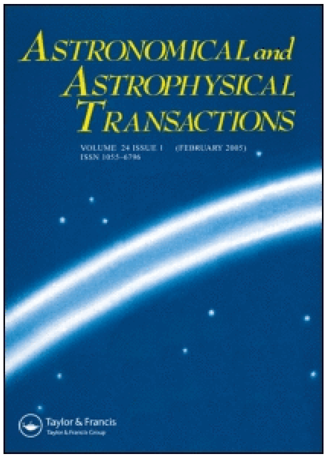

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STATIONARY MOTIONS OF A SATELLITE WITH SPHERE OF INERTIA AND THEIR STABILITIES. PLANAR MOTIONS

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In this paper, stationary motions and their stabilities are investigated for a system of two rigid bodies, one spherical and the other with a nonspherical distribution of density, but whose ellipsoid of inertia is a sphere. Under the assumption of a planar orbit, stationary solutions and stability conditions are found.

KEY WORDS Satellite dynamics, rigid body attitude, stationary solutions

1 INTRODUCTION

Generally, in the investigation of the orbital-attitude dynamics of rigid bodies, the case in which the ellipsoid of inertia of the body is a sphere is not taken into consideration, although this model may represent a wide class of celestial bodies and artificial satellites.

Most of the studies devoted to attitude dynamics of a satellite under a central force field consider only the third power of the inverse of the radial distance in the potential, and therefore, when the satellite is such that the three axes of its ellipsoid of inertia are equal (sphere), only the Keplerian potential remains in the force function. However, if we take into account higher powers of the radial distance, these terms – although small in magnitude – may change drastically the attitude of the satellite, new equilibria appear and their stability should be determined.

This paper is a first step in the way of analyzing the dynamics of a non-spherical rigid body with sphere of inertia under the gravitational attraction of a homogeneous sphere. To begin with, we study the *restricted* (we mean that the orbit is a known function of the time) and later on, the *unrestricted* (the coupled orbital-rotational motion) problem of the motion of the satellite.

The main property of the problem under consideration is that the second harmonic is equal to zero in the force function of the satellite, and the dynamics of the satellite is determined by the third, fourth and higher harmonics. Of course, the classical results about existence and stability of the stationary solutions for the satellite (Lagrange, 1870; Beletskii, 1965; Kinoshita, 1972; Barkin, 1985; etc.) are not applicable here, and this is the reason why we think that the problem considered here presents not only a theoretical interest, but also a practical interest for attitude dynamics of spacecrafts with special requirements and also for celestial bodies with such a configuration.

In Section 2 we formulate the problem, define the reference frames, and give the equations of motion in a Hamiltonian form. In Sections 3 and 4 we obtain the stationary solutions and their stabilities for the *restricted* problem when only the third and fourth harmonics are considered, respectively. Different regions of stability depending on some dynamical parameters are plotted. The *unrestricted* problem is studied in Section 5 and Subsection 5.2. The stability for these cases is analyzed in Subsection 5.1. The complete analysis considering higher harmonics and non-planar orbital motion will be the object of further studies.

2 EQUATIONS OF MOTION

Let us consider the motion of a system of two finite rigid bodies S_0 and S of masses m_0 and m which experience no other forces than the mutual gravitational attraction.

The rigid body S_0 (that will be considered as a primary) is supposed to be a homogeneous sphere, with a concentric distribution of mass and therefore may be assimilated to a mass point O_0. The other rigid body S (considered as a satellite) has one plane of the dynamical symmetry, and its ellipsoid of inertia is assumed to be a sphere, i.e., their principal central moments of inertia are equal ($A = B = C$). When only third powers of the inverse of the radial distance are taken into consideration in the development of the potential function, the problem is equivalent to the Keplerian one. However, when higher orders are considered, equations for the translational-rotational motion are to be integrated.

A further additional hypothesis is that we suppose that the orbital motion of the center of mass of the satellite O with respect to the center of mass of the primary O_0 is planar, and that the dynamical plane of symmetry of the satellite coincides with the orbital plane.

Let us consider the following references frames:

a) The space frame: $O_0\mathbf{xyz}$. A fixed system, centered at the point O_0, with the plane $O_0\mathbf{xy}$ being the orbital plane.

b) $O\mathbf{xyz}$. A system parallel to the previous one, but centered at the center of mass O of the satellite S.

c) The body frame: $O\boldsymbol{\xi\eta\zeta}$. The body frame, centered at the center of mass O of the satellite, so that the plane $O\boldsymbol{\xi\eta}$ is the dynamical plane of symmetry of the satellite, and coincides with the orbital plane $O\mathbf{xy}$.

The orbital motion of the center of mass O of the satellite S with respect to the space frame $O_0\mathbf{xyz}$ may be described by polar coordinates ρ and θ, whereas the rotational motion of S about its center of mass is given by the longitude ν of the axis $O\boldsymbol{\xi}$, measured from the axis $O\mathbf{x}$. The angle ν is defined by $\cos\nu = \mathbf{x}\cdot\boldsymbol{\xi}$. Another angle that will be useful is the angle ψ, the longitude of the axis $O\boldsymbol{\xi}$, measured from the radial direction ($\cos\psi = (\boldsymbol{\rho}\cdot\boldsymbol{\xi})/\rho$). Obviously, we have $\psi = \nu - \theta$.

The problem has three degrees of freedom, and in variables ρ, θ, ν, the kinetic energy of the system may be expressed as

$$T = \frac{1}{2}\mu(\dot{\rho}^2 + \rho^2\dot{\theta}^2) + \frac{1}{2}I\dot{\nu}^2,$$

where μ is the reduced mass $\mu = m_0 m/(m_0 + m)$ and I is the moment of inertia of the body S about any axis passing through its center of mass O.

The force function of the mutual gravitational attraction of the bodies S_0, S may be expressed as the classical spherical harmonics expansion (cf. e.g. Kaula, 1966):

$$U = \frac{\mathcal{G}m_0 m}{\rho}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{R}{\rho}\right)^n P_n^k(\phi)\left[C_{n,k}\cos k\lambda + S_{n,k}\sin k\lambda\right]. \qquad (1)$$

Here ρ, λ and ϕ are spherical coordinates of the center of mass O_0 of the body S_0 in the body frame $O\xi\eta\zeta$. The orbital motion being planar, the latitude of the point O_0 is null ($\phi = 0$), and the longitude is $\lambda = \pi - \psi$.

In (1), $\mathcal{G}$ is the gravitational constant, R is the mean radius of the body S, $P_n^k(\phi)$ are the associate Legendre polynomials and the coefficients $C_{n,k}$ and $S_{n,k}$ are the Poinsot constants of the gravitational field of the satellite. These constants are given by

$$\begin{aligned} C_{n,k} &= \frac{2}{m\delta_k}\frac{(n-k)!}{(n+k)!}\int_\tau\left(\frac{r'}{R}\right)^n P_n^k(\phi')\cos k\lambda'\delta(r',\phi',\lambda')\,d\tau, \\ S_{n,k} &= \frac{2}{m\delta_k}\frac{(n-k)!}{(n+k)!}\int_\tau\left(\frac{r'}{R}\right)^n P_n^k(\phi')\sin k\lambda'\delta(r',\phi',\lambda')\,d\tau, \end{aligned}$$

where $\delta_0 = 2$, $\delta_k = 0, (k \in N)$, and r', ϕ', λ' are spherical coordinates of a particle of elementary mass with elementary volume $d\tau$ and density $\delta(r',\phi',\lambda')$.

