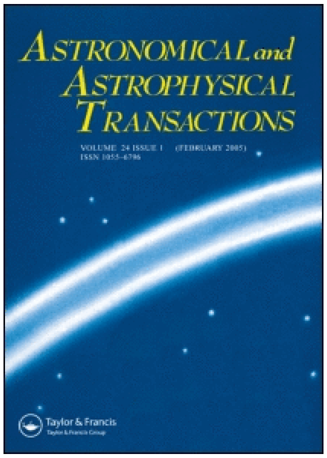

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T. A. Agekian [a]; I. I. Nikiforov [a]; V. V. Orlov [a]; N. P. Pitjev [a]
[a] Astronomical Institute, St. Petersburg State University, St.Petersburg Peterhof, Russia

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THE THIRD INTEGRAL OF MOTION†

T. A. AGEKIAN, I. I. NIKIFOROV, V. V. ORLOV, and N. P. PITJEV

Astronomical Institute, St.Petersburg State University, Bibliotechnaya pl. 2, 198904 St.Petersburg, Peterhof, Russia

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An analytical expression for a third integral of motion in an rotationally symmetric potential is derived. This expression is constant along trajectories.

KEY WORDS Stellar dynamics, integrals of motion

Rotationally symmetric potentials are the ones most frequently met in nature. The stars, planets and most of galaxies have such gravitational potentials. The problem of motion in the rotationally symmetric field is reduced to the study of the behavior of a dynamical system with two degrees of freedom, just a system of two connected oscillators on the plane. The reduction is achieved through the integral of area J.

Let as consider the motion of a point in the meridional plane (R, z) that contains the axis of symmetry and a particle, this plane is orthogonal to the plane of symmetry. The motion of the particle in the meridional plane is determined by the reduced potential $U(R, z)$. The region of possible motions in the meridional plane is described by the unequality

$$h(R,\ z) = U(R,\ z) + I \geq 0, \tag{1}$$

where I is the integral of energy.

Some numerical examples of orbits in various rotationally symmetric fields show that a trajectory does not fill the whole region of possible motions, but only its part. This part has, as a rule, a box-, or shall-, or tube-like form. This evidences an existence of a third integral of motion K. Expressions for $K(R, Z, \dot{R}, \dot{z})$ were found for some specific potentials $U(R, z)$ by Kusmin (1956), Bozis (1982), Antonov (1985) and others. However, in the general case, up to now one could not derive the third integral.

It seems that the equations of motion are insufficient for a progress. An additional factor is necessary. Agekian (1972) has proposed to consider the field of directions of the motion in the plane (R, z) as such a factor. This field is described by the angle f between the tangent to the trajectory and the axis R. In the general

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case, the field $f(R,z)$ is multiple-valued. The number m of the values of f is the multiplicity of the field of directions. The regions of the orbit where $m > 2$ are the folds of the field of directions.

In terms of the field of directions, the equation of the trajectory has the form

$$\frac{\partial f}{\partial l} = \frac{1}{2h}\frac{\partial U}{\partial n}, \tag{2}$$

where l and n are the directions along the tangent and normal to the trajectory.

Let us consider an arbitrary function

$$\Psi = \Psi(R, z, f). \tag{3}$$

The following equation takes place for Ψ:

$$\frac{\partial}{\partial l}\frac{\partial \Psi}{\partial n} - \frac{\partial}{\partial n}\frac{\partial \Psi}{\partial l} + \frac{\partial \Psi}{\partial n}\frac{\partial f}{\partial n} + \frac{\partial \Psi}{\partial l}\frac{\partial f}{\partial l} = 0. \tag{4}$$

This equation was derived by Agekian (1974). It is a base to study the structure of the velocity field.

Let us use the angle f instead of Ψ. We derive an equation for the derivative of the field f along the normal to the trajectory $\partial f/\partial n$:

$$\frac{\partial}{\partial l}\frac{\partial f}{\partial n} + \left(\frac{\partial f}{\partial n}\right)^2 + A\frac{\partial f}{\partial n} + B = 0, \tag{5}$$

where A and B are known functions of R, z and f.

Agekian (1974) has shown that $\partial f/\partial n = \pm\infty$ at the contours of the orbit and folds. Therefore, one can draw these contours with integrating equations (2) and (5) simultaneously. This is especially important when the multiplicity m is rather large.

Let us consider the function $\partial f/\partial n$ instead of Ψ. Then we have

$$\frac{\partial}{\partial l}\frac{\partial^2 f}{\partial n^2} + \left(3\frac{\partial f}{\partial n} + A\right)\frac{\partial^2 f}{\partial n^2} + M\frac{\partial f}{\partial n} + N = 0. \tag{6}$$

Here M and N are known functions of R, z and f.

Contrary to equation (5), this equation is linear with respect to the unknown function $\partial^2 f/\partial n^2$. Therefore, its solution has the form

$$\begin{aligned}\frac{\partial^2 f}{\partial n^2} &= \left\{K - \int dl\left[\left(M\frac{\partial f}{\partial n} + N\right)\exp\int\left(3\frac{\partial f}{\partial n} + A\right)dl\right]\right\}\\ &\times\ \exp\int\left(-3\frac{\partial f}{\partial n} - A\right)dl.\end{aligned} \tag{7}$$

The arbitrary constant K in this solution is a third integral of motion. It is derived from (7) using the derivatives $\partial f/\partial n$ and $\partial^2 f/\partial n^2$. The numerical integrations, which have been made for the potentials of Contopoulos (1965) and Hénon–Heiles

(1964), have shown that the value K is conserved along the trajectory. The derivatives $\partial f/\partial n$ and $\partial^2 f/\partial n^2$ strongly vary along the trajectory. At the same time, the value of K preserves a constancy.

In the future we hope to study the isolating properties of the integral K as well as to use it for studying some properties of the field of directions, e.g. to find the equations for the contours of orbit and folds and to localize periodic trajectories.

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DISCUSSION

Hunter: Is your method based on the special geometrical structure of box orbits, so that it works only for box orbits?
Orlov: No, it works also for other types of orbits, e.g. the tube orbits.
Seleznev: As I have understood, you obtained numerically the orbit of a point in some potential and numerically calculated the values K for this orbit?
Orlov: Yes, the value K has been calculated numerically.
Seleznev: Does a third integral exist in all cases when the motion of the point does not fill the whole meridional plane?
Orlov: This is only a sign of the existence of a third integral but it is not a proof of this.