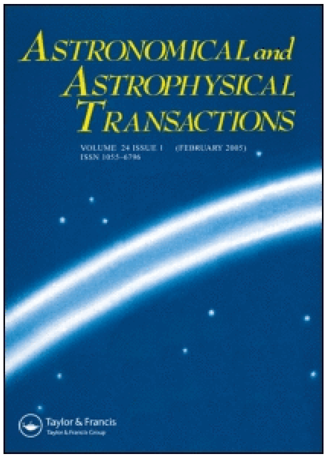

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ELECTRIC FIELDS OF NEUTRON STARS

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Theoretical aspects of the electric fields of magnetized neturon stars are reviewed. A fundamental role of the General Relativity effect of dragging inertial reference frames in generating electric fields and accelerating relativistic particles by neutron stars (pulsars) is emphasized. An allowance for this effect in the framework of the Goldreich-Julian model (for the regime of free plasma ejection from the neutron star surface) leads to the electric field enhancement by two orders of matnitude, as compared to the flat space-time case.

KEY WORDS Neutron stars, electric fields, general relativity.

1. ELECTRIC FIELD NEAR A ROTATING NEUTRON STAR IN VACUUM

A model of a rotating conducting sphere with a dipole magnetic field $\vec{B} = [3\vec{n}(\vec{m}\vec{n}) - \vec{m}]/r^3$ (where $\vec{m}$ is the magnetic dipole moment, $\vec{n} = \vec{r}/r$) is commonly used for a study of an electric field of a neutron star (NS). Deutsch (1955) calculated the electric and magnetic fields in the rest frame at any distance from the rotating sphere. However, it is convenient to have a solution in a NS corotating frame in order to study charged particle acceleration or their filling the NS magnetosphere. In the quasistatic region ($\Omega r \ll c$) of the corotating frame, the electric field is time-independent and possesses an electrostatic potential that obeys the following equation (Goldreich and Julian, 1969):

$$\Delta\phi = -4\pi(\rho + \rho_{\text{eff}}); \qquad \vec{E} = -\text{grad}\,\phi,$$
$$\rho_{\text{eff}} = \frac{1}{4\pi}\,\text{div}(\vec{g} \times \vec{B}) = \frac{\vec{\Omega}\vec{B}}{2\pi c}, \tag{1}$$

where $\vec{g} = (\vec{\Omega} \times \vec{r})/c$ and $\vec{\Omega}$ is the angular velocity of the sphere. If the sphere rotates in vacuum, then $\rho = 0$ outside the sphere, and the electric field vanishes inside the conducting sphere, $\vec{E} = 0$, $\phi = \phi_1 = \text{const}$. A solution of Eq. (1) that satisfies the boundary conditions $\phi|_{r=a} = \phi_1$; $\phi|_{r\to\infty} = 0$ is (Tsygan, 1980)

$$\phi = \phi_1 \frac{a}{r} + \frac{3(\vec{\Omega}\vec{n})(\vec{m}\vec{n}) - \vec{\Omega}\vec{m}}{3ca}\left[\frac{a}{r} - \left(\frac{a}{r}\right)^3\right] \tag{2}$$

where a is the sphere radius. The surface charge density induced on the sphere is

$$\sigma = \frac{1}{4\pi}E_n\bigg|_{r=a} = -\frac{1}{4\pi}\frac{\partial\phi}{\partial r}\bigg|_{r=a} = \frac{\phi_1}{4\pi a} - \frac{3(\vec{\Omega}\vec{n})(\vec{m}\vec{n}) - \vec{\Omega}\vec{m}}{6\pi c a^2}. \tag{3}$$

The absence of the electric field inside the sphere means that the electric charge is induced within the sphere, with the charge density of $\rho = -\rho_{\rm eff} = -(1/4\pi)\,{\rm div}(\vec{g} \times \vec{B})$. The total charge of the sphere equals

$$Q = \int \sigma\,{\rm d}S - \frac{1}{4\pi}\int {\rm div}(\vec{g} \times \vec{B})\,{\rm d}V = \int \sigma\,{\rm d}S - \frac{1}{4\pi}\int (\vec{g} \times \vec{B})\vec{n}\,{\rm d}S. \quad (4)$$

