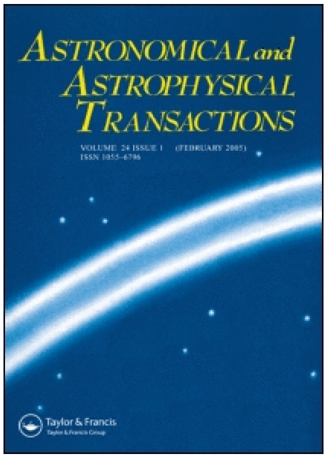

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A. G. Goundyrev; V. A. Razin

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ON THE SYNCHROTRON EMISSION OF A RELATIVISTIC ELECTRON SPIRALING IN RAREFIED PLASMA. I. THE CRITERIA OF THE PLASMA INFLUENCE

A. G. GOUNDYREV and V. A. RAZIN

Russia, Nizhni Novgorod

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The spectrum of synchrotron emission power is investigated in detail for a relativistic electron in plasma with the refractive index $n^2 = 1 - \omega_{\rm P}^2/\omega^2$. It is shown that a familiar criterion [1–3] of the plasma influence on synchrotron emission, $\omega \leq \omega^* = \omega_{\rm P}^2/(\omega_{\rm H}\sin\theta)$, is valid for $\theta \geq mc^2/E$. If $\theta < mc^2/E$, then it is necessary to take into consideration the plasma influence when $\omega_{\rm P} \geq \omega_{\rm H}$.

1 INTRODUCTION

In applications of the theory of the synchrotron radiation from relativistic electrons, there is sometimes a necessity to take into consideration the influence of a surrounding "cold plasma" on the spectrum, polarization, directivity, reabsorption and propagation of the emission [1–3]. For the electron pitch angles $\theta \gg mc^2/E$, (m is the electron mass, c is the velocity of light and E is the electron energy) in the case of a rarefied plasma and weak magnetic field, this effect may be investigated by introducing the factor [1–3]:

$$q = 1 + (1 - n^2)(E/mc^2)^2 \tag{1}$$

in the equations that are true in vacuum. Setting

$$n^2 = 1 - (\omega_{\rm P}^2/\omega^2) \tag{2}$$

(ω is the cyclic frequency of emission, $\omega_{\rm P} = (4\pi Ne^2/m)^{1/2}$ is the plasma frequency, N is the electron concentration in plasma and e is the electron charge), one finds:

$$q = 1 + (\omega_{\rm P}^2/\omega^2)(E/mc^2)^2 = 1 + (\omega_{\rm P}^2/\omega^2)\gamma^2. \tag{3}$$

Here $\gamma = E/mc^2$ is the relativistic Lorenz factor. When $\omega_{\mathrm{P}} = 0$ (emission in vacuum), we have $n = 1$ and $q = 1$. If $\omega_{\mathrm{P}} \neq 0$, then $q > 1$ and the plasma influence is essential. Supposing for definitiveness that $q = 2$, one has:

$$\omega^* = \omega_{\mathrm{P}}\gamma. \tag{4}$$

If the electron radiates mainly at frequencies $\omega \leq \omega^*$, the plasma influence will be essential. Synchrotron radiation from relativistic electrons in vacuum is known to have a wide spectrum with emission power maximum W_ω at the frequency [4]:

$$\omega_{\max} \sim \omega_{\mathrm{H}} \sin\theta\gamma^2, \tag{5}$$

where $\omega_{\mathrm{H}} = eH/mc$ is the electron gyrofrequency and H is the magnetic field strength. For $\omega \leq \omega_{\max}$, the power radiated is $W_\omega \propto \omega^{1/3}$. At frequencies $\omega \geq \omega_{\max}$, the power decreases quasi-exponentially, $W_\omega \propto \omega^{1/2}\exp(-2\omega/3\omega_{\max})$. Setting $\omega = \omega_{\max} = \omega^*$, one finds:

$$\gamma^2 = \frac{\omega_{\max}}{\omega_{\mathrm{H}}\sin\theta}, \tag{6}$$

and using (4):

$$\omega^* = \frac{\omega_{\mathrm{P}}^2}{\omega_{\mathrm{H}}\sin\theta} \approx 2\pi\cdot 30\frac{N[\mathrm{cm}^{-3}]}{H\sin\theta[\mathrm{Gs}]} \tag{7}$$

Thus at frequences

$$\omega \leq \omega^* \tag{8}$$

the surrounding plasma influence on the synchrotron emission is essential. In particular, a quasi-exponential cutoff occurs at these frequences [1–3].

A question arises about the limits of the criterion (8) applicability when the pitch angle is decreasing, because in the case when $\theta \leq mc^2/E$ the characteristics of synchrotron emission are radically changing. To examine this question, it is necessary to solve the problem of synchrotron emission of a relativistic electron gyrating in plasma with arbitrary pitch angle. This problem was considered repeatedly and general results were derived (see, for example [5–9]). However, the simplest way to derive the limits of the criterion applicability is to suppose, as in [1–3], that the plasma is rarefied and the magnetic field is weak; this will be done in the present paper.

The energy radiated will be derived by calculating the work done by the radiation friction force against the electron: $\mathbf{f}\cdot\mathbf{v} = e\cdot(\mathbf{v}\cdot\mathbf{E}) = \mathbf{j}\cdot\mathbf{E}$, where $\mathbf{E}$ is the electric field strength at the electron location, $\mathbf{v}$ is the electron velocity and $\mathbf{j}$ is the current density produced by the electron [6, 7, 9, 10]. Thus, one has to calculate the electric field $\mathbf{E}$, when the electron spirals in a rarefied plasma, and to obtain $\mathbf{j}\cdot\mathbf{E}$.

2 THE SYNCHROTRON RADIATION OF A RELATIVISTIC ELECTRON IN A RAREFIED PLASMA IN THE CASE OF A WEAK MAGNETIC FIELD

Let us assume the following conditions [1–3]: for the plasma refractive index, $n(\omega) < 1$, $1 - n(\omega) \ll 1$, and $(\omega_{\mathrm{H}}/\omega) \ll 1$. The latter inequality means that the plasma is isotropic. Under these conditions, the plasma refractive index is given by equation (2). Furthemore, we assume that 1) $n(\omega)$ does not depend on position and time; 2) the external magnetic field $\mathbf{H}_0$ is homogeneous and its value is constant; 3) the relativistic electron energy decreases very slowly with the electromagnetic wave emission and it is possible to suppose that it is constant during the period of the electron gyration in the magnetic field. Taking into account the frequency dispersion of the plasma and the superposition principle, we have the following form of Maxwell's equations for the harmonic time dependent $\exp(-i\omega t)$ electric ($\mathbf{E}$) and magnetic ($\mathbf{H}$) radiation fields (the plasma's magnetic permeability is set equal to unity):

$$\operatorname{rot}\mathbf{H} = \frac{4\pi}{c}\mathbf{j} - i\frac{n^2}{c}\omega\mathbf{E}, \tag{9}$$

$$\operatorname{rot}\mathbf{E} = i\frac{\omega}{c}\mathbf{H}\,, \tag{10}$$

$$\operatorname{div}\mathbf{E} = \frac{4\pi\rho}{n^2}\,, \tag{11}$$

$$\operatorname{div}\mathbf{H} = 0\,. \tag{12}$$

In equations (9) – (11),

$$\rho = e\cdot\delta(\mathbf{r}-\mathbf{r}_e) \tag{13}$$

is the charge density, and

$$\mathbf{j} = e\mathbf{v}\cdot\delta(\mathbf{r}-\mathbf{r}_e) \tag{14}$$

is the current density.

Here δ is Dirac's delta–function, $\mathbf{r}_e$ is the vector of the electron position, $\mathbf{v} = \partial\mathbf{r}_e/\partial t$ is the electron velocity. When the electron spirals in a homogeneous magnetic field $\mathbf{H}_0$, it is reasonable to set:

$$\mathbf{r} = (r_0\cos\omega t, r_0\sin\omega t, v_{\parallel}t), \tag{15}$$

$$\mathbf{v} = (-v_{\perp}\sin\omega t, v_{\perp}\cos\omega t, v_{\parallel}), \tag{16}$$

where $r_0 = v_{\perp}/\omega_g$ is the radius of the spiral; $\omega_g = \omega_{\mathbf{H}}/\gamma$, $v_{\perp}$ is the electron velocity component perpendicular to $\mathbf{H}_0$, and $v_{\parallel}$ is the electron velocity component parallel to $\mathbf{H}_0$. Obviously, $v_{\perp} = v\sin\theta$ and $v_{\parallel} = v\cos\theta$ ($v = |\mathbf{v}|$, θ is the angle between $\mathbf{v}$ and $\mathbf{H}_0$, the pitch angle).