In our problem, the coefficients of the second harmonic of the force function are equal to zero, since the principal moments of inertia are equal, i.e.,

$$C_{2,0} = \frac{2C - A - B}{2mR^2} = 0, \quad C_{2,2} = \frac{A - B}{4mR^2} = 0.$$

Depending on different assumptions on the symmetry of the rigid body, there follow several simplifications in the expression for the force function. In some specific parts of this paper, some of the following restrictions will be employed:

- If the coordinate plane $O\xi\eta$ is a plane of symmetry in the density distribution of the body S, the coefficients are

$$C_{n,n-2k'-1} = 0, \quad S_{n,n-2k'-1} = 0, \quad (k' = 0, 1, \ldots, [(n-1)/2]).$$

- If the rigid body S has two mutual orthogonal planes of dynamical symmetry $O\xi\eta$ and $O\xi\zeta$, then

$$\begin{aligned} S_{n,k} &= 0 \quad (n = 0, 1, \ldots; \quad k = 0, 1, \ldots, n), \\ C_{n,n-2k'-1} &= 0 \quad (k' = 0, 1, \ldots, [(n-1)/2]). \end{aligned}$$

- If all the three planes $O\xi\eta$, $O\xi\zeta$ and $O\eta\zeta$ are planes of dynamical symmetry,

$$\begin{aligned} S_{n,k} &= 0 \quad (n = 0, 1, \ldots; \quad k = 0, 1, \ldots, n), \\ C_{2n'+1,k} &= 0 \quad (n' = 0, 1, \ldots; \quad k = 0, 1, \ldots, 2n'+1). \end{aligned}$$

All over the present paper, we shall consider that the plane $O\xi\eta$ is a dynamical plane of symmetry for the body S. Therefore, by using the first property in the first terms of the expansion (1), the force function takes the form:

$$\begin{aligned} U &= \frac{\mathcal{G}m_0 m}{\rho} \\ &+ 3\frac{\mathcal{G}m_0}{2\rho^4} mR^3[(-C_{3,1}\cos\psi + S_{3,1}\sin\psi) \\ &\qquad +10(C_{3,3}\cos 3\psi - S_{3,3}\sin 3\psi)] \\ &+ \frac{\mathcal{G}m_0}{\rho^5} mR^4 \left[\frac{3}{8}C_{4,0} - \frac{15}{2}(C_{4,2}\cos 2\psi - S_{4,2}\sin 2\psi) \right. \\ &\qquad \left. +105(C_{4,4}\cos 4\psi - S_{4,4}\sin 4\psi)\right] \\ &+ \frac{\mathcal{G}m_0}{\rho^6} mR^5 \left[\frac{15}{8}(C_{5,1}\cos\psi - S_{5,1}\sin\psi) \right. \\ &\qquad -\frac{105}{2}(C_{5,3}\cos 3\psi - S_{5,3}\sin 3\psi) \\ &\qquad \left. +945(C_{5,5}\cos 5\psi - S_{5,5}\sin 5\psi)\right] \\ &+ W(\rho, \psi) \end{aligned} \tag{2}$$

where $W(\rho, \psi)$ is the remaining part in the force function (1):

$$\begin{aligned} W(\rho, \psi) &= \frac{\mathcal{G}m_0 m}{\rho} \sum_{n=6}^{\infty} \sum_{k=0}^{[n/2]} \left(\frac{R}{\rho}\right)^n P_{n,k} \Big[C_{n,n-2k}\cos[(n-2k)\psi] \\ &\qquad -S_{n,n-2k}\sin[(n-2k)\psi]\Big], \end{aligned}$$

and the coefficients are

$$P_{n,k} = (-1)^n P_n^{n-2k}(0), \quad P_n^{n-2k}(0) = \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!}.$$

Let ρ, θ, ν be the generalized coordinates and P_ρ, P_θ, P_ν their conjugate momenta. In this set of canonical variables, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2\mu}\left(P_\rho^2 + \frac{P_\theta^2}{\rho^2}\right) + \frac{1}{2I}P_\nu^2 - U(\rho, \nu - \theta). \tag{3}$$

Since the variables ν and θ appear only as the difference $\nu - \theta = \psi$, by means of the canonical transformation $(\rho, \theta, \nu, P_\rho, P_\theta, P_\nu) \longmapsto (\rho, \omega, \psi, P_\rho, P_\omega, P_\psi)$ given by

$$\psi = \nu - \theta, \quad \omega = \theta, \quad P_\psi = P_\nu, \quad P_\omega = P_\nu + P_\theta,$$

the Hamiltonian adopts the form

$$\mathcal{H} = \frac{1}{2\mu}\left(P_\rho^2 + \frac{(P_\psi - P_\omega)^2}{\rho^2}\right) + \frac{1}{2I}P_\psi^2 - U(\rho, \psi). \tag{4}$$

The coordinate ω being cyclic, its conjugate momentum P_ω is a first integral, which as a matter of fact corresponds to the integral of the angular momentum.

3 THE RESTRICTED PROBLEM. THIRD HARMONICS

To begin with, we consider the attitude dynamics of a satellite moving in a circular orbit under a central force field, when only third and fourth harmonics are retained in the force field functions. In the current section we shall take into consideration only the third harmonics for the *restricted* problem, i.e., for a Keplerian circular orbit, whereas the influence of the fourth harmonics will be studied in the next section. Under these conditions, the Hamiltonian is:

$$\mathcal{H} = \frac{1}{2I}P_\psi^2 - nP_\psi - U(\psi), \tag{5}$$

with the force function being

$$\begin{aligned} U(\psi) \quad = \quad & \frac{\mathcal{G}m_0}{a^4}(mR^3)\left[\frac{3}{2}(-C_{3,1}\cos\psi + S_{3,1}\sin\psi)\right. \\ & \left. +15(C_{3,3}\cos 3\psi - S_{3,3}\sin 3\psi)\right], \end{aligned}$$

where a is the radius of the circular orbit, and n is the orbital mean motion.

The equations of the motion corresponding to this Hamiltonian are

$$
\begin{aligned}
\frac{d\psi}{dt} = \frac{\partial\mathcal{H}}{\partial P_\psi} &= \frac{1}{I}P_\psi - n, \\
\frac{dP_\psi}{dt} = -\frac{\partial\mathcal{H}}{\partial\psi} &= \frac{3\mathcal{G}m_0}{2a^4}mR^3\Big[(C_{3,1}\sin\psi + S_{3,1}\cos\psi \\
&\qquad -30(S_{3,3}\cos 3\psi + C_{3,3}\sin 3\psi)\Big].
\end{aligned} \tag{6}
$$

The conditions for the existence and stability of stationary solutions of the Hamiltonian (5) are:

$$
\frac{\partial\mathcal{H}}{\partial P_\psi} = 0, \quad \frac{\partial\mathcal{H}}{\partial\psi} = 0, \quad \frac{\partial^2\mathcal{H}}{\partial P_\psi^2}\frac{\partial^2\mathcal{H}}{\partial\psi^2} - \left(\frac{\partial^2\mathcal{H}}{\partial P_\psi\partial\psi}\right)^2 > 0. \tag{7}
$$

That is to say, the Hessian, when evaluated at the critical points, must be a quadratic form positive definite.