For an uncharged sphere ($Q = 0$), one obtains $\phi_1 = -2(\vec{\Omega}\vec{m})/3ca$. The electric field in the rotating reference frame is

$$\vec{E} = \frac{a^2}{cr^4}\{\vec{\Omega}(\vec{m}\vec{n}) + \vec{m}(\vec{\Omega}\vec{n}) + \vec{n}[(\vec{\Omega}\vec{m}) - 5(\vec{\Omega}n)(\vec{m}\vec{n})]\}$$
$$- \frac{1}{cr^2}\{\vec{\Omega}(\vec{m}\vec{n}) + \vec{m}(\vec{\Omega}\vec{n}) + \vec{n}[(\vec{\Omega}\vec{m}) - 3(\vec{\Omega}\vec{n})(\vec{m}\vec{n})]\}. \quad (5)$$

The transition to the inertial (laboratory) reference frame is performed via the transformation

$$\vec{E} = \vec{E} + \frac{1}{c}\vec{B} \times \vec{v}; \qquad \vec{v} = \vec{\Omega} \times \vec{r}. \quad (6)$$

In this case $\vec{r} = \vec{r}'$ (below we shall omit prime assuming $\vec{r}$ to be the radius vector in the lab reference frame):

$$\vec{E} = \frac{a^2}{cr^4}\{\vec{\Omega}(\vec{m}\vec{n}) + \vec{m}(\vec{\Omega}\vec{n}) + \vec{n}[(\vec{\Omega}\vec{m}) - 5(\vec{\Omega}\vec{n})(\vec{m}\vec{n})]\}$$
$$+ \frac{1}{cr^2}\{\vec{\Omega}(\vec{m}\vec{n}) - \vec{m}(\vec{\Omega}\vec{n})\}. \quad (7)$$

This expression for $\vec{E}$ coincides with the expression given by Deutsch (1955) in the quasistatic region. For a NS with $B \sim 10^{12}$ G, $\Omega \sim 10$ s, $a \sim 10$ km, we obtain $E \sim (\Omega a/c)B \sim 3 \times 10^8$ CGSE. This electric field is capable of ejecting charged particles from the NS surface. The ejected particles will form a NS magnetosphere even if it was absent earlier.

2. NS ELECTRIC FIELD IN THE PRESENCE OF THE MAGNETOSPHERE

Let us consider a NS surrounded by a corotating magnetosphere. The magnetosphere was introduced into the physics of radiopulsars by Goldreich and Julian (1969).

For $\Omega r \ll c$, the magnetospheric electric charge density is $\rho = -\rho_{\rm eff} = -\vec{\Omega}\vec{B}/2\pi c$. The charge density defined in such a way cancels the electric field in the rotating reference frame in a closed magnetosphere. Ruderman and Sutherland (1975) assumed that the work of exit of ions (e.g., of iron) off the NS surface is about 10 keV. If the temperature of the NS polar caps is lower than 1 keV, then the thermoemission of ions in the regions of open magnetic field lines is suppressed, and these regions are not filled with the plasma. For simplicity, one assumes that magnetic field lines at the cone bottom are perpendicular to the stellar surface.

Consider the case of $\theta = \theta_0\sqrt{r/a} \ll 1$ ($\theta_0 = \sqrt{\Omega a/c} \sim 1.4 \times 10^{-2}$ for the pulsar period of $P \sim 1$ s). Inside the region of study, Eq. (1) holds, with $\rho = 0$ and with the boundary condition $\phi|_S = \phi_1 = \text{const}$ at the surface and the cone bottom; it is convenient to choose $\phi_1 = 0$. The above boundary conditions are fulfilled owing to the presence of charged particles that fill the magnetosphere, and a finite stellar conductivity. For $(r - a) \gg R_0$ or $(\eta - 1) \gg \theta_0$ ($R_0 = a\theta_0$ is the radius of the cone bottom), the electrostatic potential $\phi(\eta, \xi, \varphi)$ and the electric field component $E_{\|}$ along the magnetic field are