To solve the system of linear equations (9)–(12), we shall present $\mathbf{E}(\mathbf{r},t)$, $\mathbf{H}(\mathbf{r},t)$, $\mathbf{j}(\mathbf{r},t)$ and $\rho(\mathbf{r},t)$ as a superposition of plane waves (Fourier integrals) [6, 7, 9–12]. For example:

$$\mathbf{E}(\mathbf{r},t) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\mathbf{k}}{(2\pi)^3}\,\mathbf{E}(\mathbf{k},\omega)\exp[i(\mathbf{kr}-\omega t)], \tag{17}$$

$$\mathbf{E}(\mathbf{k},\omega) = \int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}d\mathbf{r}\,\mathbf{E}(\mathbf{r},t)\exp[-i(\mathbf{kr}-\omega t)] \tag{18}$$

and so on.

Here $\mathbf{k}$ is the wave vector, $d\mathbf{k} = dk_x dk_y dk_z$ and $d\mathbf{r} = dxdydz$. Because $\mathbf{E}(\mathbf{r}, t)$ is real, one has: $\mathbf{E}(-\mathbf{k}, -\omega) = \mathbf{E}^*(\mathbf{k}, \omega)$. From Maxwell's equations for the Fourier components of $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{H}(\mathbf{r}, t)$, $\mathbf{j}(\mathbf{r}, t)$ and $\rho(\mathbf{r}, t)$ one finds a system of algebraic equations:

$$\mathbf{k} \times \mathbf{H}(\mathbf{k}, \omega) = -\frac{4\pi i}{c}\mathbf{j}(\mathbf{k}, \omega) - n^2 \frac{\omega}{c}\mathbf{E}(\mathbf{k}, \omega), \tag{19}$$

$$\mathbf{k} \times \mathbf{E}(\mathbf{k}, \omega) = \frac{\omega}{c}\mathbf{H}(\mathbf{k}, \omega), \tag{20}$$

$$\mathbf{k} \cdot \mathbf{E}(\mathbf{k}, \omega) = -\frac{4\pi i \rho(\mathbf{k}, \omega)}{n^2}, \tag{21}$$

$$\mathbf{k} \cdot \mathbf{H}(\mathbf{k}, \omega) = 0. \tag{22}$$

Inserting $\mathbf{H}(\mathbf{k}, \omega)$ from (20) into (19) and opening the double vector product, we obtain the following equation:

$$\mathbf{k} \cdot (\mathbf{k} \cdot \mathbf{E}(\mathbf{k}, \omega)) - \mathbf{k}^2 \cdot \mathbf{E}(\mathbf{k}, \omega) + \frac{\omega^2}{c^2} n^2 \cdot \mathbf{E}(\mathbf{k}, \omega) = -\frac{4\pi i \omega}{c^2}\mathbf{j}(\mathbf{k}, \omega). \tag{23}$$

A scalar product of this equation and the wave vector $\mathbf{k}$ yields

$$\mathbf{k} \cdot \mathbf{E}(\mathbf{k}, \omega) = -\frac{4\pi i}{\omega n^2}(\mathbf{k} \cdot \mathbf{j}(\mathbf{k}, \omega)). \tag{24}$$

Relations (23) and (24) imply:

$$\begin{aligned} \mathbf{E}(\mathbf{k}, \omega) &= \left[\frac{4\pi i}{\omega n^2}\mathbf{k} \cdot (\mathbf{k} \cdot \mathbf{j}(\mathbf{k}, \omega)) - \frac{4\pi i \omega}{c^2}\mathbf{j}\right] \left(\frac{\omega^2}{c^2} n^2 - \mathbf{k}^2\right)^{-1} \\ &= \frac{4\pi i}{\omega}\left[\mathbf{j}_\perp(\mathbf{k}, \omega) + \mathbf{j}_\parallel\left(1 - \frac{c^2 k^2}{\omega^2 n^2}\right)\right]\left(\frac{c^2 k^2}{\omega^2} - n^2\right)^{-1}, \end{aligned} \tag{25}$$

where $\mathbf{j}_\perp(\mathbf{k}, \omega)$ is the current density component perpendicular to $\mathbf{k}$ and $\mathbf{j}_\parallel$ is the current density component parallel to $\mathbf{k}$. As follows from (25), the electric field $\mathbf{E}$ has two independent components. One is perpendicular to the wave vector $\mathbf{k}$:

$$\mathbf{E}_\perp(\mathbf{k}, \omega) = \frac{4\pi i}{\omega}\mathbf{j}_\perp(\mathbf{k}, \omega)\left(\frac{c^2 k^2}{\omega^2} - n^2\right)^{-1} \tag{26}$$

(transverse waves) and another is parallel to $\mathbf{k}$:

$$\mathbf{E}_\parallel(\mathbf{k}, \omega) = \frac{4\pi i}{\omega n^2}\mathbf{j}_\parallel(\mathbf{k}, \omega) \tag{27}$$

(longitudinal waves) (see also eq. (24)).

The frequency of the longitudial waves $\omega = \omega_P$ can be found from the dispersion relation $n^2 = 0$;† the plasma wave energy can be neglected in the case considered

†More precisely, one may refer to plasma oscillations only in the case of a "cold" plasma because the frequency does not depend on the wave vector $\mathbf{k}$ and the group velocity vanishes, $\partial\omega/\partial\mathbf{k} = 0$.

[6]. For this reason in the following we shall describe transverse electromagnetic waves $\mathbf{E}_\perp(\mathbf{k},\omega)$ only, the dispersion relation for which is $c^2k^2 = \omega^2 n^2(\omega)$ (see (26)).

The average power of radiation generated by the electron is:

$$W = \lim_{T\to\infty} \frac{1}{T} \int\limits_{-T/2}^{T/2} dt \int d\mathbf{r}(\mathbf{j}_\perp(\mathbf{r},t)\cdot\mathbf{E}_\perp(\mathbf{r},t)) \tag{28}$$

$$= \lim_{T\to\infty} \frac{1}{T} \int\limits_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^3} (\mathbf{j}_\perp(\mathbf{k},\omega)\cdot\mathbf{E}^*_\perp(\mathbf{k},\omega)). \tag{29}$$

Using (26), W can be written in the form:

$$W = \lim_{T\to\infty} \frac{1}{T} \int\limits_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^3} \left[-\frac{4\pi i}{\omega} |\mathbf{j}_\perp(\mathbf{k},\omega)|^2 \left(\frac{c^2k^2}{\omega^2} - n^2\right)^{-1} \right]. \tag{30}$$

To perform integration in (30), we use the following standard way: one introduces a small imaginary part of the refractive index. In this case the following equations are true [6]:

$$\mathrm{Re}\left(\frac{-i}{k^2c^2/\omega^2 - n^2}\right) \to \pi\delta\left(\frac{k^2c^2}{\omega^2} - n^2\right). \tag{31}$$

Thus, one has

$$W = \frac{1}{4\pi^2 T} \int\limits_{-\infty}^{+\infty} \frac{d\omega}{\omega} \int d\mathbf{k} |\mathbf{j}_\perp(\mathbf{k},\omega)|^2 \delta\left(\frac{c^2k^2}{\omega^2} - n^2\right). \tag{32}$$

But then (see, e.g., [13])

$$\delta\left(\frac{c^2k^2}{\omega^2} - n^2\right) = \frac{\omega}{c} \frac{\delta[k - (\omega/c)n] + \delta[k + (\omega/c)n]}{2n}; \tag{33}$$

and

$$d\mathbf{k} = dk_x dk_y dk_z = k^2 \sin\alpha d\alpha d\beta dk = k^2 d\Omega dk, \tag{34}$$

where α is the angle between the z-axis (directed along the external magnetic field) and $\mathbf{k}$; β is the angle between the projection of $\mathbf{k}$ on the (XY) plane and the x-axis; and $d\Omega = \sin\alpha d\alpha d\beta$ is the solid angle element. Besides, because positive and negative frequencies are equivalent physically and $\mathbf{j}(-\mathbf{k},-\omega) = \mathbf{j}^*(\mathbf{k},\omega)$, the integral (32) can be replaced by the following integral:

$$W = \frac{1}{4\pi^2 c^3 T} \int\limits_0^{+\infty} \omega^2 n(\omega) d\omega \int |\mathbf{j}_\perp(\mathbf{k},\omega)|^2_{k=(\omega/c)n} d\Omega. \tag{35}$$

Here the subscript $k = (\omega/c)n$ at the current modulus squared means that it is necessary to set $|\mathbf{k}|$ equal to $(\omega/c)n$.