3.1 Lagrange's Solution and Their Stabilities

In a problem similar to ours, but considering only the second harmonics for a triaxial rigid body, Lagrange (1870) obtained several particular solutions such that the principal axes of inertia are either in radial direction, or tangential or normal to the orbit. For the unrestricted problem and considering an axisymmetric body, analogous cases were named by Duboshin (1960) as *arrow*, *spoke* and *float* solutions. In our problem, every body axis passing through the body's center of mass O (and in particular $O\xi$, $O\eta$ and $O\zeta$) is a principal axis of inertia. Therefore, we named, formally, as the *Lagrangian* solutions those for which the body axes ($O\xi$, $O\eta$, $O\zeta$) have the Lagrangian orientations, characterized by the following values of the angle ψ:

$$
\psi = 0, \pi/2, \pi, 3\pi/2.
$$

For $\psi = 0$ and $\psi = \pi$, the body axis $O\xi$ coincides with the radius vector, whereas for the other two values, the axis is tangent to the orbit.

For the existence and stability of these particular stationary solutions some additional restrictions to the dynamical structure of the body should be added, such as shown in the following table:

ψ	*Existence*	*Stability*
0	$s = 1$	$c < 3$
π	$s = 1$	$c > 3$
$\pi/2$	$c = -1$	$s > -3$
$3\pi/2$	$c = -1$	$s < -3$

where the parameters c and s are

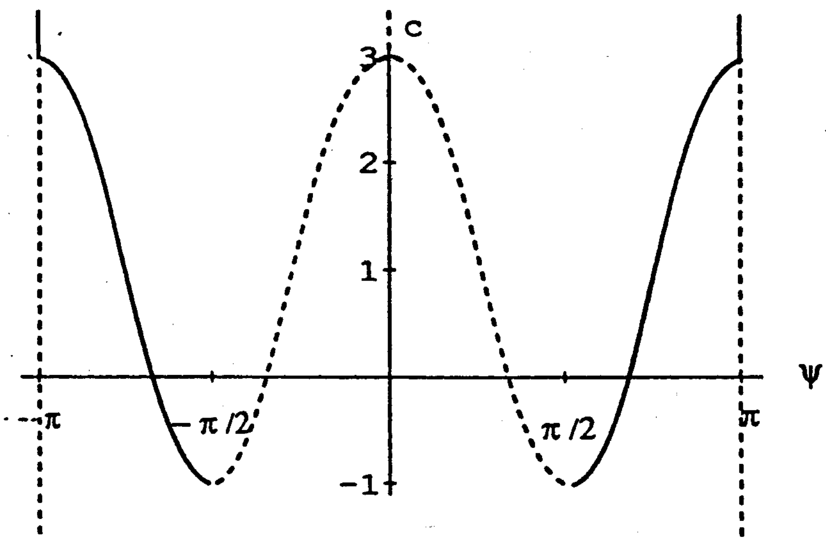

Figure 1 The existence of stationary solutions for the third harmonics when the rigid body has two orthogonal planes of dynamic symmetry. Dashed lines correspond to unstable solutions and solid lines, to stable points.

$$c = \frac{C_{3,1}}{30C_{3,3}}, \quad s = \frac{S_{3,1}}{30S_{3,3}}.$$

It is worth to note the difference with the Lagrangian solutions in the classical case. Indeed, in the classical one, the conditions for the stability of the points $\psi = 0$, $\psi = \pi$ (analogously for $\psi = \pi/2$, $\psi = 3\pi/2$) are identical, just the contrary that in the present case.

3.2 A Rigid Body Having Two Orthogonal Planes of Dynamical Symmetry

Let us consider a body S having two planes of dynamical symmetry: $O\xi\eta$ and $O\xi\zeta$. For this particular case, we use some of the properties of the Poinsot coefficients noted in Section 2. In this case, we have $S_{3,1} = S_{3,3} = 0$, and the conditions (7) for existence and stability are:

$$\begin{array}{ll} \text{Existence:} & \sin\psi(3 - c - 4\sin^2\psi) = 0, \\ \text{Stability:} & \cos\psi(3 - c - 12\sin^2\psi) > 0; \end{array}$$

therefore, for $\psi = 0$ and $\psi = \pi$ are critical solutions for whatever value of the coefficients $C_{3,1}$ and $C_{3,3}$. Besides these, when c is in the range $-1 \leq c \leq 3$, there is a critical solution for

$$\sin\psi = \pm\frac{1}{2}\sqrt{3 - c}.$$

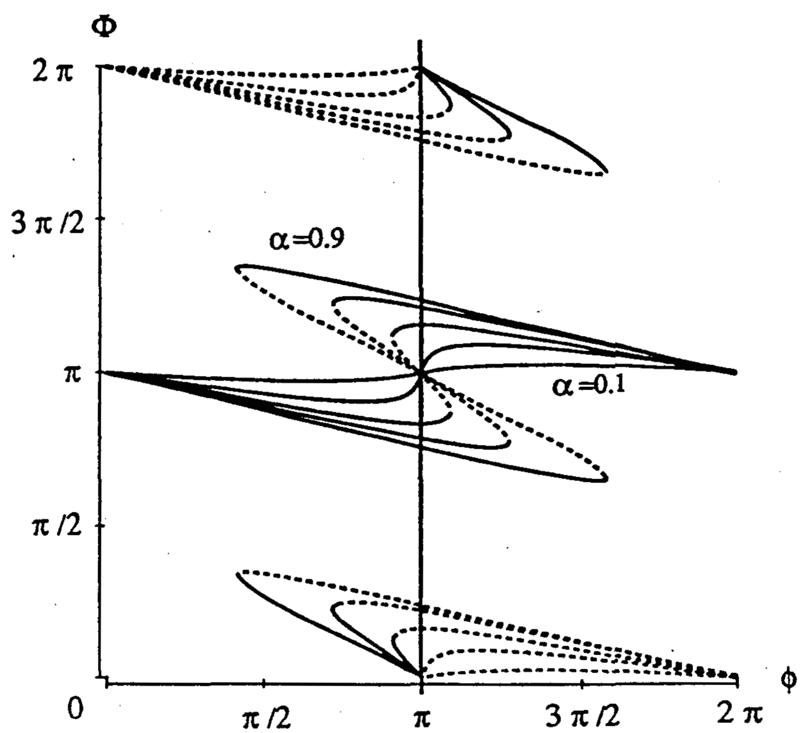

Figure 2 The existence of stationary solutions for $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$ in the general case, considering only the third harmonics. Dashed lines correspond to unstable solution and solid lines, to stable points.

The critical points and their stability for this particular case are given in Figure 1. Continuous lines represent sufficient conditions for stability, whereas points on dashed are unstable.

3.3 A Rigid Body Having a Single Plane of Dynamical Symmetry

Let us consider now a more general model: the body S has only one plane of dynamical symmetry $(O\xi\eta)$, which coincides with the orbital plane. Defining the parameter

$$\alpha = 30\sqrt{\frac{C_{3,3}^2 + S_{3,3}^2}{C_{3,1}^2 + S_{3,1}^2}},$$

the angles Φ and ϕ

$$\Phi = \psi + \phi_1, \quad \phi = \phi_3 - 3\phi_1,$$

and two auxiliary angles ϕ_1, ϕ_3 by

$$\sin\phi_1 = \frac{S_{3,1}}{\sqrt{C_{3,1}^2+S_{3,1}^2}}, \qquad \cos\phi_1 = \frac{C_{3,1}}{\sqrt{C_{3,1}^2+S_{3,1}^2}},$$
$$\sin\phi_3 = -\frac{S_{3,3}}{\sqrt{C_{3,3}^2+S_{3,3}^2}}, \qquad \cos\phi_3 = -\frac{C_{3,3}}{\sqrt{C_{3,3}^2+S_{3,3}^2}},$$

after some elemental manipulations, the conditions for existence and stability (7) become

$$\begin{array}{ll} \text{Existence:} & \sin\Phi + \alpha\sin(3\Phi+\phi) = 0, \\ \text{Stability:} & \cos\Phi + 3\alpha\cos(3\Phi+\phi) < 0. \end{array} \tag{8}$$

Curves of stationary solutions (ϕ versus Φ) for different values of parameter α, and their stability given by the conditions (8) appear in Figure 2.

