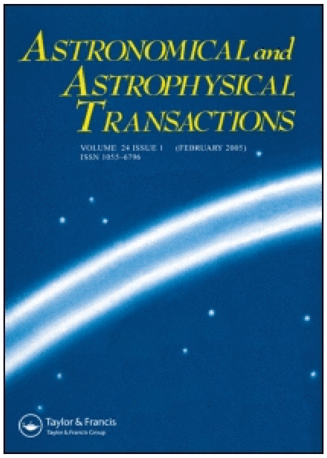

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THE SEARCH FOR EXACT INTEGRALS OF STAR MOTION†

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A brief review of the search for exact analytical integrals of motion for stationary potential fields is presented. New examples and theorems of non-existence are given.

KEY WORDS Integral of motion, potentials, trajectories, regularity

The search for the third integral of motion may seem a common-place question. If astronomer have found two integrals, what interferes to search failing one? In reality, it is a profound and complex problem. We should distinguish between exact and approximate (truncated series, etc.) forms of the third integral. Besides, there are potential fields which never permit even rough forms of the third integral. An exact or an approximate integral is almost the same for the nature, but not for the theory. The theory uses specific methods to search for exact integrals of motion. Many took up the question. A relative crowning was carried out thanks to investigations of Kuzmin [1]. Because it is convenient to classify integrals of motion according to their degree with respect to the velocity components, Kuzmin's integral can be called the integral of the second degree, as well as the classical integral of energy (and the angular momentum is an integral of the first degree). Attempts of concrete applications of Kuzmin's integral to the galaxy structure and dynamics are very numerous, e.g. [2, 3, 4]. However, nothing forces galaxies to have potentials which adhere to Kuzmin's theory. The differences exist undoubtedly. Indeed, we can often model some regions of galaxies with these potentials but not a galaxy as whole. Thus, a search for integrals of a higher degree is expedient. It has been done for the last 20–30 years by many authors from different standpoints, apparently, without a systematic approach. Usually, some form of the integral is chosen and then suitable potentials are deduced. A review is given by Hietarinta [5] (see also [6]).

An important difference arises. Potentials that admit integrals of the second degree form "a continent": they depend on an arbitrary function of an argument.

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Potentials which admit integrals of a higher degree form "islands": they depend on a finite set of parameters only. Some potentials have, nevertheless, characteristic properties and induce an orbital star motion with a more complicated topology than for Kuzmin's integral; thus, they are very important in stellar dynamics. I analyzed certain forms of the integrals of motion to cover the related potential all without exception. This problem leads to distinctive functional equations that yield at happy approaches.

1. An integral of the fourth degree was proposed by Bozis [7] in the following form:

$$J = u^2v^2 + Au^2 + 2Buv + Gv^2 + E, \tag{1}$$

where A, B, G and E are functions of both coordinates x and y and u and v are related velocities. The related two-dimensional potential, as the author notes should look as

$$U = \frac{1}{2}[A(y) + G(x)] + f(x+y) - g(x-y), \tag{2}$$

with a certain form binding of coefficients (1):

$$A = A(y), \quad G = G(x), \quad B = f(x+y) - g(x-y) \tag{3}$$

The four of the basic functions $A(\tau), G(\tau), f(\tau), g(\tau)$ need to have a concrete expression. Bozis gave a particular example afterwards, further ones were found and expressed in terms of elliptical functions.

I deduced that the question is exhausted: a search for other solutions of the functional equations for A, G, f, g, besides the following together with their limiting (elementary) variants were waste. Also A, G, f, g at trivial transformations assume either a form:

$$G(\tau) = A(\tau) = c_1P(\tau) + C_2P(\tau - w) + C_3P(\tau - w') + C_4P(\tau - w - w'),$$

$$f(\tau) = -g(\tau) = C_5P(\tau), \tag{4}$$

or another form,

$$G(\tau) = A(\tau) = C_1P(\tau) + C_2P(\tau - w'), \tag{5}$$

$$f(\tau) = -g(\tau) = C_3[P(\tau/2) + P(\tau/2 - w')] + C_4[P(\tau/2 - w) + P(\tau/2 - w - w')],$$

where P is Weierstrass' elliptical function with any pair of periods $2w, 2w'$ and $C_1, \ldots$ are arbitrary constants. Of course, the potential should be generated as real-valued.