Let us now clarify the character of a current $\mathbf{j}(\mathbf{r}, t)$, producing the electromagnetic emission. Using (14) one finds:

$$\mathbf{j}(\mathbf{k}, \omega) = e \int_{-\infty}^{+\infty} dt\, \mathbf{v}(t) \exp[-i(\mathbf{k}\mathbf{r}_e - \omega t)]. \tag{36}$$

Taking into account the cylindrical symmetry of the emission angular distribution, we shall choose a coordinate system in which $\mathbf{k}$ is parallel to the (YZ) plane. Under this assumption: $k_x = 0$, $k_y = k \sin\alpha$ and $k_z = k\cos\alpha$. Using (36), (15) and (16) we can define the projections of $\mathbf{j}(\mathbf{k}, \omega)$ on the axes:

$$j_x(\mathbf{k}, \omega) = -ev_\perp \int_{-\infty}^{+\infty} dt \sin\omega_g t \exp[-i(x \sin\omega_g t + kv_\parallel t \cos\alpha - \omega t)], \tag{37}$$

$$j_y(\mathbf{k}, \omega) = ev_\perp \int_{-\infty}^{+\infty} dt \cos\omega_g t \exp[-i(x \sin\omega_g t + kv_\parallel t \cos\alpha - \omega t)], \tag{38}$$

$$j_z(\mathbf{k}, \omega) = ev_\parallel \int_{-\infty}^{+\infty} dt \exp[-i(x \sin\omega_g t + kv_\parallel t \cos\alpha - \omega t)], \tag{39}$$

where $x = kr_0 \sin\alpha$.

The following calculations will be made by using the relations [13]:

$$\exp(-ix\sin\varphi) = \sum_{s=-\infty}^{+\infty} J_s(x) \exp(-is\varphi), \tag{40}$$

$$\sin\varphi \exp(-ix\sin\varphi) = i \sum_{s=-\infty}^{+\infty} J_s'(x) \exp(-is\varphi), \tag{41}$$

$$\cos\varphi \exp(-ix\sin\varphi) = \sum_{s=-\infty}^{+\infty} \frac{s}{x} J_s(x) \exp(-is\varphi), \tag{42}$$

$$\int_{-\infty}^{+\infty} dt \exp(i\omega t) = 2\pi\delta(\omega), \tag{43}$$

where $J_s(x)$ and $J_s'(x)$ are the Bessel function and its derivative with respect to x, respectively. After a simple reduction one finds:

$$j_x(\mathbf{k}, \omega) = -2\pi i e v_\perp \sum_{s=-\infty}^{+\infty} J_s'(x)\delta(y), \tag{44}$$

$$j_y(\mathbf{k},\omega) = 2\pi e v_\perp \sum_{s=-\infty}^{+\infty} (s/x) J_s(x)\delta(y), \tag{45}$$

$$j_z(\mathbf{k},\omega) = 2\pi e v_\parallel \sum_{s=-\infty}^{+\infty} J_s(x)\delta(y), \tag{46}$$

$x = kr_0 \sin\alpha$, $y = \omega - s\omega_g - kv_\parallel \cos\alpha$.

The electric vector $\mathbf{E}_\perp(\mathbf{k},\omega)$ can be written as a sum of two orthogonal linearly polarized components: the parallel and perpendicular to the (YZ) plane, i.e., as $\mathbf{e}_\alpha E_{\perp\alpha}(\mathbf{k},\omega) + \mathbf{e}_\beta E_{\perp\beta}(\mathbf{k},\omega)$, where $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ are unit vectors corresponding to the directions of electric field polarizations. Analogously, $\mathbf{j}_\perp(\mathbf{k},\omega)$ has two components:

$$\mathbf{j}_\perp(\mathbf{k},\omega) = \mathbf{e}_\beta j_x(\mathbf{k},\omega) + \mathbf{e}_\alpha \left[j_y(\mathbf{k},\omega)\cos\alpha - j_z(\mathbf{k},\omega)\sin\alpha\right], \tag{47}$$

$$|\mathbf{j}_\perp(\mathbf{k},\omega)|^2 = |j_x(\mathbf{k},\omega)|^2 + |j_y(\mathbf{k},\omega)\cos\alpha - j_z(\mathbf{k},\omega)\sin\alpha|^2. \tag{48}$$

Using (44)-(46), we may rewrite $\mathbf{j}_\perp(\mathbf{k},\omega)$ and $|\mathbf{j}_\perp(\mathbf{k},\omega)|^2$ in the form

$$\mathbf{j}_\perp(\mathbf{k},\omega) = 2\pi e \sum_{s=-\infty}^{+\infty} \left[\mathbf{e}_\beta(-iv_\perp J_s'(x)) + \mathbf{e}_\alpha(v_\perp \frac{s}{x}\cos\alpha - v_\parallel \sin\alpha) J_s(x)\right]\delta(y); \tag{49}$$

$$|\mathbf{j}_\perp(\mathbf{k},\omega)|^2 = \frac{4\pi^2 e^2 T}{2\pi} \sum_{s=-\infty}^{+\infty} \left[(v_\perp J_s'(x))^2 + (v_\perp \frac{s}{x}\cos\alpha - v_\parallel \sin\alpha)^2 J_s^2(x)\right]\delta(y), \tag{50}$$

where the following operator equality [7, 9] was used:

$$\delta^2(y) = \frac{T}{2\pi}\delta(y)$$

(here T is the total time of the electron emission).

Using (35), (49) and (50) we have:

$$W = \sum_{s=-\infty}^{+\infty} W_s \tag{51}$$

where $W_s = W_s^\alpha + W_s^\beta$, W_s^α and W_s^β are the emission powers associated with the components of $\mathbf{E}(\mathbf{k},\omega)$ polarized in the directions $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$, respectively. Thus,

$$W_s = \frac{e^2}{2\pi c^3} \int \omega^2 n(\omega) \left[(v_\perp J_s'(x))^2 + (v_\perp \frac{s}{x}\cos\alpha - v_\parallel \sin\alpha)^2 J_s^2(x)\right]\delta(y)\, d\omega\, d\Omega. \tag{52}$$

Here

$$x = kr_0 \sin\alpha = \frac{\sin\beta_\perp \sin\alpha}{1 - \beta_\parallel n\cos\alpha}, \tag{53}$$

$$y = \omega(1 - \beta_\parallel n \cos\alpha) - s\omega_g, \tag{54}$$

$$v_\perp \frac{s}{x} \cos\alpha - v_\| \sin\alpha = \frac{c(\cos\alpha - \beta_\| n)}{n \sin\alpha}, \tag{55}$$
$$\beta_\| = v_\|/c\ ,\ \beta_\perp = v_\perp/c\ ,\ k = (\omega/c)n.$$

It is easy to perform integration over frequencies in (52). One rewrites $\delta(y)$ in the form [13]:

$$\delta(y) = \frac{\delta(\omega - \omega_1)}{|y'(\omega_1)|} + \frac{\delta(\omega - \omega_2)}{|y'(\omega_2)|},$$

where y' denotes the derivative of y with respect to ω, and ω_1 and ω_2 are the roots of equation $y = 0$:

$$\omega_{1,2} = \frac{s\omega_g}{2(1 - \beta_\| \cos\alpha)} \left[1 \mp \left(1 - \frac{2\beta_\| \omega_{\rm P}^2 \cos\alpha}{s^2\omega_g^2}(1 - \beta_\| \cos\alpha)\right)^{1/2}\right]. \tag{56}$$

If the expression under the root tends to unity, then:

$$\omega_1 \approx \frac{\beta_\| \omega_{\rm P} \cos\alpha}{2s\omega_g},$$

$$\omega_2 \approx \frac{s\omega_g}{1 - \beta_\| \cos\alpha} - \frac{\beta_\| \omega_{\rm P}^2 \cos\alpha}{2s\omega_g}. \tag{56'}$$

Thus,

$$\begin{aligned} W_s &= \frac{e^2 s^2 \omega_g^2 n^2(\omega_2)}{2\pi c(1 - \beta_\| \cos\alpha)^2 |n(\omega_2) - \beta_\| \cos\alpha|} \\ &\times \left\{\beta_\perp^2 J_s'^2(x) + \frac{(\cos\alpha - \beta_\| n)^2}{n^2 \sin^2\alpha} J_s^2(x)\right\} d\Omega. \end{aligned} \tag{57}$$