4 THE RESTRICTED PROBLEM. FOURTH HARMONICS

In this section we consider the restricted problem, but assuming that the force function contains only the fourth harmonics. The Hamiltonian now is:

$$\begin{aligned} \mathcal{H} &= \frac{1}{2I}P_\psi^2 - nP_\psi + \frac{15\mathcal{G}m_0}{2a^5}mR^4\left[(C_{4,2}\cos 2\psi - S_{4,2}\sin 2\psi)\right. \\ &\qquad \left. +14(S_{4,4}\sin 4\psi - C_{4,4}\cos 4\psi)\right]. \end{aligned} \tag{9}$$

From this Hamiltonian, we shall obtain the stationary solutions and their stability. As in Section 3, we shall obtain the Lagrangian solutions and also the stationary solutions when some restrictive hypotheses about the symmetry of the rigid body S are made.

The equations of motion corresponding to this Hamiltonian are

$$\begin{aligned} \frac{d\psi}{dt} &= \frac{1}{I}P_\psi - n, \\ \frac{dP_\psi}{dt} &= -\frac{15\mathcal{G}m_0}{a^5}mR^4\left[(-C_{4,2}\sin 2\psi - S_{4,2}\cos 2\psi)\right. \\ &\qquad \left. +28(S_{4,4}\cos 4\psi + C_{4,4}\sin 4\psi)\right]. \end{aligned} \tag{10}$$

4.1 Lagrange's Solutions

Besides the classical Lagrange solutions $\psi = 0, \pi/2, \pi, 3\pi/2$, the following values of the rotation angle:

$$\psi = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4,$$

are stationary solutions when some restrictions on the coefficient $C_{n,m}$ and $S_{n,m}$ of the fourth harmonics ($n = 4$) in the force function are implemented. The conditions for their existence and stability are given in the following table:

ψ	*Existence*	*Stability*
$0, \pi$	$ss = 1$	$cc > 2$
$\pi/4, 5\pi/4$	$cs = -1$	$sc < 2$
$\pi/2, 3\pi/2$	$ss = -1$	$cc < -2$
$3\pi/4, 7\pi/4$	$cs = 1$	$sc > -2$

with

$$cc = \frac{C_{4,2}}{28C_{4,4}}, \quad ss = \frac{S_{4,2}}{28S_{4,4}}, \quad cs = \frac{C_{4,2}}{28S_{4,4}}, \quad sc = \frac{S_{4,2}}{28C_{4,4}}.$$

It is worth noting that, in distinction to the case $n = 3$ (Section 3), the condition of stability of one solution (ψ_0) is the same as for its opposite value ($-\psi_0$).

4.2 A Body Having Two Orthogonal Planes of Dynamical Symmetry

When a rigid body S has two orthogonal planes of dynamical symmetry, such as mentioned in Section 2, coefficients $S_{4,2} = S_{4,4}$ vanish and the force function adopts the form:

$$U(\psi) = \frac{15}{2}\frac{\mathcal{G}m_0}{a^5}(mR^4)(-C_{4,2}\cos 2\psi + 14C_{4,4}\cos 4\psi),$$

and now, the conditions for the existence of equilibrium solutions and stability (7) are, respectively:

$$\begin{array}{ll} \text{Existence:} & \sin 2\psi(cc - 2\cos 2\psi) = 0, \\ \text{Stability:} & 2 + cc\cos 2\psi - 4\cos^2 2\psi < 0, \end{array} \tag{11}$$

where $cc = C_{4,2}/(28C_{4,4})$.

The first equation (11) is satisfied when

a) either $\psi = \{0, \pm\pi/2, \pm\pi\}, \forall cc$,

b) or $\cos 2\psi = cc/2$.

From the second equation (11), we deduce that the solutions $\psi = 0, \pm\pi$ are stable for $cc < 2$, and $\psi = \pm\pi/2$ are stable for $cc > -2$

The second set of solutions $\cos 2\psi = cc/2$ does not satisfy the sufficient conditions of stability (11), and therefore, they are always unstable in the Lyapunov sense.

The plot of the stationary solutions and their stable or unstable character are illustrated in Figure 3.

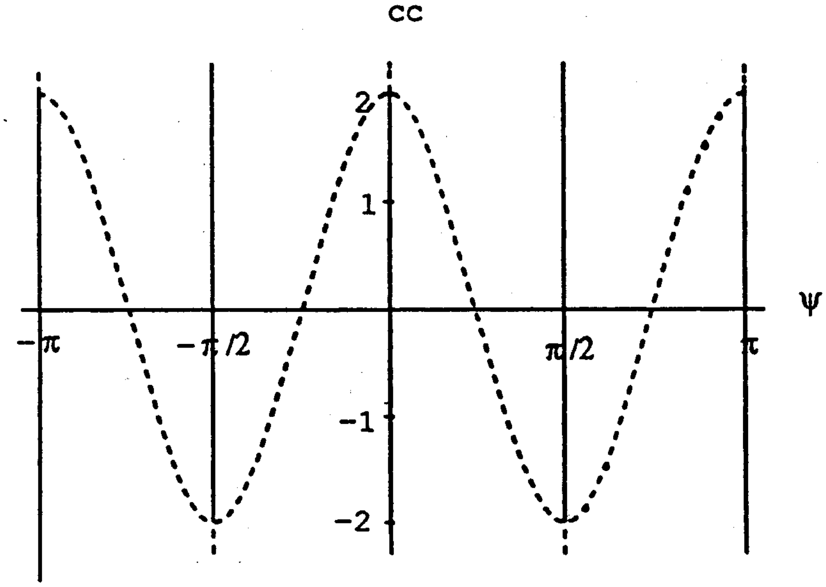

Figure 3 The existence of stationary solutions considering only the fourth harmonics when the body S has two orthogonal planes of symmetry. Dashed lines correspond to unstable solutions and solid lines, to stable points.

4.3 A Body with a Single Plane of Dynamical Symmetry

Let us suppose now that the body has the plane $O\xi\eta$ as the only plane of dynamical symmetry, and that it coincides with the orbital plane. Under this condition, from the equations of motion (10), and after introducing parameter $\tilde{\alpha}$ as

$$\tilde{\alpha} = 28\sqrt{\frac{C_{4,4}^2 + S_{4,4}^2}{C_{4,2}^2 + S_{4,2}^2}},$$

two auxiliary angles ϕ_2 and ϕ_4 defined by

$$\sin\phi_2 = \frac{S_{4,2}}{\sqrt{C_{4,2}^2 + S_{4,2}^2}}, \quad \cos\phi_2 = \frac{C_{4,2}}{\sqrt{C_{4,2}^2 + S_{4,2}^2}},$$
$$\sin\phi_4 = -\frac{S_{4,4}}{\sqrt{C_{4,4}^2 + S_{4,4}^2}}, \quad \cos\phi_4 = -\frac{C_{4,4}}{\sqrt{C_{4,4}^2 + S_{4,4}^2}},$$

and angles Φ, ϕ as

$$\Phi = 2\psi + \phi_2, \quad \phi = \phi_4 - 2\phi_2,$$

after some manipulations, the conditions for the existence of the critical solutions and their stability are given by, respectively:

