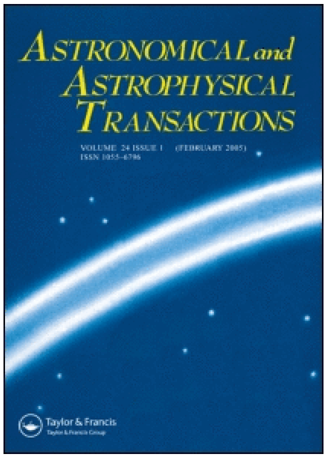

Astronomical & Astrophysical Transactions
The Journal of the Eurasian Astronomical Society
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713453505

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Astronomical and Astrophysical Transactions, 1994,
Vol. 5, pp. 85–89
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THE CATASTROPHE THEORY AND SOME PROBLEMS OF GRAVITY THEORY: FROM ZELDOVICH'S PANCAKES TO GRAVITATIONAL LENSES AND GEODESICS IN THE KERR METRIC

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(15 December 1992)

We use the catastrophe theory (the theory of the singularities of smooth functions) for an analysis of structurally stable and unstable images in gravitational lenses and for computations of the asymptotics of some rapidly oscillating integrals. We used also the methods of the catastrophe theory for the problem of the classification of particle motion type in the Kerr metric.

KEY WORDS Gravitational lenses, black holes, geodesics.

The formation of pancakes in the framework of the adiabatic theory of the large-scale structure formation was predicted in a remarkable paper of Zeldovich (1970). A possible scenario of the formation of pancakes (folds, in the catastrophe theory language) in the framework of nonlinear one-dimensional approach was presented in this paper. A widely accepted theory of large-scale structure formation was presented in a seminal paper of Arnold, Zeldovich and Shandarin (1981).

Following Bliokh and Minakov (1989), we use the following equations of gravitational lensing when the source is at infinity:

$$\rho(\mathbf{p}, D_{\text{do}}) = \mathbf{p} + D_{\text{do}}\Theta_g(\mathbf{p}) = \mathbf{p} - 2r_g D_{\text{do}}\mathbf{p}/p^2, \tag{1}$$

where $\mathbf{p}$ is the impact parameter (vector), ρ is the vector that describes the deviation of the observer from the symmetry axis (for the given value of the impact parameter), D_{do} is the distance between the lense and the observer. We can find the impact parameter for a given ρ:

$$\mathbf{p}_{1,2}(\rho) = \rho[1/2 \pm \sqrt{1/4 + 2r_g D_{\text{do}}/\rho^2}]. \tag{2}$$

We have, from the energy conservation,

$$J^0\Sigma_{\text{in}} = J(D_{\text{do}}, \rho)2\pi\rho\, d\rho, \tag{3}$$

where Σ_{in} is the introductory aperture, J^0 is the intensity, $J(D_{\text{do}})$ is the intensity at the distance D_{do},

$$\Sigma_{\text{in}} = \{2\pi(p_1\, dp_1 + p_2\, |dp_2|) \tag{4}$$

Then the energy amplification coefficient is given by

$$q(D_{\rm do}, \rho) = J(D_{\rm do}, \rho)/J^0 = (p_1/\rho)(dp_1/d\rho) + (p_2/\rho)\,|dp_2/\rho|, \tag{5}$$

and for gravitational lenses we have (Bliokh and Minakov, 1989)

$$q(D_{\rm do}, \rho) = \{l/\rho(\rho^2/(2l^2) + 1)/\sqrt{\rho/(4l^2) + 1}, \tag{6}$$

where $l = \sqrt{2r_g D_{\rm do}}$. It is clear that $q \to \infty$ for $\rho \to 0$. Similarly, if the source is at the distance $D_{\rm sd}$ from the lens, we introduce the following parameters:

$$\begin{aligned} \tilde{\rho} &= D_{\rm sd}\rho/(D_{\rm sd} + D_{\rm so}), \\ \tilde{D}_{\rm do} &= D_{\rm do}D_{\rm sd}/(D_{\rm do} + D_{\rm sd}), \\ \tilde{l} &= \sqrt{2r_g \tilde{D}_{\rm do}} \end{aligned} \tag{7}$$

Then

$$q(\tilde{D}_{\rm do}, \rho) = \tilde{l}/\tilde{\rho}\frac{(\tilde{\rho}/(2\tilde{l}^2) + 1)}{\sqrt{\tilde{\rho}^2/(4\tilde{l}^2) + 1}} \tag{8}$$