$$\begin{aligned}
\phi(\eta, \xi, \varphi) &= \tfrac{1}{2}\phi_0\theta_0^2[(1-\xi^2)\cos\chi + \tfrac{3}{4}\theta_0\sqrt{\eta}\,\xi(1-\xi^2)\sin\chi\cos\varphi],\\
E_{\|} &= -\frac{3}{16}\frac{\phi_0}{a}\theta_0^3\frac{1}{\sqrt{\eta}}\xi(1-\xi^2)\sin\chi\cos\varphi,\\
\rho_{\text{eff}} &= \frac{\Omega B_0}{2\pi c}\frac{1}{\eta^3}\{\cos\chi + \tfrac{3}{2}\xi\theta(\eta)\sin\chi\cos\varphi\}
\end{aligned} \tag{8}$$

where $\eta = r/a$; $\xi = \vartheta/\theta(\eta) = \vartheta/(\theta_0\sqrt{\eta})$ (ξ is constant along magnetic lines); $\phi_0 = (\Omega a/c)B_0 a$; B_0 is the magnetic field at the NS pole, χ is the angle between Ω and $\vec{m}$.

The solution near the stellar surface ($(r - a) \lesssim a$ or $(\eta - 1) \lesssim 1$) satisfying the boundary conditions $\phi|_{\eta=1} = \phi|_{\xi=1} = 0$ can be written as

$$\begin{aligned}
\phi(\eta, \xi, \varphi) = 4\phi_0\theta_0^2\Bigg\{&\cos\chi\sum_{i=1}^{\infty}\left[1 - \exp\left(-\frac{k_i}{\theta_0}(\eta-1)\right)\right]\frac{J_0(k_i\xi)}{k_i^3 J_1(k_i)}\\
&+ \tfrac{3}{2}\theta_0\sin\chi\cos\varphi\sum_{i=1}^{\infty}\left[1 - \exp\left(-\frac{\tilde{k}_i}{\theta_0}(\eta-1)\right)\right]\frac{J_1(\tilde{k}_i\xi)}{\tilde{k}_i^3 J_2(\tilde{k}_i)},
\end{aligned} \tag{9}$$

where k_i are positive solutions of the equation $J_0(k_i) = 0$, $\tilde{k}_i$ are positive solutions of $J_1(\tilde{k}_i) = 0$, and $J_\nu(z)$ is the Bessel function.

Solutions (8) and (9) are matched at $\theta_0 \ll (\eta - 1) \ll 1$. To prove this, one should take into account the relationships

$$\sum_{i=1}^{\infty}\frac{J_0(k_i\xi)}{k_i^3 J_1(k_i)} = \tfrac{1}{8}(1-\xi^2); \qquad \sum_{i=1}^{\infty}\frac{J_1(\tilde{k}_i\xi)}{\tilde{k}_i^3 J_2(\tilde{k}_i)} = \tfrac{1}{16}\xi(1-\xi^2). \tag{10}$$

Let us notice that the electric field $E \sim (\Omega a/c)^{3/2}B_0$ reaches maximum in a small region $(r - a) \lesssim R_0$ near the neutron star, and the maximum magnitude differs from that in the absence of the magnetosphere by a factor of $\sqrt{\Omega a/c}$. For a puslar with $P \lesssim 1$ s, the magnitude of this electric field is sufficient to generate an electron-positron avalanche (Ruderman and Sutherland, 1975, Sturrok, 1971). This produces a "vacuum" electric discharge whose top edge is formed by the electron-positron plasma, and the bottom edge is the neutron star surface. The electrostatic potential and electric field within the discharge of the height $az_0 = a(\eta_0 - 1)$ are

$$\begin{aligned}
\phi &= 2\phi_0 z\left(z_0 - \frac{z}{2}\right)(\cos\chi + \tfrac{3}{2}\xi\theta_0\sin\chi\cos\phi),\\
E_{\|} &= -\frac{2\phi_0}{a}(z_0 - z)(\cos\chi + \tfrac{3}{2}\xi\theta_0\sin\chi\cos\varphi),
\end{aligned} \tag{11}$$

with $z \equiv (\eta - 1) \ll 1$; $(\xi - 1) \gg z_0$, $E_{\|}|_{z=z_0} = 0$.