$$\sin\Phi + \tilde{\alpha}\sin(2\Phi + \phi) = 0,$$

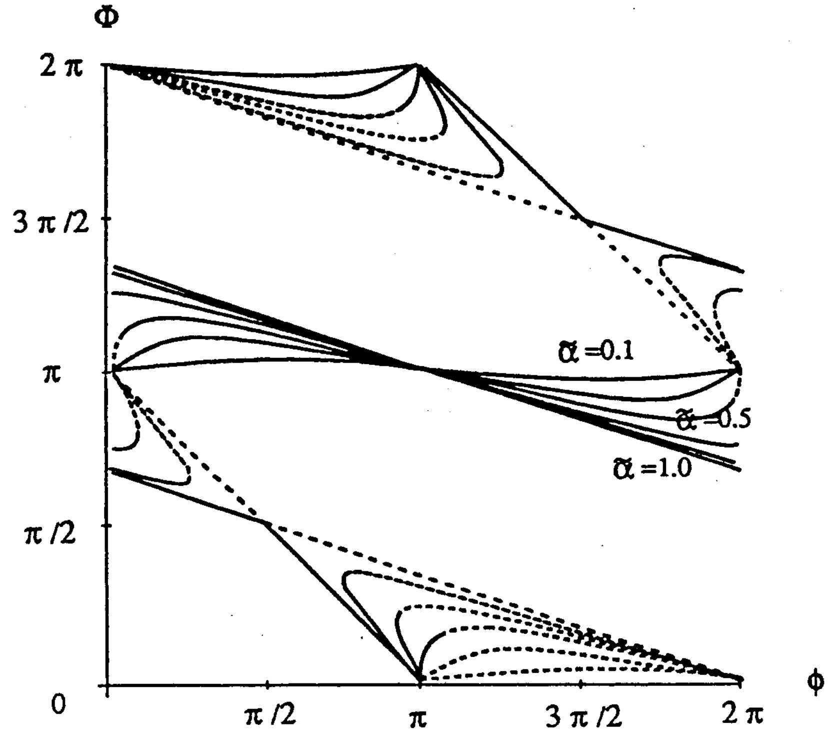

Figure 4 The existence of stationary solution for $\tilde{\alpha}$ = 0.1, 0.3, 0.5, 0.7, 0.9, 1, when only the fourth harmonics is considered. Dashed lines correspond to unstable solutions and solid lines, to stable points.

$$\cos\Phi + 2\tilde{\alpha}\cos(2\Phi + \phi) \quad < \quad 0. \tag{12}$$

We have solved the first of equations (12) for several values of parameter $\tilde{\alpha}$, and found the regions of stability in the parameter plane ϕ, Φ. The result appears in Figure 4.

5 THE UNRESTRICTED PROBLEM

In the two previous sections, we assumed that the orbit is a known function of time, namely a Keplerian circle. When we consider that the attitude and the orbit are not independent, but coupled, the problem is much more involved, since the number of degrees of freedom has increased. In a first step, we will suppose that the orbit is planar, although it depends on the rotational motion. The Hamiltonian of the coupled orbital-rotational motion of the two rigid bodies S_0 and S, is given by (4). The stationary solutions to this Hamiltonian will be obtained by solving the system of equations:

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial P_\rho} = 0, \quad \frac{\partial \mathcal{H}}{\partial \rho} = 0, \\
\frac{\partial \mathcal{H}}{\partial P_\psi} = 0, \quad \frac{\partial \mathcal{H}}{\partial \psi} = 0.
\end{aligned} \tag{13}
$$

From the first one, we have only the trivial solution $P_\rho = 0$. The rest of equations are:

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial P_\psi} &= -\frac{P_\omega - P_\psi}{\mu \rho^2} + \frac{P_\psi}{I} = 0, \\
\frac{\partial \mathcal{H}}{\partial \rho} &= -\frac{(P_\omega - P_\psi)^2}{\mu \rho^3} + \frac{\mathcal{G} m m_0}{\rho^2} \\
&+ \frac{6\mathcal{G} m_0 (mR^3)}{\rho^5} \Big[(-C_{3,1} \cos\psi + S_{3,1} \sin\psi) \\
&\qquad + 10 (C_{3,3} \cos 3\psi - S_{3,3} \sin 3\psi) \Big] \\
&+ \frac{5\mathcal{G} m_0 (mR^4)}{\rho^6} \Big[\frac{3}{8} C_{4,0} - \frac{15}{2} (C_{4,2} \cos 2\psi - S_{4,2} \sin 2\psi) \\
&\qquad + 105 (C_{4,4} \cos 4\psi - S_{4,4} \sin 4\psi) \Big] \\
&+ \frac{6\mathcal{G} m_0 (mR^5)}{\rho^7} \Big[\frac{15}{8} (C_{5,1} \cos\psi - S_{5,1} \sin\psi) \\
&\qquad - \frac{105}{2} (C_{5,3} \cos 3\psi - S_{5,3} \sin 3\psi) \\
&\qquad + 945 (C_{5,5} \cos 5\psi - S_{5,5} \sin 5\psi) \Big] - \frac{\partial W}{\partial \rho} = 0, \\
\frac{\partial \mathcal{H}}{\partial \psi} &= -\frac{3\mathcal{G} m_0 (mR^3)}{2\rho^4} \Big[(C_{3,1} \sin\psi + S_{3,1} \cos\psi) \\
&\qquad - 30 (C_{3,3} \sin 3\psi + S_{3,3} \cos 3\psi) \Big] \\
&- \frac{15\mathcal{G} m_0 (mR^4)}{\rho^5} \Big[(C_{4,2} \sin 2\psi + S_{4,2} \cos 2\psi) \\
&\qquad - 28 (C_{4,4} \sin 4\psi + S_{4,4} \cos 4\psi) \Big] \\
&- \frac{15\mathcal{G} m_0 (mR^5)}{\rho^6} \Big[-\frac{1}{8} (C_{5,1} \sin\psi + S_{5,1} \cos\psi) \\
&\qquad + \frac{21}{2} (C_{5,3} \sin 3\psi + S_{5,3} \cos 3\psi)
\end{aligned}
$$

$$\left.-315(C_{5,5}\sin 5\psi + S_{5,5}\cos 5\psi)\right] - \frac{\partial W}{\partial \psi} = 0, \tag{14}$$

where the function $W(\rho, \psi)$ was defined in Section 2 and contains powers of order 7 and higher of the radial distance.

As usual, we may assume that the size of the satellite is small in comparison with the orbit radius. This assumption allows us to introduce a small parameter

$$\varepsilon = \frac{R}{a} \ll 1,$$

with R the mean radius of the satellite (or the radius of the smallest sphere in which we can put the satellite); and a the radius of the unperturbed circular orbit.

Equations (14) form a system of three algebraic equations in the unknowns P_ω, P_ψ, ψ and ρ, but P_ω may be chosen as a constant value, equal to that in the unperturbed Keplerian motion of the body S, considered as a material point on a circular orbit of radius a and the mean motion $n = (\mathcal{G}(m+m_0)/a^3)^{1/2}$. Then $P_\omega = \mu n a^2$.