Here the facts that $|y'(\omega_2)| = |n - \beta_\| \cos\alpha|/n$ and $\omega_1 \ll \omega_{\rm P}$ were taken into account. With an error not exceeding $\omega_{\rm P}^2/\omega^2$, (57) may be rewritten in the form:†

$$W_s = \frac{e^2 s^2 \omega_g^2}{2\pi c \sin^2\alpha (1 - \beta_\| n \cos\alpha)} \left[x_1^2 J_s'^2(sx_1) + z^2 J_s^2(sx_1)\right] d\Omega, \tag{57'}$$

where x_1 is the value of x for $s = 1$.

$$z = \frac{\cos\alpha - \beta_\| n}{1 - \beta_\| n \cos\alpha}, \tag{58}$$

$$\omega = \frac{s\omega_g}{1 - \beta_\| n \cos\alpha}. \tag{59}$$

This result can be also easily derived if we present $\mathbf{E}(\mathbf{k}, \omega)$ as a sum of two circularly polarized waves. Let us introduce the following orthogonal complex vectors:

$$\mathbf{e}_+ = (\mathbf{e}_\alpha + i\mathbf{e}\beta)/2^{1/2}, \tag{60}$$

†Eq. (57′) can be obtained directly from (52) if we assume that $n = \text{const}$.

$$\mathbf{e}_- = (\mathbf{e}_\alpha - i\mathbf{e}_\beta)/2^{1/2}. \tag{61}$$

Vector $\mathbf{e}_+$ corresponds to the wave with the left polarization (if one looks towards the wave, the electric vector turns counter-clockwise) and $\mathbf{e}_-$ corresponds to the wave with the right polarization. As follows from (60) and (61),

$$\mathbf{e}_\alpha = (\mathbf{e}_+ + \mathbf{e}_-)/2^{1/2}, \tag{62}$$

$$\mathbf{e}_\beta = (\mathbf{e}_+ - \mathbf{e}_-)/2^{1/2}. \tag{63}$$

Expression (49) is transformed, with the help of (62) and (63), to the form:

$$\begin{aligned} \mathbf{j}_\perp(\mathbf{k},\omega) &= \sum_{s=-\infty}^{+\infty} \{\mathbf{e}_+ \cdot [-v_\perp J_s'(x) + ((v_\perp s/x)\cos\alpha - v_\parallel \sin\alpha)\, J_s(x)] \\ &+ \mathbf{e}_- \cdot [v_\perp J_s'(x) + ((v_\perp s/x)\cos\alpha - v_\parallel \sin\alpha)\, J_s(x)]\} \times \delta(y). \end{aligned} \tag{64}$$

Using (35), one has:

$$\begin{aligned} W_s^\pm &= \frac{e^2 d\Omega}{4\pi c^3} \int_0^\infty \omega^2 n(\omega) \left[\mp v_\perp J_s'(x) + (v_\perp \frac{s}{x}\cos\alpha - v_\parallel \sin\alpha) J_s\right]^2 \\ &\times \delta\left[\omega(1 - \beta_\parallel n \cos\alpha) - s\omega_g\right] d\omega \end{aligned} \tag{65}$$

where $W_s^\pm$ is the emission power of the harmonics with the left and right polarization. Obviously:

$$W_s = W_s^+ + W_s^- \tag{66}$$

and we have again expressions (52), (57) and (57′).

A few words about the summation over s in (52) are timely. As it was pointed out, positive and negative frequences are physically equivalent. Besides, $J_{-s}(x) = (-1)^s J_s(x)$ and the formulae discussed contain these functions squared, hence for the terms with $s > 0$ we have equal terms with $s < 0$. Hence:

$$W = \sum_{s=-\infty}^{+\infty} W_s = W_{s=0} + 2\sum_{s=1}^{+\infty} W_s. \tag{67}$$

We should mention that in the case considered ($n(\omega) < 1$) $1 - \beta_\parallel n \cos\alpha$ cannot vanish. When $\omega > 0$ and $s = 0$, the argument of the δ-function in (52) also cannot be zero, and hence $W_{s=0} = 0$.†

Taking into account these remarks, we may rewrite (57′) in the form (see [7, 9]):

$$W_s = \frac{e^2 s^2 \omega_g^2}{\pi c \sin^2\alpha(1 - \beta_\parallel n \cos\alpha)} \left[x_1^2 J_s'^2(sx_1) + z^2 J_s^2(sx_1)\right] d\Omega, \tag{68}$$

where $s = 1, 2, 3, \ldots$

†If the electron radiates in a medium with $n > 1$ and $\beta_\parallel n > 1$, then $W_{s=0} \neq 0$ and the Vavilov–Čerenkov emission appears.

3 THE EMISSION SPECTRUM FOR LARGE PITCH ANGLES

To investigate the spectral distribution of synchrotron emission power, we perform the integration of (52) over a solid angle. Due to cylindrical symmetry of the emission angle distribution, one can set $d\Omega = 2\pi \sin\alpha d\alpha$.

We show first how the plasma acts on the synchrotron emission for large pitch angles ($\theta \sim 1$). In this case, one can use the following asymptotic form of the Bessel functions [18, 19]:

$$J_s(s + \zeta s^{1/3}) \approx (2/s)^{1/3} Ai(-2^{1/3}\zeta), \tag{69}$$

$$J_s'(s + \zeta s^{1/3}) \approx -(2/s)^{2/3} Ai'(-2^{1/3}\zeta). \tag{70}$$

Here

$$Ai(x) = \frac{1}{\pi} \int\limits_0^{+\infty} \cos(t/3 + xt)\, dt \tag{71}$$

is the Airy function† of first kind and $s \gg 1$; the prime denotes a derivative with respect to the argument. Let us replace the variable x by introducing a new variable ζ in the following way: $s + \zeta s^{1/3} = sx_1$ and $\zeta = -s^{2/3}(1 - x_1)$. Now

$$J_s(sx_1) \approx (2/s)^{1/3} Ai\left[2^{1/3} s^{2/3}(1 - x_1)\right], \tag{69'}$$

$$J_s'(sx_1) \approx -(2/s)^{2/3} Ai'\left[2^{1/3} s^{2/3}(1 - x_1)\right]. \tag{70'}$$

We shall set x_1 equal to (53) for $s = 1$, and use (69′) and (70′) for calculations with the help of (68). To simplify calculations, we introduce the instantaneous coordinate system (x', y', z'), so that $x' = x$, y' is parallel to the electron velocity at $t = 0$. In the new coordinate system, angle α is replaced by $\alpha' = \pi/2$, where ξ is the angle between $\mathbf{k}$ (wave vector) and $\mathbf{v}$ (electron velocity) ($\xi \ll 1$). Further, $v_\parallel' = v_z' = 0$, $v_\perp' = v_y' = v$; $x_1 \to n\beta\cos\xi$, $z \to \sin\xi \approx \xi$, $\omega_g \to \omega_0 = \omega_g \sin\theta$, $\omega = s\omega_0$, and we obtain the modified Shott formula, allowing for the influence of the surrounding plasma on synchrotron radiation:

$$W_s = \frac{e^2\omega^2}{2\pi c}\left[J_s'^2(sn\beta\cos\xi) + \xi^2 J_s^2(sn\beta\cos\xi)\right] d\Omega. \tag{72}$$

Using (69′) and (70′), one finds from (72):

$$W_s \approx \frac{2e^2}{\pi^2 c}\omega_0^2 \cdot (\omega/\omega_0)^{2/3}\left[\nu'^2(\tau) + \eta^2\nu^2(\tau)\right] d\Omega, \tag{73}$$

†The Macdonald functions $K_\nu(x)$, related to the Airy functions are also widely used [18, 19]:

$$Ai(\tau) = \frac{1}{\pi}(\tau/3)^{1/2} K_{1/3}(2\tau^{3/2}/3),$$

$$Ai'(\tau) = -\frac{1}{\pi}(\tau/3^{1/2}) K_{2/3}(2\tau^{3/2}/3).$$

where as in [19] notation $v(\tau) = Ai(\tau) \cdot \pi^{1/2}$ is introduced and

$$\tau = 2^{1/3}(\omega/\omega_0)^{2/3} \cdot (1 - n\beta \cos\xi) \ , \ \eta = \xi \cdot (\omega/2\omega_0)^{1/3}.$$

Further, we obtain for

$$\tau = 2^{1/3}(\omega/\omega_0)^{2/3} \cdot (1-\beta n+\xi^2/2) = 2^{-2/3}(\omega/\omega_0)^{2/3} \cdot ((mc^2/E) \cdot q + \zeta^2) = q\chi + \eta^2,$$

$$\chi = (\omega/2\omega_1)^{2/3} \ , \ \omega_1 = (eH_{0\perp}/mc) \cdot (E/mc^2)^2.$$

Here $H_{0\perp} = H_0 \sin\theta$ and $q = 1 + (1-n^2)(E/mc^2)^2$.