Recent calculations (Jones, 1985, Neuhauser *et al.*, 1987) yield a rather low work of exit of ions off the NS surface, about 200–300 keV. For a surface temperature of $\sim 10^6$ K in the polar regions, thermoemission of ions occurs, and a "free-emission" regime of charged particles becomes possible (Arons and Scharlemann, 1979). In the latter regime, the charged particle ejection rate in the region of open magnetic lines is close to that corresponding to the total screening of the electric field near the surface ($E_r|_{r=a} = 0$). Since charged particles near the NS surface are accelerated to relativistic energies and move along magnetic field lines, the electric current density can be written as

$$\vec{j} = c\rho \frac{\vec{B}}{B}; \qquad \operatorname{div} \vec{j} = c\vec{B} \operatorname{grad} \frac{\rho}{B} = 0. \tag{12}$$

For $(\eta - 1) \gg \theta_0$, the electrostatic potential is given by

$$\phi(\eta, \xi, \varphi) = \tfrac{3}{8}\phi_0\theta_0^3(\sqrt{\eta} - 1)\xi(1 - \xi^2) \sin\chi \cos\varphi, \tag{13}$$

and

$$\rho = -\frac{\Omega B_0}{2\pi c}\frac{1}{\eta^3}(\cos\chi + \tfrac{3}{2}\theta_0\sqrt{\eta}\,\xi \sin\chi \cos\varphi).$$

For $(\eta - 1) \ll 1$, using for the boundary conditions $\phi|_{\eta=1} = \phi|_{\xi=1} = 0$; $(\partial\phi/\partial\eta)|_{\eta=1} = 0$, we obtain the solution

$$\phi(\eta, \xi, \varphi) = 3\phi_0\theta_0^4 \sin\chi \cos\varphi \times \sum_{i=1}^{\infty} \left[\exp\left(-\frac{\bar{k}_i}{\theta_0}(\eta - 1)\right) - 1 + \frac{\bar{k}_i(\eta - 1)}{\theta_0}\right] \frac{J_1(\bar{k}_i\xi)}{\bar{k}_i^4 J_2(\bar{k}_i)}. \tag{14}$$

Solutions (13) and (14) match at $\theta_0 \ll (\eta - 1) \ll 1$. Near the star, the longitudinal electric field component is given by

$$E_{\parallel} \approx -\frac{3}{16}\frac{\phi_0}{a}\theta_0^3 \frac{1}{\sqrt{\eta}}\xi(1 - \xi^2)\sin\chi\cos\varphi, \tag{15}$$

and $E_{\parallel} = 0$ at the surface. For typical parameters $P \sim 1$ s and $B_0 \sim 10^{12}$ G, this field is insufficient for generating an electron-positron avalanche.

3. NS ELECTRIC FIELDS WITH ACCOUNT FOR GENERAL RELATIVELY EFFECTS

In the polar coordinate frame $\tilde{x}^1 = r$; $\tilde{x}^2 = \tilde{\vartheta}$; $\tilde{x}^3 = \tilde{\varphi}$ (with the polar axis directed along Ω) corotating with the neutron star the gravitational field is described by the metric

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2\,dt^2 - \left[\frac{dr^2}{1 - \frac{r_g}{r}} + r^2(d\tilde{\vartheta}^2 + \sin^2\tilde{\vartheta}\,d\tilde{\vartheta})\right] + 2\tilde{g}_{03}c\,dt\,d\tilde{\varphi},$$
$$\tilde{g}_{03} = \frac{[\Omega - \omega(r)]}{c}r^2\sin^2\tilde{\vartheta} = -\left[1 - \kappa\left(\frac{a}{r}\right)^3\right]\frac{\Omega r^2}{c}\sin^2\tilde{\vartheta}, \tag{16}$$

where $\omega(r) = 2GIc^{-2}r^{-3}\Omega$; $\Omega r \ll c$; $\kappa = \varepsilon\beta$, $r_g = 2GM/c^2$ is the gravitational radius, M is the stellar mass, $\varepsilon = r_g/a$ is the compactness parameter, $\beta = I/I_0$ is the stellar moment of inertia in units of $I_0 = Ma^2$, and a is the radial coordinate of the stellar surface. The nondiagonal component $\tilde{g}_{03}$ of the metric tensor contains the term $(-\Omega r^2 \sin^2\tilde{\vartheta})/c$ associated with the transition to the corotating reference frame, and also the term $[\omega(r)r^2\sin^2\tilde{\vartheta}]/c$ that describes the drag of inertial reference in the linear approximation. The dragging leads to the well known Lense–Thirring effect. The metric (16) can easily be obtained by linearizing the Kerr metric with respect to the angular drag velocity ω, and by subsequent transition to the corotating reference frame. Notice that in the linear approximation in Ω, the rotating star can be considered as being spherical (the rotational oblateness is an effect of second order).