For the sake of simplicity, let us introduce three dimensionless quantities p, r and c defined as:

$$p = \frac{P_\psi}{In}, \quad r = \frac{\rho}{a}, \quad c = \frac{I}{mR^2}.$$

After simple algebraic manipulations and making use of the dimensionless quantities defined above, equations (13) for the existence of the critical points adopt the form

$$\begin{aligned}
f &= f_0(p,r) + \varepsilon^2 f_2(p,r) = 0,\\
g &= g_0(p,r) + \varepsilon^3 g_3(r,\psi) + \varepsilon^4 g_4(r,\psi) + \ldots = 0,\\
h &= h_0(\psi) + \varepsilon h_1(r,\psi) + \varepsilon^2 h_2(r,\psi) + \varepsilon^3 h_3(r,\psi) + \ldots = 0,
\end{aligned} \tag{15}$$

where functions f_i, g_i, h_i are

$$\begin{aligned}
f_0 &= p - \frac{1}{r^2},\\
f_2 &= \frac{\beta p}{r^2}, \quad \text{with } \beta = \frac{cm}{\mu},\\
g_0 &= -p^2 + \frac{1}{r^3},\\
g_3 &= \frac{6}{r^6}\left[(-C_{3,1}\cos\psi + S_{3,1}\sin\psi) + 10(C_{3,3}\cos 3\psi - S_{3,3}\sin 3\psi)\right],\\
g_4 &= \frac{5}{r^7}\left[\frac{3}{8}C_{4,0} - \frac{15}{2}(C_{4,2}\cos 2\psi - S_{4,2}\sin 2\psi)\right.\\
&\qquad \left. + 105(C_{4,4}\cos 4\psi - S_{4,4}\sin 4\psi)\right],
\end{aligned}$$

$$
\begin{aligned}
h_0 &= (C_{3,1}\sin\psi + S_{3,1}\cos\psi) - 30(C_{3,3}\sin 3\psi + S_{3,3}\cos 3\psi), \\
h_1 &= \frac{10}{r}\left[(C_{4,2}\sin 2\psi + S_{4,2}\cos 2\psi) - 28(C_{4,4}\sin 4\psi + S_{4,4}\cos 4\psi)\right], \\
h_2 &= \frac{10}{r^2}\left[-\frac{1}{8}(C_{5,1}\sin\psi + S_{5,1}\cos\psi) + \frac{21}{2}(C_{5,3}\sin 3\psi + S_{5,3}\cos 3\psi)\right. \\
&\qquad \left. -315(C_{5,5}\sin 5\psi + S_{5,5}\cos 5\psi)\right], \\
h_3 &= -\frac{140}{r^3}\left[\frac{1}{8}(C_{6,2}\sin 2\psi + S_{6,2}\cos 2\psi) - 9(C_{6,4}\sin 4\psi + S_{6,4}\cos 4\psi)\right. \\
&\qquad \left. +297(C_{6,6}\sin 6\psi + S_{6,6}\cos 6\psi)\right].
\end{aligned}
$$

In order to obtain the critical points, we have to solve system (15). To perform that, we follow a scheme given in Barkin (1985) and based on recurrent power series (Steffensen, 1956): we try solutions in the form of power series in the small parameter ε,

$$
\begin{aligned}
r &= r_0 + \varepsilon r_1 + \varepsilon^2 r_2 + \varepsilon^3 r_3 + \ldots, \\
\psi &= \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \ldots, \\
p &= p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \varepsilon^3 p_3 + \ldots,
\end{aligned} \tag{16}
$$

and first of all, we compute the zero order approximation (r_0, ψ_0, p_0) to the solution. With this approximation, we determine the first order (r_1, ψ_1, p_1), and so on.

The equations for the zero order approximation ($\varepsilon = 0$) are

$$
\begin{aligned}
f_0 &= p_0 - \frac{1}{r_0^2} = 0, \\
g_0 &= -p_0^2 + \frac{1}{r_0^3} = 0, \\
h_0 &= (C_{3,1}\sin\psi_0 + S_{3,1}\cos\psi_0) - 30(C_{3,3}\sin 3\psi_0 + S_{3,3}\cos 3\psi_0) = 0.
\end{aligned}
$$

A solution of these equantions is

$$
r_0 = 1, \quad p_0 = 1, \quad \psi_0 = \psi_0(\alpha, \phi), \tag{17}
$$

where the angle $\psi_0(\alpha, \phi)$ depends on the coefficients of the third harmonics, and its value is obtained by solving the first of equations (8). Taking into account the above definition of the parameters, we obtain the values of the variables at the zero order approximation:

$$
\rho_0 = a, \quad \psi = \psi_0, \quad P_\psi^0 = In.
$$

Consequently, the zero order approximation coincides with the solution of the analogous restricted problem, previously considered in Section 3.

By using the implicit function theorem, we can confirm the existence of a stationary solution of the unrestricted problem in the neighborhood of solutions (17), and they may be expressed in terms of power series in small parameter ε, which are convergent for sufficiently small values of the parameter when the following condition is satisfied:

$$\Delta = \det \frac{\partial(f_0, g_0, h_0)}{\partial(p, r, \psi)} \neq 0.$$

This determinant must be computed with the unperturbed values of the variables (17). In our problem, we have

$$\frac{\partial f_0}{\partial \psi_0} = \frac{\partial g_0}{\partial \psi_0} = \frac{\partial h_0}{\partial p_0} = \frac{\partial h_0}{\partial r_0} = 0,$$

and therefore

$$\Delta = \Delta_1 \frac{\partial h_0}{\partial \psi_0},$$

where

$$\Delta_1 = \frac{\partial f_0}{\partial p_0}\frac{\partial g_0}{\partial r_0} - \frac{\partial f_0}{\partial r_0}\frac{\partial g_0}{\partial p_0} = 1,$$

and therefore, the sufficient condition for the existence is reduced to

$$\frac{\partial h_0}{\partial \psi_0} = (C_{3,1}\cos\psi_0 - S_{3,1}\sin\psi_0) - 90(C_{3,3}\cos 3\psi_0 - S_{3,3}\sin 3\psi_0) \neq 0;$$

and proceeding as in Section 3, we have

$$\cos\Phi + 3\alpha\cos(3\Phi + \phi) \neq 0,$$

where $\Phi = \psi_0 + \phi_1$, $\phi = \phi_3 - 3\phi_1$, and the angles ϕ_1, ϕ_3 were defined in section 3. Since a bifurcation point appears when the determinant vanishes, this situation has place when the angles Φ and ϕ are in the relation

$$3\tan\Phi = \tan(3\Phi + \phi).$$

Before proceeding to higher orders, let us recall that if we have a function $F(x, y)$, such that its arguments are power series in ε,

$$\begin{aligned} x &= x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots, \\ y &= y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots, \end{aligned}$$

the expansion of F in power series is

$$
\begin{aligned}
\boldsymbol{F} &= F\Big|_{x_0,y_0} + \varepsilon\left(\frac{\partial F}{\partial x}\Big|_{x_0,y_0} x_1 + \frac{\partial F}{\partial y}\Big|_{x_0,y_0} y_1\right) \\
&+ \varepsilon^2\left(\frac{\partial F}{\partial x}\Big|_{x_0,y_0} x_2 + \frac{\partial F}{\partial y}\Big|_{x_0,y_0} y_2\right. \\
&\quad\left. +\frac{1}{2}\frac{\partial^2 F}{\partial x^2}\Big|_{x_0,y_0} x_1^2 + \frac{\partial^2 F}{\partial x \partial y}\Big|_{x_0,y_0} x_1 y_1 + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}\Big|_{x_0,y_0} y_1^2\right) + \ldots. \qquad (18)
\end{aligned}
$$

Substituting series (16) into equations (15) and making use of the expansion (18), we will have higher order approximations for the stationary solution. We need only to solve the system obtained by making zero the coefficients ε, ε^2, etc.