With the new notation (73) reduces to

$$W_s \approx \frac{2e^2}{\pi^2 c}\omega_0^2 \cdot (\omega/2\omega_0)^{2/3} \left[\nu'^2(q\chi + \eta^2) + \eta^2\nu^2(q\chi + \eta^2)\right] d\Omega. \qquad (74)$$

Let us perform the integration of (74) over the solid angle $d\Omega = 2\pi d\xi$, taking into account that the range of essential values of W_s is restricted by very small values of ξ. Besides one replaces the integration variable ξ by η. Due to the Doppler effect, every harmonic number is connected with frequency range $(\Delta\omega)_\xi \approx \omega_g\gamma^2 \cos\theta/\sin^3\theta$ ("the line width")† when the angle ξ is changing within the beam $\sim mc^2/E = \gamma^{-1}$. On the other hand, the change of frequency due to a change of the harmonic number s to $s+1$ is: $(\Delta\omega)_s \approx \omega_g/\sin^2\theta$.

The result is $(\Delta\omega)_\xi/(\Delta\omega)_s \approx \gamma^2 \cot\theta$. When $\theta \sim 1$, this ratio is of order γ^2 ($\gg 1$), i.e. the spectral lines overlap. Hence, it is reasonable to sum (74) over s in the range $d\omega$ and instead of W_s we shall have the spectral density of the power emitted, W_ω. That is, $W_s ds = W_\omega d\omega$, and $ds = d\omega/\omega_0$, one has $W_\omega = W_s/\omega_0$. Finally, one has the following equation for the spectrum of the power emitted:

$$W_\omega = \frac{8e^2}{\pi c}\omega_0 \cdot (\omega/2\omega_0)^{1/3} I(q\chi), \qquad (75)$$

where

$$I(q\chi) = \int_0^{+\infty} \left\{\left[\nu'(q\chi + \eta^2)\right]^2 + \eta^2 \left[\nu(q\chi + \eta^2)\right]^2\right\} d\eta. \qquad (76)$$

These formulae were derived earlier in [1–3] and were investigated in detail in [16]. When $q = 1$, it is transformed to Vladimirsky's formulae [15].

It is convinient to rewrite (75) as a function of the parameter $f = 2(q\chi)^{3/2} = f_1 q^{3/2}$, $f_1 = \omega/\omega_1$.

$$W_\omega = \frac{8}{\pi}\frac{e^3 H_{0\perp}}{mc^2} q^{-1/2} Y(f), \qquad (77)$$

$$Y(f) = (f/2)^{1/3} I\left[(f/2)^{2/3}\right] \qquad (78)$$

†The relative "width of line" is $(\Delta\omega)_\xi/\omega \approx \gamma^{-1}\cot\theta$.

is a universal function which characterizes the power spectrum of an ultrarelativistic electron moving in magnetic field [15]. When $q = 1$, in the limit cases of low and high frequencies $Y(f)$ can be approximated as [15]:

$$0.256(f_1/2)^{1/3} \ (f_1 \ll 1); \tag{79}$$

$$\frac{1}{16}(\pi f_1)^{1/2} e^{-(2f_1/3)} \ (f_1 \gg 1). \tag{80}$$

Thus, we have derived a well known result: at low frequencies the synchrotron emission intensity increases with frequency as $\omega^{1/3}$; at high frequencies, it decreases quasi-exponentially. The functions $Y(f)$ and $I(\chi)$ were calculated in [15]. $Y(f)$ has a maximum for $f_1 \approx 0.5$, $Y(0.5) \approx 0.1$. Hence a maximum of synchrotron emission in vacuum occurs at frequencies $\nu = \omega/2\pi$ around

$$\nu_{\max} \approx 0.5(\omega_1/2\pi) \approx 1.4 \cdot 10^6 H_{0\perp}(E/mc^2)^2. \tag{81}$$

Here ω is expressed in Hz and H, in Gs. The power at the emission maximum in vacuum is

$$W_{\nu_{\max}} = 2\pi W_{\omega_{\max}} \approx 3 \cdot 10^{-22} H_{0\perp}[\text{erg}\, s^{-1} Hz^{-1}]. \tag{82}$$

When $q > 1$ (i.e., when the plasma influence cannot be neglected), the synchrotron spectrum changes radically. The spectral distribution of emission is determined, mainly, by the function $Y(f)$. At the frequency [2, 16]:

$$\omega_{f_{min}} = 2^{1/2}\omega_P \gamma, \tag{83}$$

the parameter f has a minimum value:

$$f_{\min} = \frac{3^{3/2}}{2}\frac{\omega_P}{\omega_1}\gamma \approx \frac{\omega_{f_{\min}}}{\omega_{\max}}, \tag{84}$$

where $\omega_{\max} \approx 0.5\omega_1$ is the frequency at the emission maximum in vacuum. If $f_{\min} \ll 1$ ($\omega^* \ll \omega_1$), then due to the presence of nonrelativistic plasma only low-frequency harmonics (for which $f > 1$) are suppressed. If $(\omega^*/\omega)^2 \gg 1$, then $f \approx (\omega^*/\omega)^2$ and

$$W_\nu \approx \pi^{1/2}\frac{e^3 H_{0\perp}}{mc^2}\exp[-(2/3)\cdot(\omega^*/\omega)^2], \tag{85}$$

i.e., the spectrum sharply cuts off at frequencies $\omega \leq \omega^*$. The total power of the synchrotron emission decreases in this case to

$$\int_0^{\omega^*} W_\omega \, d\omega \approx \frac{1.22}{\pi}\frac{e^3 H_{0\perp}}{mc^2}\omega^* \left(\frac{\omega^*}{\omega_1}\right)^{1/3}. \tag{86}$$

In the frequency range $\omega^* < \omega < 0.5 \cdot \omega_1$ the emission power $W_\omega \propto \omega^{1/3}$, as in vacuum. If $f_{\min} > 1$, then the synchrotron spectrum changes radically also at high frequencies: the emission maximum is shifted to high frequencies ($\sim \omega_{f_{\min}}$) and the emission power is strongly reduced. For $\omega^* > \omega_1$, the parameter $f(\omega) = (\omega/\omega_1)(1+\omega^{*2}/\omega^2)^{3/2}$ can be approximated by two terms of a series obtained by expanding $f(\omega)$ near $\omega = \omega_{f_{\min}} = 2^{1/2}\omega^*$:

$$\begin{aligned} f(\omega) &\approx f(\omega_{f_{\min}}) + (1/2)f''_{\omega}(\omega_{f_{\min}})(\omega - \omega_{f_{\min}})^2 \\ &= \frac{3^{3/2}}{2}\frac{\omega^*}{\omega_1} + \frac{3^{1/2}}{2}\frac{1}{\omega_1\omega^*}(\omega - 2^{1/2}\omega^*)^2. \end{aligned} \tag{87}$$

Using (77), (80) and (87) one has (we replaced the slowly varying factor at the exponent by its value for $\omega = \omega_{f_{\min}}$):

$$\begin{aligned} W_\nu &\approx 3^{1/4}\pi^{1/2}\frac{e^3 H_{0\perp}}{mc^2}(\omega^*/\omega_1)^{1/2}\exp[-3^{1/2}(\omega^*/\omega_1)] \\ &\times \exp\left(-\frac{(\omega - 2^{1/2}\omega^*)^2}{3^{1/2}\omega_1\omega^*}\right). \end{aligned} \tag{88}$$

Thus, when the plasma influence is strong, the synchrotron spectrum becomes quasi-gaussian with the line-half-width $(3^{1/2}\omega_1\omega^*)^{1/2}$ at the e^{-1} level. It is not difficult to show that in this case the total emission power decreases quasi-exponentially with ω^*:

$$W \approx \frac{3^{1/2}e^3 H_{0\perp}}{2mc^2}\omega^* \exp(-3^{1/2}\cdot\omega^*/\omega_1). \tag{89}$$

The spectral power of synchrotron emission of the ensemble of relativistic electrons with a power law energy distribution decreases also quasi-exponentially at frequencies $\omega \leq \omega^*$ [1–3, 16]:

$$W \propto (\omega/\omega^*)^{1-\kappa}\cdot\exp[-3^{1/2}(\omega^*/\omega)] \tag{89'}$$

(κ is the relativistic electron energy spectrum ($N(E) \propto E^{-\kappa}$) exponent).

