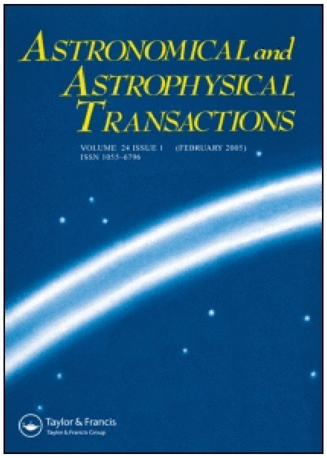

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ON THE LINEAR LOCAL INTEGRAL OF MOTION IN A ROTATING SYSTEM†

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A particle motion in the four-dimensional phase space is considered. The system is supposed to be time independent in a given frame rotating with a constant angular velocity. A particular case of the local integral linear in velocities is discussed.

KEY WORDS Stellar dynamics, plane motion, potential, local integral

1 INTRODUCTION

Let us consider the plane motion of a star in a nonaxisymmetric stellar system which is in a steady state in a reference frame rotating with the angular velocity $\Omega = \text{const}$. But there is no general method to know whether or not a given dynamical system possesses an additional integral (besides the Jacobi integral). Still, the additional integral problem is of great interest in celestial mechanics and stellar dynamics.

On the other hand, it is known that some of the hypersurfaces of the additional integral in the phase space are isolating (Arnold, 1962) and if an individual hypersurface is known, it gives a chance to construct a model with a triaxial velocity ellipsoid. It would be useful to find at least such an individual hypersurface.

In our recent paper (Antonov and Shamshiev, 1993), this individual hypersurface, i.e., a linear local integral in the sense of Antonov (1981) have been found.

In this paper we continue to study this local integral of motion.

2 CONSTRUCTION OF THE LOCAL INTEGRAL

Let x and y be Cartesian coordinates in a plane perpendicular to the axis of rotation, and let u and v be components of the velocity, let U denote the time independent

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gravitational potential, and $\Omega = \text{const}$ the angular velocity of the rotating reference frame. In our paper Antonov, Shamshiev (1993) we found the class of potentials

$$U = \frac{1}{2}\left[S(\psi) + \varphi^2 - \Omega^2(x^2+y^2)\right] , \tag{1}$$

with the linear local integral of motion

$$J = Au + Bv - \varphi , \tag{2}$$

for the case when $A = \sin\sigma$, $B = -\cos\sigma$. In Eq.(1) and (2) the function $\sigma(x,y)$ is implicitly defined from

$$u\sin\sigma - v\cos\sigma = f'(\sigma) , \tag{3}$$

and the functions φ and ψ are

$$\varphi = \frac{\Omega\left[x^2+y^2-2\psi f''(\sigma)\right] + N(\sigma)}{x\cos\sigma + y\sin\sigma - f''(\sigma)} , \tag{4}$$

$$\psi = x\cos\sigma + y\sin\sigma + f(\sigma) , \tag{5}$$

where $N(\sigma)$, $S(\psi)$ and $f(\sigma)$ are arbitrary functions.

It would be noted that our local integral concept is used in another sense then Lynden-Bell's third local integral (Lynden-Bell, 1962), however both theories can give similar results.

Now we will demonstrate that (2) is also a local integral when $A = -y$ and $B = x$, i.e.,

$$J = xv - yu - \varphi \tag{6}$$

is a local integral of motion, along with the Jacobi integral

$$\frac{u^2+v^2}{2} - \frac{\Omega^2(x^2+y^2)}{2} = U + h . \tag{7}$$

Using (6) together with (7) we obtain a double solution for the velocity components:

$$u = \frac{\pm x\sqrt{T}}{r^2} - \frac{y\varphi}{r^2} ,$$

$$v = \frac{\pm y\sqrt{T}}{r^2} + \frac{x\varphi}{r^2} , \tag{8}$$

where

$$T = 2(U+h)r^2 + \Omega^2 r^4 - \varphi^2 , \; (r = \sqrt{x^2+y^2}) . \tag{9}$$

In the case of rotation, x and y are canonically conjugate with the momenta $u - \Omega y$ and $v + \Omega x$. According to the general rules of analytical mechanics, the expression

$$(u - \Omega y)dx + (v + \Omega x)dy , \tag{10}$$

where u and v are defined by (8) for any particular trajectory, should be a total differential. It is easy to see that (10) is true for the univalent parts of (7) and for parts containing the square root of T, i.e.,

$$\frac{\partial}{\partial x}\left(\frac{y\sqrt{T}}{r^2}\right) - \frac{\partial}{\partial y}\left(\frac{x\sqrt{T}}{r^2}\right) = 0\,, \tag{11}$$

$$\frac{\partial}{\partial x}\left(\frac{x\varphi}{r^2} + \Omega x\right) + \frac{\partial}{\partial y}\left(\frac{y\varphi}{r^2} + \Omega y\right) = 0\,. \tag{12}$$