The task appears to be very tedious, but it is very simple to perform with the help of an algebraic manipulator. After making the substitutions, we have

$$
\begin{aligned}
\boldsymbol{f} &= \left(p_0 - \frac{1}{r_0^2}\right) + \varepsilon\left(p_1 + \frac{2r_1}{r_0^3}\right) \\
&+ \varepsilon^2\left(p_2 + \frac{\beta p_0}{r_0^2} - \frac{3r_1^2}{r_0^4} + \frac{2r_2}{r_0^3}\right) \\
&+ \varepsilon^3\left(p_3 + \frac{\beta p_1}{r_0^2} - \frac{2\beta p_0 r_1}{r_0^3} + \frac{4r_1^3}{r_0^5} - \frac{6r_1 r_2}{r_0^4} + \frac{2r_3}{r_0^3}\right) \\
&+ O(\varepsilon^4),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{g} &= \left(\frac{1}{r_0^3} - p_0^2\right) + \varepsilon\left(-2p_0p_1 - \frac{3r_1}{r_0^4}\right) \\
&+ \varepsilon^2\left(-2p_0p_2 - p_1^2 + \frac{6r_1^2}{r_0^5} - \frac{3r_2}{r_0^4}\right) \\
&+ \varepsilon^3\left(-2p_1p_2 - 2p_0p_3 - \frac{10r_1^3}{r_0^6} + \frac{12r_1r_2}{r_0^5} - \frac{3r_3}{r_0^4} + \frac{6D_3}{r_0^6}\right) \\
&+ O(\varepsilon^4),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{h} &= (E_3) + \varepsilon\left(\frac{10E_4}{r_0} + \psi_1\frac{dE_3}{d\psi_0}\right) \\
&+ \varepsilon^2\left(-\frac{10r_1E_4}{r_0^2} + \frac{10E_5}{r_0^2} + \psi_2\frac{dE_3}{d\psi_0} + \frac{10\psi_1}{r_0}\frac{dE_4}{d\psi_0} + \frac{1}{2}\psi_1^2\frac{d^2E_3}{d\psi_0^2}\right) \\
&+ \varepsilon^3\left(-\frac{10r_2E_4}{r_0^2} + \frac{10r_1^2E_4}{r_0^3} - \frac{20r_1E_5}{r_0^3}\right. \\
&\quad +\psi_3\frac{dE_3}{d\psi_0} + \frac{10\psi_2}{r_0}\frac{dE_4}{d\psi_0} - \frac{10\psi_1 r_1}{r_0^2}\frac{dE_4}{d\psi_0} + \frac{10\psi_1}{r_0^2}\frac{dE_5}{d\psi_0}
\end{aligned}
$$

$$+\psi_1\psi_2\frac{dE_3}{d\psi_0}+\frac{5\psi_1^2}{r_0}\frac{d^2E_4}{d\psi_0^2}+\frac{1}{6}\psi_1^3\frac{d^3E_3}{d\psi_0^3}+\frac{140}{r_0^3}E_6\Big)$$
$$+\quad O(\varepsilon^4),$$

where functions $D_i(\psi)$ and $E_i(\psi)$ are the coefficients appearing in the definition of functions g and h, depending of the i-th harmonics, that is

$$D_3=\frac{r^6}{6}g_3,\quad E_3=h_0,\quad E_4=\frac{r}{10}h_1,\quad E_5=\frac{r^2}{10}h_2,\quad E_6=\frac{r^3}{140}h_3.$$

When we replace r_0, ψ_0, and p_0 by the zero-order solutions (17), functions f, g and h become

$$\begin{aligned}
f &= \varepsilon(p_1+2r_1)+\varepsilon^2(\beta+p_2-3r_1^2+2r_2)\\
&+ \varepsilon^3(\beta p_1+p_3-2\beta r_1+4r_1^3-6r_1r_2+2r_3)\\
&+ O(\varepsilon^4),
\end{aligned}$$

$$\begin{aligned}
g &= \varepsilon(-2p_1-3r_1)+\varepsilon^2(-p_1^2-2p_2+6r_1^2-3r_2)\\
&+ \varepsilon^3(-2p_1p_2-2p_3-10r_1^3+12r_1r_2-3r_3+6D_3)\\
&+ O(\varepsilon^4),
\end{aligned}$$

$$\begin{aligned}
h &= \varepsilon\left(10E_4+\psi_1\frac{dE_3}{d\psi_0}\right)\\
&+ \varepsilon^2\left(-10r_1E_4+10E_5+\psi_2\frac{dE_3}{d\psi_0}+10\psi_1\frac{dE_4}{d\psi_0}+\frac{1}{2}\psi_1^2\frac{d^2E_3}{d\psi_0^2}\right)\\
&+ \varepsilon^3\Big(-10r_2E_4+10r_1^2E_4-20r_1E_5+\psi_3\frac{dE_3}{d\psi_0}+10\psi_2\frac{dE_4}{d\psi_0}\\
&\qquad -10\psi_1r_1\frac{dE_4}{d\psi_0}+10\psi_1\frac{dE_5}{d\psi_0}\psi_1\psi_2\frac{dE_3}{d\psi_0}+5\psi_1^2\frac{d^2E_4}{d\psi_0^2}\\
&\qquad +\frac{1}{6}\psi_1^3\frac{d^3E_3}{d\psi_0^3}+140E_6\Big)\\
&+ O(\varepsilon^4).
\end{aligned}$$

The first-order approximation is obtained by equating to zero the coefficients of ε in f, g and h, and solving the system in the unknowns p_1, r_1 and ψ_1. The solution is:

$$\begin{aligned}
p_1 &= 0,\\
r_1 &= 0,\\
\psi_1 &= -10E_4/\frac{dE_3}{d\psi_0}.
\end{aligned}\tag{19}$$

Once having determined the values p_1, r_1 and ψ_1, we may obtain the second-order approximation in an analogous way. The system to solve is

$$\begin{aligned}\beta + p_2 + 2r_2 &= 0,\\ -2p_2 - 3r_2 &= 0,\\ 10E_5 + \psi_2\frac{dE_3}{d\psi_0} + 10\psi_1\frac{dE_4}{d\psi_0} + \frac{1}{2}\psi_1^2\frac{d^2E_3}{d\psi_0^2} &= 0,\end{aligned}$$

and its solution is

$$\begin{aligned}p_2 &= 3\beta,\\ r_2 &= -2\beta,\\ \psi_2 &= -\left(10E_5\left(\frac{dE_3}{d\psi_0}\right)^2 - 100E_4\frac{dE_3}{d\psi_0}\frac{dE_4}{d\psi_0} + 50E_4^2\frac{d^2E_3}{d\psi_0^2}\right) / \left(\frac{dE_3}{d\psi_0}\right)^3\end{aligned}$$

From these values, we see that the second-order perturbations of the orbital variables – r and p – do not contain terms due to the shape of the body S. The direct influence of gravitational attraction of the bodies S_0 and S appears in the third approximation:

$$\begin{aligned}p_3 &= 12D_3,\\ r_3 &= -6D_3,\\ \psi_3 &= \frac{-10}{3\left(\frac{dE_3}{d\psi_0}\right)^5}\left[12\beta E_4\left(\frac{dE_3}{d\psi_0}\right)^4 - 30E_5\left(\frac{dE_3}{d\psi_0}\right)^3\left(\frac{dE_4}{d\psi_0}\right)\right.\\ &+ 300E_4\left(\frac{dE_3}{d\psi_0}\right)^2\left(\frac{dE_4}{d\psi_0}\right)^2 - 30E_4\left(\frac{dE_3}{d\psi_0}\right)^3\left(\frac{dE_5}{d\psi_0}\right)\\ &+ 30E_4E_5\left(\frac{dE_3}{d\psi_0}\right)^2\left(\frac{d^2E_3}{d\psi_0^2}\right) - 450E_4^2\left(\frac{dE_3}{d\psi_0}\right)\left(\frac{dE_4}{d\psi_0}\right)\left(\frac{d^2E_3}{d\psi_0^2}\right)\\ &+ 150E_4^3\left(\frac{d^2E_3}{d\psi_0^2}\right)^2 + 150E_4^2\left(\frac{dE_3}{d\psi_0}\right)^2\left(\frac{d^2E_4}{d\psi_0^2}\right)\\ &\left.- 50E_4^3\left(\frac{dE_3}{d\psi_0}\right)\left(\frac{d^3E_3}{d\psi_0^3}\right)\right].\end{aligned}$$