4 THE EMISSION SPECTRUM FOR SMALL PITCH ANGLES

Consider the synchrotron radiation spectrum for small pitch angles ($\theta \ll 1$). First, perform the integration of (52) over solid angle $d\Omega = 2\pi\sin\alpha d\alpha$, and further perform the sum over s. Introducing the new variable $\mu = \cos\alpha$, one has $d\Omega = -2\pi d\mu$, $\sin\alpha = (1-\mu^2)^{1/2}$,

$$\delta(y) = \delta(-y) = \delta(\omega\beta_{\parallel}n\mu - \omega + s\omega_g) = \frac{\delta(\mu-\mu_0)}{\omega\beta_{\parallel}n},$$

$$\mu_0 = \frac{\omega - s\omega_g}{\omega\beta_{\parallel}n}\ ,\ \beta_{\parallel} \neq 0.$$

After simple calculations one finds the following expression for power emitted in the frequency range $\omega, \omega + d\omega$:

$$W(\omega,\theta)d\omega = \frac{2e^2\omega d\omega}{c\beta\cos\theta}\sum_s\left\{[\beta\sin\theta J_s'(x)]^2 + \frac{(\mu_0 - \beta n\cos\theta)^2}{n^2(1-\mu_0)^2}J_s^2(x)\right\}, \tag{90}$$

where

$$x = \frac{sn\beta\sin\theta(1-\mu_0^2)^{1/2}}{1 - n\beta\cos\theta\cdot\mu_0}. \tag{91}$$

The values of the harmonics number over which the summation runs are determined by the following inequalities:

$$\frac{\omega}{\omega_g}(1 - n\beta\cos\theta) \leq s \leq \frac{\omega}{\omega_g}(1 + n\beta\cos\theta), \tag{92}$$

which follows from $-1 \leq \mu_0 \leq 1$. Finally, inserting μ_0 in (90) one has:

$$\begin{aligned} W(\omega,\theta)d\omega &= \frac{2e^2\omega d\omega}{cn^2\beta\cos\theta}\sum_s\Big\{[n\beta\sin\theta J_s'(x)]^2 \\ &+ \frac{(\omega - s\omega_g - \omega n^2\beta^2\cos^2\theta)^2}{\omega^2 n^2\beta^2\cos^2\theta - (\omega - s\omega_g)^2}J_s^2(x)\Big\}, \end{aligned} \tag{93}$$

$$x = \frac{\tan\theta}{\omega_g}(\omega^2 n^2\beta^2\cos^2\theta - (\omega - s\omega_g)^2)^{1/2}. \tag{94}$$

The frequency range $\Delta\omega$ in (93) is restricted by $\omega_1 \leq \omega \leq \omega_2$, where ω_1 and ω_2 are the roots of the equation $\mu_0 = 1$ for $\alpha = 0$

$$\Delta\omega = \frac{s\omega_g}{1 - \beta\cos\theta}\left(1 - \frac{2\beta\cos\theta\omega_{\mathrm{P}}^2}{s^2\omega_g^2}(1 - \beta\cos\theta)\right)^{1/2}. \tag{95}$$

Formula (93) gives a possibility to calculate the synchrotron emission spectral density from relativistic electrons spiraling at arbitrary velocities and pitch angles in a rarefied plasma with the refractive index $n^2 = 1 - \omega_{\mathrm{P}}^2/\omega^2$.

This formula is rather complicated due to the presence of the Bessel function sum. But in the case of small pitch angles only the Bessel functions of low orders are essential (i.e. some lower harmonics contribute to the emission). If $\theta \leq (mc^2/E) \ll 1$, then retaining only the terms up to second order in θ, one finds:

$$\begin{aligned} W(\omega,\theta)d\omega &= \frac{2e^2}{c}\theta^2\omega d\omega\sum_s\Big\{J_s'^2(x) \\ &+ \frac{[\omega(q + \theta^2\gamma^2) - s\omega_{\mathrm{H}}\gamma]\, J_s^2(x)}{\omega\gamma^2\theta^2\,[2s\omega_{\mathrm{H}}\gamma(q + \theta^2\gamma^2)^{-1} - \omega]}\Big\}, \end{aligned} \tag{96}$$

$$x = \frac{\theta\omega^{1/2}}{\omega_{\mathrm{H}}}\left[2s\omega_{\mathrm{H}}\gamma - \omega(q + \theta^2\gamma^2)\right]^{1/2}, \tag{97}$$

where the parameter $q = 1 + (\omega_P^2/\omega^2)\gamma^2$ is introduced. For $\theta \ll (mc^2/E)$, it is possible to set [18]:

$$J_s(sx) \approx \frac{(sx)^s}{2^s s!}\ ;\ J_s'(sx) \approx \frac{(sx)^{s-1}}{2^s s!}. \tag{98}$$

Obviously, the emission at the first harmonic is dominant in this case and

$$W(\omega,\theta)d\omega = \frac{e^2}{c}\theta^2\omega d\omega\left\{1 - \frac{q\omega}{\omega_H\gamma} + \frac{1}{2}\left(\frac{q\omega}{\omega_H\gamma}\right)^2\right\}. \tag{99}$$

The radiated frequency interval is:

$$\Delta\omega = \omega_2 - \omega_1 = 2\omega_H\gamma\left(1 - (\omega_P/\omega)^2\right)^{1/2}, \tag{100}$$

$W(\omega,\theta) = 0$ beyond this range.

In the absence of plasma we have to put $q = 1$ in (96) and (99), and this yields the formulae obtained in [17].

It is convenient to use for numerical calculation in this case the Wild and Hill's approximation for the Bessel function:

$$\begin{aligned} J_s(sx) &= \frac{\exp(-s/2n_0)}{(2\pi s)^{1/2}}\left[(1-x^2)^{3/2} + \frac{0.5033}{s}\right]^{1/6}, \\ J_s'(sx) &= \frac{\exp(-s/2n_0)}{(2\pi s)^{1/2}x}\left[(1-x^2)^{3/2} + \frac{1.193}{s}\right]^{-1/6}\left(1 - \frac{1}{5s^{2/3}}\right), \end{aligned} \tag{101}$$

where

$$(1/2n_0) = \ln\left[1 + (1-x^2)^{3/2}\right] - \ln x - (1-x^2)^{1/2}.$$

This approximation has the accuracy 5% for $0 < x < 0.999$ and $s < \infty$ [14].

In Figure 1 the emission spectra corresponding to individual harmonics in vacuum and their sum are presented in the case $\theta = mc^2/E = 10^{-3}$. Calculations were made with the help of (93) and (101).

It can be seen that really a wide frequency range between ω_1 and ω_2 is wider than separation between two adjacent harmonics for any harmonics and we have a continuous spectrum. In Figures 2–6, the emission spectra for $\theta = 5mc^2/E$, $\theta = 2mc^2/E$, $\theta = mc^2/E$, $\theta = 0.5mc^2/E$ and $\theta = 0.1mc^2/E$ ($mc^2/E = 10^{-3}$) are presented for vacuum and plasma and different plasma frequencies. When the pitch angle is sufficiently small ($\theta \le 0.1mc^2/E$; so that the inequality $\theta\gamma \ll 1$ is true), then as we have already mentioned, the emission occurs, mainly, at the first harmonics and the spectrum falls off at the frequency

$$\omega = \omega_{\max} = \frac{\omega_H\gamma^{-1}}{1-\beta\cos\theta} \approx \frac{\omega_H\gamma^{-1}}{1-[1-1/(2\gamma^2)](1-\theta^2/2)} = \frac{2\omega_H\gamma}{1+\theta^2\gamma^2} \approx 2\omega_H\gamma \tag{102}$$

(see Figure 6). At low frequencies $\omega \ll \omega_H\gamma$ $W(\omega) \propto \omega$.