According to the general theory of partial differential equations, the general solution of (11) and (12) is

$$T = 2r^2[F(r) + h]\,, \tag{13}$$

$$\varphi = -\Omega r^2 + g(\Theta)\,,\ (\Theta = \arctan(y/x))\,, \tag{14}$$

where $F(r)$ and $g(\Theta)$ are arbitrary functions. Then using (9), (13) and (14), it is not difficult to write out the class of potentials

$$U = F(r) - g(\Theta)\Omega + g^2(\Omega)/(2r^2)\,. \tag{15}$$

As an example we can take $g(\Theta) = \kappa\cos\Theta$, $F(r) = \chi/r$, (κ and χ are constants). It appears from this that the motion domain will be a circle for all the real values of $T \geq 0$.

We can illustrate a more interesting example with the local integral (3). The straight lines σ = const have an envelope. Let us pay attention to the fact that the system of straight lines σ = const is very similar to the system of the tangents of an asteroid. Indeed, if we construct these tangents taking into account only one direction then the whole exterior will be hatched by single tangents (Figure 1). Using them makes it possible to represent "box" orbits (in our previous (Antonov and Shamshiev, 1993) paper a pear-shaped orbits have been indicated).

For an astroid of the form

$$y^{2/3} + x^{2/3} = R^{2/3} \qquad \text{or} \qquad R\cos^3\Theta\,,\ y = R\sin^3\Theta\,, \tag{16}$$

the family of tangents is taking

$$x + \frac{1}{\sqrt{(R/x_0)^{2/3} - 1}}\,y = x_0(R/x_0)^{2/3}\,, \tag{17}$$

where x is a parameter.

Comparing (3) with (17), it is easy to show that

$$\begin{aligned}
\sin\sigma &= \sqrt{1 - (x_0/R)^{2/3}}\,,\\
\cos\sigma &= -\sqrt{\left[1 - x_0/R^{2/3}\right]\Big/\left[R/{x_0}^{2/3} - 1\right]}\,,\\
f'(\sigma) &= (R/x)^{2/3}x_0\sqrt{1 - (x_0/R)^{2/3}}\,.
\end{aligned}$$

This makes it possible to express other functions, $f(\sigma)$ and $f''(\sigma)$. Antonov and Shamshiev (1993) have shown that the domain of the particle motion is defined by

$$\frac{d\psi}{dt} = \pm\sqrt{S(\psi) + 2h}\,, \; S(\psi_1) = S(\psi_2)\,. \tag{18}$$

As far as the evolvent of the envelope plays the role of the domain, we find it as

$$\left.\begin{array}{rcl} \xi & = & R\cos\Theta(1/2 - C + 5/4\cos 2\Theta) \\ \zeta & = & R\sin\Theta(1/2 + C + 5/4\cos 2\Theta) \end{array}\right\} \tag{19}$$

The evolvent has singular points when $C = 5/2$ and $C = -7/4$. So, the domain must be taken for $C > 5/2$ and $C < -7/4$. Constructing the domain shows box-shaped orbits.

3 CONCLUSION

In any case, as it is known, the existence of an additional integral of motion in its exact sense, allows to find be quadratures all the trajectories. Although the local integral, in contrast to a real integral of motion, does not give us an opportunity to find all the trajectories by means of quadratures, but it allows to reduce the order of the system of differential equations and to facilitate their solution. In some intermediate cases, the existence of a local integral operating on a certain surface of the phase space, leads to a total integrability of the motion on this surface. Such a case has been discussed by Antonov and Shamshiev (1994).

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DISCUSSION

Ossipkov: In connection with Professor Kondrat'ev's question†. I wish to ask you what does Jeans theorem mean for non-integrable system.

†Professor B. Kondrat'ev asked about the possibility of using local integrals as arguments of the distribution function. We have no exact text of the question.

Antonov: The phase density is $f = f(E)$, where E is the energy integral.
Ossipkov: But if the circular orbit is stable but Birkhoff's ergodic zones exist then the distribution function may depend not only on the energy integral, is not it true?
Antonov: When a family of circular orbits exists, it can be considered as an initial (embryonic) form for local integrals.