2. An integral of the third degree,

$$J = U^3 - 3uv^2 + A(x,y)u + B(x,y)v, \tag{6}$$

is in a very similar situation. The potential should have following form:

$$U = \varphi_1(x) + \varphi_2(-x/2 + \sqrt{3}y/2) + \varphi_3(-x/2 - \sqrt{3}y/2). \tag{7}$$

The basic solution is

$$\varphi_1(\tau) = \varphi_2(\tau) = \varphi_3(\tau) = P(\tau), \tag{8}$$

besides, as I showed, there are a few limiting and exceptional solutions.

As another approach, we fix the potential itself. A great interest is associated with the one proposed by Contopoulos *et al.*,

$$U = \frac{1}{2}(x^2 + \lambda y^2) + \alpha x^3 + xy^2. \tag{9}$$

Every integral of a finite degree N is thus a polynomial in x, y, u, v. At which values of λ and α does this additional (independent on energy) integral exist? We have solved this question without an upper for N. We begin with an expansion with respect to the total degree on both y and $v = \dot{y}$:

$$J = J_n + J_{n+2} + \ldots + J_n \tag{10}$$

(the terms of the same degree are combined). The apparition of J_n is a necessary stage for forming J as whole, though we took into account that one can stumble at following terms too. The decisive idea is to express J_n in terms of the both particular solutions y_1 and y_2 of the linearized equation for small oscillations,

$$\dot{y} + (2x(t) + \lambda)y = 0, \tag{11}$$

where $x(t)$ describes the motion strictly along the x-axis, i.e. $x(t)$ satisfies the equation

$$\ddot{x} + x + 3\alpha x^2 = 0. \tag{12}$$

Thus,

$$J_n = \sum_{k=0}^{n} q_k (y_1\dot{y} - \dot{y}_1 y)(y_2\dot{y} - \dot{y}_2 y) \quad (q_k = \text{const}) \tag{13}$$

and a main problem concentrates on constructing a single- valued (so long as $x(t)$ is a single valued function) even in the complex plane of t and at every values of the energy constant h for (12), coefficient

$$\sum_{k=0}^{n} q_k y_1^k y_2^{n-k}. \tag{14}$$

The group theory helps to make (14) a single-valued expression at all rounds about singularities of the equation (11). The final result runs that there are three pairs only of parameters λ, α, for which a polynomial integral is permitted:

$$\begin{aligned} \alpha &= 16/3, \quad \lambda = 1/16 \\ \alpha &= 2, \text{ any } \lambda \\ \alpha &= 1/3, \quad \lambda = 1 \end{aligned}$$

All cases were known.

We should note that, in the common case without integrals of a finite degree, invariant surfaces in the phase space do not disappear but acquire a more complicated topology and only keep the isolating property. These chinky invariant tori were investigated by me simultaneously with and independently of Aubry.

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DISCUSSION

Ossipkov: Is the Arnold diffusion possible in two dimensions?
Antonov: Yes, it is possible for some cases of a non-smooth potential.
Ossipkov: As you know, there were attempts to connect the integrability with the Painleve property (that is the case when the only singularity on the complex time plane is a pole) or the weak Painleve property. Now we know that having these properties does not mean integrability. Then what does is mean physically that the corresponding potentials are exceptional?
Antonov: There is a parallelism between the integrability and the Painleve property. However, I know that is not an automatic identity.
Bekov: What can you say concerning the case of a non-stationary potential, and can these results be used for non-autonomous systems?
Antonov: Non-stationary potentials give a too wide manifold and the related integrals of motion are seldom investigated. An example: an analog of Kuzmin's quadratic integral the non-stationary integral of Genkin. I worked the things episodically in some cases.