Let us perform the transformation to the polar coordinate frame with the polar axis z directed along the magnetic moment $\vec{m}$. The x-axis will be chosen to lie in the $(\vec{\Omega}, \vec{m})$-plane. The relationship between the angles $\tilde{\vartheta}$ and $\tilde{\phi}$ and new angles ϑ and φ is given by

$$\begin{aligned} \cos\tilde{\vartheta} &= \cos\chi\cos\vartheta + \sin\chi\sin\vartheta\cos\varphi, \\ \sin\tilde{\vartheta}\sin\tilde{\vartheta} &= \sin\vartheta\sin\varphi, \end{aligned} \tag{17}$$

where χ is the angle between $\vec{\Omega}$ and $\vec{m}$. In the new reference frame, the metric (16) is

$$\begin{gathered} ds^2 = \left(1 - \frac{r_g}{r}\right)c^2\,dt^2 - \left[\frac{dr^2}{1 - \dfrac{r_g}{r}} + r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2)\right] + 2g_{0\mu}c\,dt\,dx^\mu, \\ x^\mu \to (r, \vartheta, \varphi); \qquad \mu = 1, 2, 3; \qquad x^0 = ct, \qquad g_{01} = 0, \\ g_{02} = \left[1 - \kappa\left(\frac{a}{r}\right)^3\right]\frac{\Omega r^2}{c}\sin\chi\sin\varphi, \\ g_{03} = -\left[1 - \kappa\left(\frac{a}{r}\right)^3\right]\frac{\Omega r^2}{c}\sin\vartheta\,(\cos\chi\sin\vartheta - \sin\chi\cos\vartheta\cos\varphi). \end{gathered} \tag{18}$$

Maxwell's equations for electromagnetic field read (Landau and Lifshitz, 1975)

$$\begin{gathered} \operatorname{div}\vec{B} = 0; \qquad \operatorname{curl}\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}; \\ \operatorname{div}\vec{D} = 4\pi\rho; \qquad \operatorname{curl}\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t} + \frac{4\pi}{c}\vec{s}; \\ \vec{D} = \frac{\vec{E}}{\sqrt{h}} + \vec{H}\times\vec{g}; \qquad \vec{B} = \frac{\vec{H}}{\sqrt{h}} - \vec{E}\times\vec{g}; \\ \frac{\partial\rho}{\partial t} + \operatorname{div}\vec{s} = 0; \qquad s^\mu = \rho\frac{dx^\mu}{dt}; \qquad j^i = \frac{\rho c}{\sqrt{h}}\frac{dx^i}{dx^0}; \qquad \vec{s} = \sqrt{h}\,\vec{j}. \end{gathered} \tag{19}$$

In this case $h = g_{00} = 1 - (r_g/r)$; $g_\mu = -g_{0\mu}/g_{00}$, or

$$\vec{g} = \frac{1}{hc}\left[1 - \kappa\left(\frac{a}{r}\right)^3\right]\vec{\Omega}\times\vec{r}. \tag{20}$$

In the problem of study, the electric field is generated from the magnetic one due to the stellar rotation, $E \sim gB$. The $gE \sim g^2B$ represents a second order term, or $\vec{B} \approx \vec{H}/\sqrt{h}$. The electric and magnetic fields, $\vec{E}$ and $\vec{H}$, are stationary in the NS corotating reference frame. The term $4\pi\vec{s}/c$ describes a magnetospheric current of charged particles. It is small and can be neglected. Then we obtain:

$$\begin{aligned} &\operatorname{div}\vec{B} = 0, \quad \operatorname{curl}(\sqrt{h}\,\vec{D} + h\vec{g}\times\vec{B}) = 0; \\ &\operatorname{div}\vec{D} = 4\pi\rho, \quad \operatorname{curl}(\sqrt{h}\,\vec{B}) = 0; \quad \operatorname{div}(\sqrt{h}\,\vec{j}) = 0. \end{aligned} \tag{21}$$