If $\rho = 0$, then $p = \tilde{l}$ and the observer sees a ring of the radius $\tilde{l}$ (the Einstein ring). The observer sees the Einstein ring only at distances $D_{\rm do} > D_{\rm do,min}$, where

$$D_{\rm do,min} = D_{\rm sd}R^2/(2r_g D_{\rm sd} - R^2), \tag{9}$$

with R, the lens radius. We do not see sources with $D_{\rm do} < R^2/(2r_g)$. A Czech engineer Mandle once asked Einstein about the computations and Einstein published them in 1936 in "Science" (Einstein, 1936). Einstein wrote that "There is not much hope of observing this phenomenon directly". If a source has some displacement from the symmetry line, then one observes a double star (Chwolson, 1924). Therefore the Einstein ring is structurally unstable and it is more correct to say about the impossibility of the Einstein ring image but not about a small probability. However most authors repeated Einstein's phase (see, for example Wamsganss, 1990a). Another example of the application of the catastrophe theory methods to gravitational lenses is a description of the section of a caustics surface by a plane through the symmetry axis of a gravitational lens (Bliokh and Minakov, 1989):

$$\rho = a(x/x_F)(1 - x_F/x)^{3/2}, \tag{10}$$

where x is the distance from a fixed point x_F (x is the symmetry axis, x_F is the focus of the gravitational lens). It is well known that this function describes the singularity known as the pleat (cusp) (Poston and Stewart, 1978). It is clear that if we consider small perturbations, then the caustics surface is similar to that described by Poston and Stewart (1978) as the perturbational section of the caustics surface is different from the circle. It is well known that caustics surfaces in 3-dimensional space (structurally stable) may be only "folds" (codimension 1), pleats (cusps) (codimension 2), swallowtails, hyperbolical and elliptical umbilics (codimension 3), and, in 4-dimensional space (the case may be for moving caustics surfaces) me have also "butterflies" and parabolic umbilics (codimension 4) (Arnol'd, 1984a). The calculation of the intensity near a singularity of a known type reduces to the evaluation of the asymptotics of integrals of rapidly ossilating functions. It is known that the intensity near the singularities is related to the wave vector by $I \sim k^{\alpha}$; $\alpha = 1/6$ for the fold singularity A_2 (Airy), $\alpha = 1/4$ for the pleat (cusp) singularity A_3 (Persey), $\alpha = 3/10$ for the singularity A_4 (swallowtail), $\alpha = 1/3$ for the elliptic and hyperbolic umbilics (D_4) (Arnol'd *et al.*, 1984). Knowing the asymptotics we can determine the type of the caustics surface (if the enhanced image brightness is caused by crossing a caustics surface) since we may observe the intensity in different spectrum intervals.

Recently, two groups of German astrophysicists from Hamburg Sternwarte (Kayser, 1989) and from Garching Institute for Astrophysics (Wamsganss, 1990b) presented the results of numerical simulations of caustics surfaces from microlensing for concrete astrophysics objects. The asymptotic analysis adds to the numerical simulations.

The equation of motion for the radial variable in the Kerr metric is (Chanrasekhar, 1983):

$$\rho^4(dr/dr)^2 = R(r),$$

$$R(r) = r^4 + (a^2 - \xi^2 - \eta)r^2 + 2M[\eta + (\xi - a)^2]r - a^2\eta \text{ (Photons)}, \qquad (11)$$

$$R(r) = r^4 + (a^2 - \xi^2 - \eta)r^2 + 2M[\eta + (\xi - a)^2]r - a^2\eta - r^2\Delta/E \text{ (Particles)},$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$ and $a = S/M$. The constants S and M refer to the black hole, namely S is the angular momentum and M is the mass of the black hole. The constants E, ξ and η refer to the particle, namely E is its energy at infinity, $\xi = L_z/E$ (L is the angular momentum of the particle about the axis of rotation of the black hole), and $\eta = Q/E$ (Q is given by $Q = p_\theta^2 + \cos^2\theta[a^2(\xi^2 - E^2) + L_z^2 \sin^{-2}\theta]$ and μ is the mass of the particle). It is readily verified that the radial motion of the particle depends on the following constants: $\hat{a} = a/M$, $\hat{E} = E/\mu$, $\hat{\xi} = \xi/M$ and $\hat{\eta} = \eta/M$. The radial motion of photons does not depend on the constant E. Instead of the coordinate r, we now introduce $\hat{r} = r/M$. (The Λ-symbol will be omitted henceforth). Thus, the character of motion in the r-coordinate for a given value of a is determined by the three constants E, ξ, η in the case of a moving particle, and by the two constants ξ and η in the case of photons.

