800	3.628596
6	16.02088	1.59269×10^{-5}	2.8790	3.623321
7	18.97938	1.45735×10^{-6}	2.9585	4.782784
8	20.27663	–	1.2973	–

5 DISCUSSION AND CONCLUSION

We have obtained the solution of the problem in the case $\alpha_0(1) \neq 0$. We constructed the asymptotic solution as a sum of a degenerate one and a boundary layer. The latter is shown in Figure 1 as a solid line, dashed line shows the degenerate solution.

Let us discuss the case of the mean helicity being equal to zero at the surface of the disk: $\alpha_0(1) = 0$. Then we can approximate $\alpha_0(x)$ near $z = 1$ as

$$\alpha_0(z) = \frac{d\alpha_0}{dz}|_{z=1}(z-1)$$

with accuracy up to first-order terms. Substituting this into Eq. (1) and using much the same argument as in Section 2, we obtain that $\kappa = \frac{1}{4}$, and the following differential equation for the boundary layer function $\Psi(t)$ emerges:

$$\frac{d^3}{dt^3}\Psi = -t(\psi^3 - 3\Psi^2 + 2\Psi), \tag{25}$$

with boundary conditions

$$\begin{aligned} \Psi(0) &= 1, \\ \frac{d^2}{dt^2}\Psi(0) &= 0, \\ \Psi(\infty) &= 0. \end{aligned} \tag{26}$$

A qualitative study and numerical investigation of this problem, in a manner similar to that of Section 3 show, that there is no solution decaying at infinity and, thus, fulfilling condition (26). The reason for the lack of decay is the factor t on the right-hand side of Eq. (25). This causes the solution to oscillate with the amplitude growing more rapidly with t. This means that in this case the boundary layer solution does not exist near $z = 1$.

It is necessary to discuss here the accuracy of the result obtained. In the present paper we considered the simple one-dimensional dynamo model given by Vainstein and Ruzmaikin (1972). When deriving Eq. (1), one neglected the terms which correspond to the radial dependence of the magnetic field. These terms are of order $\lambda^{1/2}$ (see, e.g., Ruzmaikin *et al.*, 1988), where $\lambda = h/R$ is the ratio of the half-thickness of the galactic disk h to its characteristic radius R, or the so-called aspect ratio of the galactic disk. Thus, for real galaxies, where λ is of order 0.04, the influence of these terms is about 20%. Therefore, our investigation may add some qualitative information on the structure of the boundary layer in galaxies. However, that this accuracy has no relation to the very high accuracy of the determination of the so-called "shooting" parameter p in Section 4, caused by the necessity to study the qualitative behavior of the boundary layer solution. Note that this parameter was used only to solve the problem of the existence of the boundary layer solution. Its accuracy cannot be interpreted as the accuracy of the model.

In the present paper, we used the assumption of a sharp boundary of the galactic disk. However, this is rather crude idealization for many real galaxies where such boundaries are not well defined. Nevertheless, this approach may be reasonable for some physical and astrophysical objects. The algorithm constructed and the basic results can be applied to other models. The structure of the boundary layer solution obtained characterizes a qualitative behavior of the magnetic field near the boundary of the galactic disk.

We have shown that the asymptotic representation given by Kvasz *et al.* (1992) is valid only if $\alpha_0(1) \neq 0$. If $\alpha_0(1) = 0$, it is not possible to construct a solution of Eq. (1) for $|D| \to \infty$ as a sum of a slowly changing solution and a boundary layer solution. A similar asymptotic behavior near the point where the field smoothly tends to zero is studied in detail by Belyanin *et al.* (1993) for a nonlinear fluctuation dynamo problem. Then one should either consider the problem by the use of differential inequalities, or, since the boundaries are not well defined, modify the helicity function $\alpha(z)$ by an addition of order $|D|^{-1/2}$ as it was done by Belyanin *et al.* (1994) and further assume that $\alpha_0(1) \neq 0$ and solve the problem using the representation of Kvasz *et al.* (1992) in much the same way as above.

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