This allows us to introduce the electrostatic potential ϕ:

$$\begin{aligned} &\vec{E} = \sqrt{h}\,\vec{D} + h\vec{g}\times\vec{B} = -\operatorname{grad}\phi, \\ &\operatorname{div}\left(\frac{1}{\sqrt{h}}\operatorname{grad}\phi\right) = -4\pi(\rho + \rho_{\text{eff}}), \\ &\frac{\sqrt{h}}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{\sqrt{h}\,r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\phi}{\partial\vartheta}\right) + \frac{1}{\sqrt{h}\,r^2\sin^2\vartheta}\frac{\partial^2\phi}{\partial\varphi^2} = -4\pi(\rho + \rho_{\text{eff}}), \\ &\rho_{\text{eff}} = \frac{1}{4\pi}\operatorname{div}(\sqrt{h}\,\vec{g}\times\vec{B}) = \frac{\sqrt{h}\,\vec{B}}{4\pi}\operatorname{curl}\vec{g}. \end{aligned} \tag{22}$$

Equations for the electrostatic potential have been studied by Muslimov and Tsygan (1990a, 1990b, 1991). They generalize equations (1) to the case of a strong gravitational field.

Notice that, in this section, vector operators are defined as

$$\begin{aligned} &\operatorname{div}\vec{A} = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\mu}(\sqrt{\gamma}A^\mu); \qquad (\operatorname{curl}\vec{A})^\mu = \frac{1}{\sqrt{\gamma}}e^{\mu\nu\lambda}\frac{\partial A_\lambda}{\partial x^\nu}; \\ &(\operatorname{grad}\phi)_\mu = \frac{\partial\phi}{\partial x^\mu}; \qquad (\operatorname{grad}\phi)^\mu = \gamma^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}; \\ &\gamma_{\mu\nu} = -g_{\mu\nu} + \frac{g_{0\mu}g_{0\nu}}{g_{00}} \approx -g_{\mu\nu}. \end{aligned} \tag{23}$$

unit length, the metric at a given point is Galilean: $\hat{g}_{00} = 1$; $\hat{g}_{11} = \hat{g}_{22} = \hat{g}_{33} = -1$; $\hat{g}_{ik} = 0$ for $i \neq k$, and $\hat{\gamma}_{\mu\nu} = \delta_{\mu\nu}$. In the Galilean frame with the metric tensor $\hat{\gamma}_{\mu\nu} = \delta_{\mu\nu}$, the 3D vector components are "physical" quantities, e.g., $\hat{E}_\mu = \hat{E}^\mu$ ($\hat{E}_\mu = E_\mu/\sqrt{\gamma_{\mu\mu}}$). In the linear approximation in $\Omega r/c$ (we are interested in), stellar rotation does not affect the magnetic field. In the spherical coordinate frame with the polar axis directed along the NS magnetic moment, the dipole magnetic field $\vec{B}$ reads (Ginzburg and Ozernoy, 1964)

$$\begin{aligned} &\hat{B}_1 = B_0\frac{f(\eta)}{f(1)}\frac{1}{\eta^3}\cos\vartheta; \qquad \hat{B}_2 = -B_0\frac{\sqrt{h}}{f(1)}\left[f(\eta) - \frac{3}{2}\frac{1}{1-\dfrac{\varepsilon}{\eta}}\right]\frac{1}{\eta^3}\sin\vartheta; \\ &f(\eta) = -3\left(\frac{\eta}{\varepsilon}\right)^3\left[\ln\left(1-\frac{\varepsilon}{\eta}\right) + \frac{\varepsilon}{\eta}\left(1+\frac{\varepsilon}{2\eta}\right)\right]; \qquad f(\eta) \approx 1 + \frac{3\varepsilon}{4\eta}, \qquad \varepsilon \ll \eta. \end{aligned} \tag{24}$$