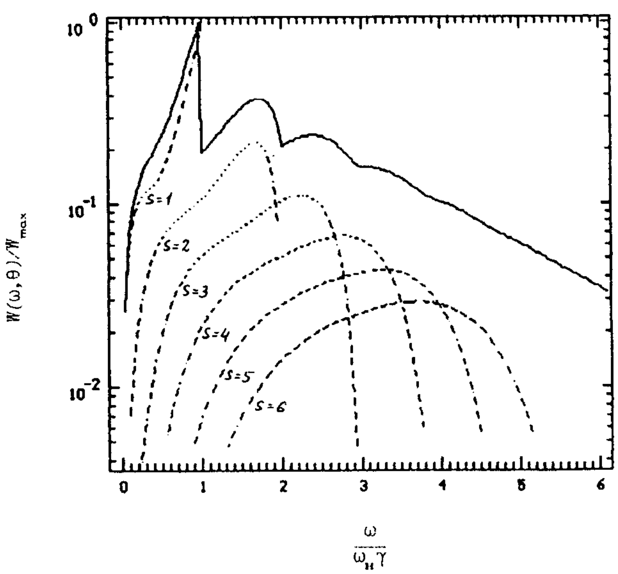

Figure 1 The synchrotron emission power spectral density of the relativistic electron with the pitch angle $\theta = mc^2/E$ in vacuum (solid line). The spectra of separate harmonics ($s = 1, 2, \ldots, 6$) are shown dashed.

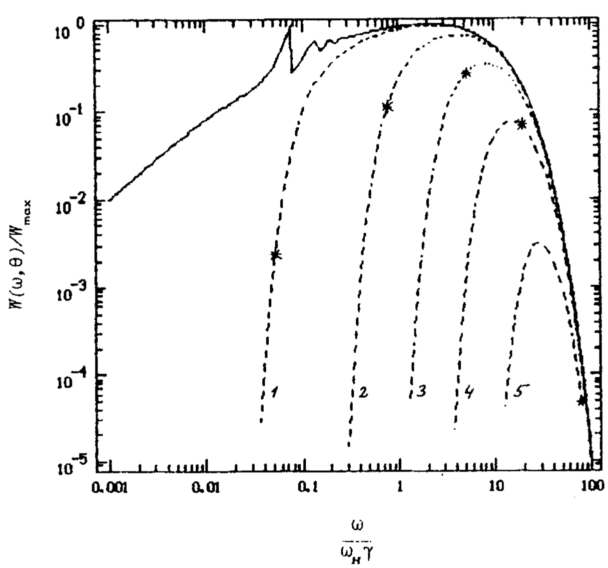

Figure 2 The synchrotron emission power spectral density of the relativistic electron with the pitch angle $\theta = 5mc^2/E$ in vacuum (solid line) and in plasma (dashed lines): 1. $\omega_P = 0.5\omega_H$; 2. $\omega_P = 2\omega_H$; 3. $\omega_P = 5\omega_H$; 4. $\omega_P = 10\omega_H$; 5. $\omega_P = 20\omega_H$.
The frequency $\omega^* = \omega_P^2/(\omega_H \sin\theta)$ is marked here and in the following figures by asterisk.

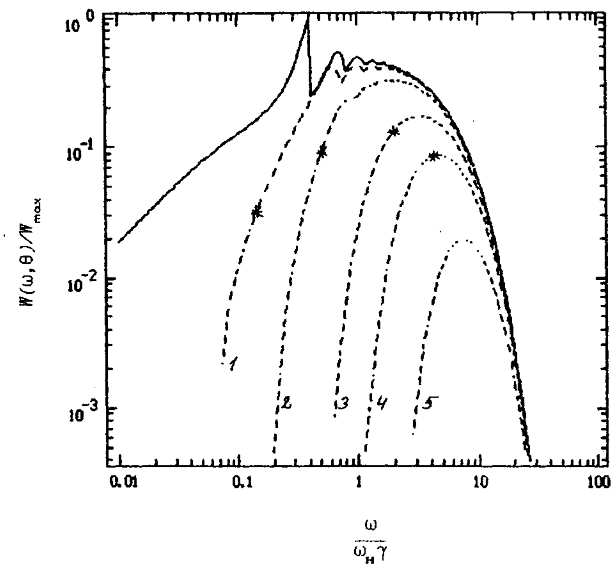

Figure 3 The synchrotron emission power spectral density of the relativistic electron with the pitch angle $\theta = 2mc^2/E$ in vacuum (solid line) and in plasma (dashed lines): 1. $\omega_P = 0.5\omega_H$; 2. $\omega_P = \omega_H$; 3. $\omega_P = 2\omega_H$; 4. $\omega_P = 3\omega_H$; 5. $\omega_P = 5\omega_H$.

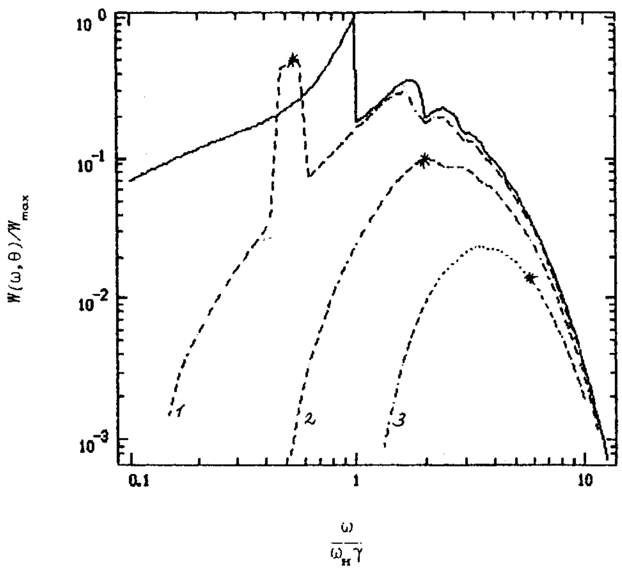

Figure 4 The synchrotron emission power spectral density of the relativistic electron with the pitch angle $\theta = mc^2/E$ in vacuum (solid line) and in plasma (dashed lines): 1. $\omega_P = 0.7\omega_H$; 2. $\omega_P = 1.5\omega_H$; 3. $\omega_P = 2.5\omega_H$.

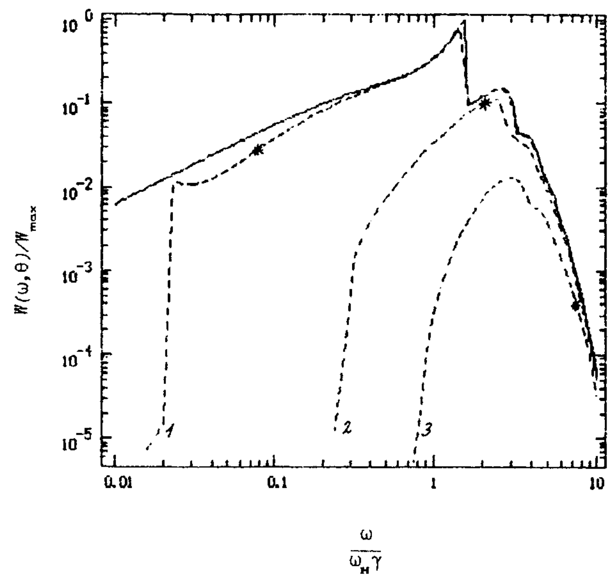

Figure 5 The synchrotron emission power spectral density of the relativistic electron with the pitch angle $\theta = 0.5mc^2/E$ in vacuum (solid line) and in plasma (dashed lines): 1. $\omega_P = 0.2\omega_H$; 2. $\omega_P = \omega_H$; 3. $\omega_P = 2\omega_H$.

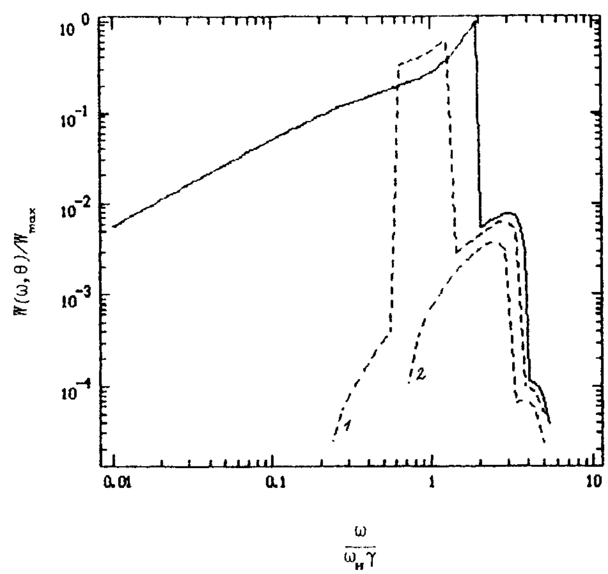

Figure 6 The synchrotron emission power spectral density of the relativistic electron with the pitch angle $\theta = 0.1mc^2/E$ in vacuum (solid line) and in plasma (dashed lines): 1. $\omega_P = 0.9\omega_H$ ($\omega^* = 8.1\omega_H\gamma$); 2. $\omega_P = 1.5\omega_H$ ($\omega^* = 22.5\omega_H\gamma$).