Therefore, coming back to the original variables, we have the solution

$$\begin{aligned}\rho &= a(1 + \varepsilon^2 r_2 + \varepsilon^3 r_3 + \ldots),\\ P_\psi &= In(1 + \varepsilon^2 p_2 + \varepsilon^3 p_3 + \ldots),\\ \psi &= \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \ldots.\end{aligned}$$

All these formulae describe the effects of the nonsphericity of the satellite on its altitude at a circular orbit. Besides, it follows that the orbit of the satellite is non-Keplerian, and its radius vector is defined by parameters of the gravitational field of the satellite and angle ψ_0. The orientation of the satellite is defined by angle ψ_0 and it depends on higher harmonics of the force function.

5.1 Stability in the Unrestricted Problem

Let us analyze the stability of the stationary solutions just obtained. To perform this task, we need the equations of motion in the neighborhood of the equilibria. The Hamiltonian is a first integral, and it will be used as the Lyapunov function. As it is well known, sufficient conditions for the stability in the Lyapunov sense coincide with the requirement for the Hamiltonian to be positive definite. After simple manipulations, we obtain the following conditions for the stability of the stationary solutions:

$$A_{11} > 0, \quad A_{11}A_{22} - A_{12}^2 > 0, \tag{20}$$

where the coefficients A_{ij} are the quadratic part of the transformed Hamiltonian,

$$A_{11} = \frac{\mu P_\psi^2}{I^2}(3m\rho^2 - I) - \frac{\partial^2 U}{\partial \rho^2}, \quad A_{12} = -\frac{\partial^2 U}{\partial \rho \partial \psi}, \quad A_{22} = -\frac{\partial^2 U}{\partial \psi^2}, \tag{21}$$

and the second derivatives of the force function (1),

$$\begin{aligned}
\frac{\partial^2 U}{\partial \rho^2} &= \frac{\mathcal{G} m m_0}{\rho^3} \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \left(\frac{R}{\rho}\right)^n (n+2)(n+1) P_{n,k} \\
&\times \left(C_{n,n-2k} \cos(n-2k)\psi - S_{n,n-2k} \sin(n-2k)\psi\right), \\
\frac{\partial^2 U}{\partial \psi^2} &= \frac{\mathcal{G} m m_0}{\rho} \sum_{n=3}^{\infty} \sum_{k=0}^{[n/2]} \left(\frac{R}{\rho}\right)^n (n-2k)^2 P_{n,k} \\
&\times \left(-C_{n,n-2k} \cos(n-2k)\psi + S_{n,n-2k} \sin(n-2k)\psi\right), \\
\frac{\partial^2 U}{\partial \rho \partial \psi} &= \frac{\mathcal{G} m m_0}{\rho^2} \sum_{n=3}^{\infty} \sum_{k=0}^{[n/2]} \left(\frac{R}{\rho}\right)^n (n+1)(n-2k) P_{n,k} \\
&\times \left(C_{n,n-2k} \sin(n-2k)\psi + S_{n,n-2k} \cos(n-2k)\psi\right).
\end{aligned} \tag{22}$$

All these derivatives must be evaluated at the stationary solutions, that have been obtained as series expansions in small parameter ε.

When $\varepsilon = 0$, conditions (20) reduce to

$$\begin{aligned}
3\mu n^2 - \frac{\mathcal{G} m m_0}{\rho^3} = \mu n^2 &> 0, \\
(C_{3,1}\cos\psi_0 - S_{3,1}\sin_{\psi_0}) - 90\,(C_{3,3}\cos 3\psi_0 - S_{3,3}\sin 3\psi_0) &> 0.
\end{aligned} \tag{23}$$

As it might be expected, conditions (23) coincide with the stability conditions (8) obtained for the unrestricted problem.

The detailed analysis of the influence of higher orders of perturbation in the small parameter ε are not given here. Let us mention only that these additional terms in the conditions of stability lead to an unimportant deformation in the boundary of the region of stability of the stationary solutions.

5.2 A Particular Case ($S_{3,i} = C_{3,i} = 0$)

We have seen that in the zero order approximation, the equilibria and the stability conditions coincide with those obtained in the restricted problem, when all harmonics but the third are null. Now, we shall consider a particular case of the unrestricted problem in which the coefficients of the third harmonics are null ($S_{3,i} = C_{3,i} = 0$). In this situation, equations (5) for the equilibria become

$$\begin{aligned} f_0(p,r) + \varepsilon^2 f_2(p,r) &= 0, \\ g_0(r) + \varepsilon^4 g_4(r,\psi) + \dots &= 0, \\ h_1(\psi) + \varepsilon h_2(r,\psi) + \varepsilon^2 h_3(r,\psi) + \dots &= 0, \end{aligned} \tag{24}$$

where f_0, f_2, g_0, g_4, h_1, h_2 and h_3 have been already defined. By setting $\varepsilon = 0$ in equations (24), we obtain the zero order approximation:

$$r_0 = 1, \quad p_0 = 1, \quad \psi = \psi_0(\tilde{\alpha}, \phi),$$

where $\psi_0(\tilde{\alpha}, \theta)$ was defined in equation (12). As expected, the zero order approximation coincides with the exact solution of the restricted problem.

A sufficient condition for the existence of solutions of equations (24) as expansions in series in small parameter ε is that the Jacobian

$$\Delta = \det \frac{\partial(f_0, g_0, h_1)}{\partial(p, r, \psi)} \neq 0,$$

or equivalently

$$(C_{4,2} \cos 2\psi_0 - S_{4,2} \sin 2\psi_0) - 56(C_{4,4} \cos 4\psi_0 - S_{4,4} \sin 4\psi_0) \neq 0,$$

the condition which may be written in the form

$$\cos \Phi + 2\tilde{\alpha} \cos(2\Phi + \phi) \neq 0.$$

As in Section 4, a bifurcation point appears when the parameters Φ and ϕ are in the relation

$$2 \tan \Phi = \tan(2\Phi + \phi).$$

The procedure to obtain different elements of the series is completely analogous to the previous part, and we do no repeat it. We only give the final formula for the first order perturbation of the angle of rotation ψ_1:

$$\begin{aligned}\psi_1 &= -\Big[-\frac{1}{8}(C_{5,1}\sin\psi_0 + S_{5,1}\cos\psi_0) + \frac{21}{2}(C_{5,3}\sin 3\psi_0 + S_{5,3}\cos 3\psi_0) \\ &- 315(C_{5,5}\sin 5\psi_0 + S_{5,5}\cos 5\psi_0)\Big] \\ &\times \frac{1}{2\left[(C_{4,2}\cos 2\psi_0 - S_{4,2}\sin 2\psi_0) - 56(C_{4,4}\cos 4\psi_0 - S_{4,4}\sin 4\psi_0)\right]}.\end{aligned}$$

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