In this part it will be shown that a black hole in extreme rotation can have stable geodesics with any energy (in units of particle mass) in the range $0 \le E \le 3^{-1/2}$ although it is well known that for a particle moving along a circular geodesics in the field of a black hole in the state of extreme rotation, the binding energy has been found capable of reaching the value $3^{-1/2}$. One readily finds (Carter, 1968) that if the particle is moving in the equatorial plane ($\theta = \pi/2$), $Q = 0$. The particle will then travel along a circular orbit if the relations

$$R(r) = 0, \qquad \partial R/\partial r = 0 \qquad (12)$$

are satisfied, as well as the orbital stability criteria $\partial^2 R/\partial^2 r < 0$. Let us consider a black hole with an extreme value of the rotation parameter ($a = 1$). Assume that the particle orbits lie in the equatorial plane, with the following values for the constants of motion: $L_z = 2E$, $E < (1/\sqrt{3})$, $r = 1$. It is not difficult to see that the stability criteria will then hold true. Accordingly, as a particle moves through successive near-circular orbits, an energy of order 1 can be liberated. It is also noteworthy that for any value of the constant $Q (Q > 0)$ and the constants of particle motion that satisfy the conditions, the criteria will remain valid; that is stable orbits will exist that correspond to the motion of a particle along the surface $r = \text{const}$ with a given energy and given angular momentum, but which are not circular. The maximum value of θ for such orbits will be determined by the value Q. Outside the equatorial plane one finds, as Wilkins (1972) has argued, that stable nonequatorial orbits also can exist throughout the energy range $3^{-1/2} < E < 1$: $L_z = 2E$, $Q > 3E^2 - 1$. One should recognize, however, that if the black hole departs slightly from a state of extreme rotation, there can be no stable circular orbits with energy in the range $0 < E < 3^{-1/2}$. In just the same way, the stable orbits (16) will disappear if the rotation falls short of the extreme case. This result can be demonstrated either by considering the potentials $V_\pm$ as done by Wilkins (1972) (the potentials V_+ and V_- will merge into a "knife edge" (Christoudoulou, 1971)), or by turning to Eqs (5), which implicity specifies, say, the parameters L_z and E as functions of the rotation parameter (Zakharov, 1986). The property that we have here is analogous to the straight-line part of the relation between $\rho_\parallel(\rho_\perp)$ and the capture cross-section in the case where $a = 1$ and photons or particles (Zakharov, 1986) are incident on the black hole. Thus the presence of stable circular orbits is not sole property that distinguishes extreme ($a = 1$) from nonextreme holes. Since the Wilkins potentials coinside ($V_- = V_+$) for parameter whose values confirm the criteria (6), one can readily see that the value 1 for the rotation parameter a represents a bifurcation point corresponding to the fold singularity (Bröcker, 1975; Gilmore, 1984; Arnol'd, 1984b) and if a departs from that value the stable orbits (7) will disappear.

It is easy to see the connection of the problem of the classification of particle motion in the Kerr metric with the singularities of the smooth functions (with the catastrophe theory), particularly, those sets are connected with semialgebraical submanifold of the algebraic manifold D_f (swallowtail) (Zakharov, 1991).

Now we present conclusions. Some properties which we considered in astro-physical gravitational problems are structurally unstable (for example, the "Einstein ring" and some properties of geodesics in the extreme Kerr metric) and we may use some results of mathematical theory for astrophysical problems as Zeldovich and others made for the large-scale-structure formation and we made above for gravitational lenses.

I express my deep gratitude for their attention to this work to Prof. V. S. Imshennik, Prof. D. K. Nadyozhin, Prof. P. V. Bliokh, A. A. Minakov, A. V. Mandzhos. I want to express also my deep gratitude to Prof. P. Schneider for sending his papers, especially a part (that is connected with catastrophe theory) of his monograph "Gravitational Lensing" and R. Kayser and V. Faraoini for sending their papers. Especially I want to thank the Organizing Committee of Zeldovich's Conference for invitation and hospitality, and S. Kopejkin for attention to my paper and useful remarks.

This work was supported by the grant of American Astronomical Society.

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