The presence of plasma leads to the narrowing of the frequency range corresponding to the first harmonics number; the power emitted is also reduced (see Figure 6 for $\omega_{\mathbf{P}} = 0.9\omega_{\mathbf{H}}$ and Figure 4 for $\omega_{\mathbf{P}} = 0.7\omega_{\mathbf{H}}$).

When the plasma frequency increases to a certain value, the emission at the first harmonics is suppressed. This happens when the expression under the root in (95) becomes negative. The condition of the absence of emission at the s-harmonics is the following:

$$s < s^* = \frac{\omega_{\mathbf{P}}\gamma}{\omega_{\mathbf{H}}}[2\beta\cos\theta \cdot (1-\beta\cos\theta)]^{1/2} \approx \frac{\omega_{\mathbf{P}}}{\omega_{\mathbf{H}}}[1+\theta^2\gamma^2]^{1/2}. \qquad (103)$$

For example, when $\theta = mc^2/E$ and $\omega_{\mathbf{P}} = 1.5\omega_{\mathbf{H}}$ (see Figure 4), the first harmonics is not emitted and we have a considerably reduced emission power at low frequencies. For larger values of $\omega_{\mathbf{P}}$, the higher harmonics will not be emitted. In the case of $\theta \gg mc^2/E$, plasma leads to a significant modification of the spectrum at frequencies $\omega \le \omega^* = \omega_{\mathbf{P}}^2/(\omega_{\mathbf{H}}\sin\theta)$.

When the pitch angle decreases and $\theta \to 0$, the frequency ω^* increases too and tends to infinity. But, on the other hand, the power spectrum is restricted at high frequencies by $\omega_{\max} = 2\omega_{\mathbf{H}}\gamma$; hence, ω^* becomes an inadequate characteristic of the plasma influence on the radiation at small pitch angles. Hence, it is necessary to find the minimum value of θ, for which one can use the criterion (8).

As can be seen from Figure 6, in the case $\theta < mc^2/E$ the criterion discussed is not accurate enough because it gives a too large value for ω^* (for example, for $\theta = 0.1mc^2/E$ and $\omega_{\mathbf{P}} = 0.9\omega_{\mathbf{H}}$, the criterion gives $\omega^* = 8.1\omega_{\mathbf{H}}\gamma$, but the spectrum is restricted by $\omega_{\max} = 2\omega_{\mathbf{H}}\gamma$). For $\theta \ge mc^2/E$, the criterion (8) is valid (see Figures 2–4).

For example, for $\theta = mc^2/E$ and $\omega_{\mathbf{P}} = 1.5\omega_{\mathbf{H}}$ the power emitted in plasma at the frequency $\omega^* = 2.25\omega_{\mathbf{H}}\gamma$ (marked by asterisk in the figures) decreases by a factor of 2.5 in comparison with vacuum.

At frequencies $\omega < \omega^*$ one can see a quasi-exponential fall–off of the power emitted: $W_\omega \propto \exp[-(2/3)(\omega^*/\omega)^2]$; at frequencies $\omega > \omega^*$, plasma does not practically change the synchrotron spectrum.

Thus, criterion (8) is valid when $\theta \ge mc^2/E$.

The inapplicability of (8) for $\theta < mc^2/E$ is due to the following reasons. When obtaining the criterion, the following expression for the maximum frequency for $\theta \gg mc^2/E$ was used: $\omega_{\max} \sim \omega_{\mathbf{H}}\gamma^2\sin\theta$ ($\omega_{\max}$ is proportional to the square of γ).

But for $\theta \le mc^2/E$ the maximum frequency is given by: $\omega_{\max} \sim 2\omega_{\mathbf{H}}\gamma$ ($\omega_{\max}$ is proportional to γ).

Inserting $\gamma \sim \omega_{\max}/\omega_{\mathbf{H}}$ in (4), one finds for $\theta < mc^2/E$ that plasma influence is essential if

$$\omega_{\mathbf{P}} \ge \omega_{\mathbf{H}} \qquad (104)$$

(which is obvious also from eq. (100) for the emitted frequency interval).

For $\theta \sim mc^2/E$, the criteria (8) and (104) are equivalent. Actually, writing (104) as $\omega_{\mathbf{P}}^2/\omega_{\mathbf{H}} > \omega_{\mathbf{H}}$ or $\omega_{\mathbf{P}}^2/(\omega_{\mathbf{H}}\sin\theta) > \omega_{\mathbf{H}}/\sin\theta$, and taking into account that for $\theta \sim mc^2/E \sin\theta \sim \theta \sim \gamma^{-1} = mc^2/E$ (for relativistic electrons) one has

$$\omega_{\mathbf{P}}^2/(\omega_{\mathbf{H}}\sin\theta) = \omega^* > \omega_{\mathbf{H}}\gamma, \qquad (105)$$

Table 1 The frequency $\nu^*_{\theta=\pi/2}$ for several cosmic sources.

Object	*Electron density, cm^{-3}*	*Magnetic field strength, Gs*	*$\nu^*_{\theta=\pi/2}$, Hz*
Solar bursts	10^9	10^2	3×10^8
Supernova remnants (filaments)	10^3	10^{-5}	3×10^9
Molecular clouds	10^2	10^{-5}	3×10^8
Interstellar medium (HII–regions)	1	3×10^{-6}	10^7
Gaseous nebulae	10^3	10^{-5}	3×10^9
Quasars	$10^6 \div 10^8$	$10^{-3} \div 10^{-2}$	$3 \times 10^{10} \div 3 \times 10^{11}$
Active galactic nuclei	10^{12}	5×10^2	6×10^{10}

i.e., the criterion (8), because for $\theta \sim mc^2/E$ the observable frequencies are $\omega \sim \omega_H \gamma$ (see 102).

5 CONCLUSION

A strict solution of the problem of the synchrotron emission of relativistic electrons spiraling in a rarefied plasma with the refractive index $n^2 = 1 - \omega_P^2/\omega^2$, shows that for $\theta \geq mc^2/E$ the plasma influence is essential at frequencies $\omega \leq \omega^* = \omega_P^2/(\omega_H \sin\theta)$.

For $\theta < mc^2/E$, the observable frequencies are less or of the order of $2\omega_H\gamma$; and, because $\omega^* = \gamma\omega_P$, the plasma influence is essential if $\omega_P \geq \omega_H$.

The frequency $\nu^*_{\theta=\pi/2} = (1/2\pi) \cdot \omega_P^2/\omega_H$ (for $\theta = \pi/2$) is shown in Table 1 for several cosmic sources [20–24]. As can be seen, this frequency belongs to the frequency range of observations of these objects. If the electron pitch angle distribution is strongly anisotropic in some object and the average pitch angle is small (say, $\theta \sim mc^2/E$), then the frequency ν^* will be larger by the factor $\gamma = E/mc^2$ in comparison with the large pitch angle case:

$$\nu^* = \frac{1}{2\pi}\frac{\omega_P^2}{\omega_H \sin\theta} \approx \frac{1}{2\pi}\frac{\omega_P^2}{\omega_H}\gamma = \nu^*_{\theta=\pi/2} \cdot \gamma.$$

This means that ν^* increases hundred ar thousand times, because the value $\gamma = E/mc^2$ is large for relativistic electrons and, consequently, the plasma influence is essential practically always.

Let us point out that the following equation may be convenient for approximate calculations of ν^*:

$$\nu^*[\mathrm{Hz}] \approx 1.5 \cdot 10^7 N^{0.6}[\mathrm{cm}^{-3}]. \tag{106}$$

It follows from (7) by using the following expression [25]:

$$H[\mathrm{Gs}] = 2 \cdot 10^{-6} N[\mathrm{cm}^{-3}], \tag{107}$$

which is valid for cosmic sources with linear dimensions $10^{-3} \div 10$ pc (under the assumption that the gas is completely ionized).

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