This field corresponds to the effective charge density

$$\rho_{\text{eff}} = \frac{\Omega B_0}{2\pi c} \frac{1}{\sqrt{h}\,\eta^3} \frac{f(\eta)}{f(1)}$$
$$\times \left\{ \left(1 - \frac{\chi}{\eta^3}\right) \cos\chi + \tfrac{3}{2} T(\eta) [\sin\chi \sin\vartheta \cos\vartheta \cos\varphi - \sin^2\vartheta \cos\chi] \right\},$$
$$T(\eta) = \left(\frac{\varepsilon}{\eta} - \frac{\kappa}{\eta^3}\right) + \left(1 - \frac{3\varepsilon}{2\eta} + \frac{\kappa}{2\eta^3}\right) \left[f(\eta)\left(1 - \frac{\varepsilon}{\eta}\right)\right]^{-1}. \tag{25}$$

Using (24), one can obtain an equation for the magnetic lines of the field $\vec{B}$. The equation for the outermost open line in the small-angle approximation is (Muslimov and Tsygan, 1990b)

$$\theta(\eta) \approx \theta_0 \left(\eta \frac{f(\eta)}{f(1)}\right)^{1/2}; \qquad \theta_0 = \left(\frac{\Omega a}{c} \frac{1}{f(1)}\right)^{1/2}. \tag{26}$$

Let us consider the electric field of a neutron star surrounded by the Goldreich-Julian magnetosphere for the case of free emission of charged particles in the region of open magnetic field lines. Then the boundary condition at the surface and the cone bottom, $\phi|_S = 0$, should be supplemented by the condition $E_1|_{r=a} = 0$. Charged particles supplied at the hot NS surface are efficiently accelerated up to relativistic energies and move at the speed of light along magnetic field lines. The electric current density of these particles is

$$\hat{j}^\mu = c\hat{\rho} \frac{\hat{H}^\mu}{H} = c\hat{\rho} \frac{\hat{B}^\mu}{B} \quad \text{(in ZAMO)};$$
$$j^\mu = \frac{1}{\sqrt{\gamma_{\mu\mu}}} (\hat{j}^\mu - \sqrt{h}\, c\hat{\rho}\hat{g}^\mu) \approx \frac{1}{\sqrt{\gamma_{\mu\mu}}} \hat{j}^\mu = \frac{c\rho}{\sqrt{\gamma_{\mu\mu}}} \frac{\hat{B}^\mu}{B} = c\rho \frac{B^\mu}{B}, \tag{27}$$
$$\rho = \hat{\rho}, \qquad \vec{j} = c\rho \vec{B}/B.$$

From the continuity equation, one obtains an equation for the charge density ρ of relativistic particles:

$$\operatorname{div}(\sqrt{h}\,\vec{j}) = \operatorname{div}\left(\sqrt{h}\, c\rho \frac{\vec{B}}{B}\right) = 0, \qquad \vec{B} \operatorname{grad}\left(\frac{\sqrt{h}\,\rho}{B}\right) = 0. \tag{28}$$

The solution of Eq. (28) compatible with the boundary condition for the potential can be written as

$$\rho(\eta, \xi, \varphi) = \frac{\Omega B_0}{2\pi c} \frac{1}{\sqrt{h}\,\eta^3} \frac{f(\eta)}{f(1)} [A(\xi) \cos\chi + \tfrac{3}{2} D(\xi) \sin\chi \cos\varphi], \tag{29}$$

where $A(\xi)$ and $D(\xi)$ are functions of $\xi = \vartheta/\theta$ $(0 \le \xi \le 1)$ to be determined together with ϕ. A general Eq. (22) for the electrostatic potential in the small-angle approximation is

$$\frac{1}{a^2}\left\{\frac{\sqrt{h}}{\eta^2}\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial\phi}{\partial\eta}\right)+\frac{1}{\sqrt{h}\,\eta^2\theta^2(\eta)}\frac{1}{\xi}\left[\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi}{\partial\xi}\right)+\frac{1}{\xi}\frac{\partial^2\phi}{\partial\varphi^2}\right]\right\}=-4\pi(\rho+\rho_{\rm eff}),$$

$$\rho_{\rm eff}=\frac{\Omega B_0}{2\pi c}\frac{1}{\sqrt{h}\,\eta^3}\frac{f(h)}{f(1)}\left[\left(1-\frac{x}{\eta^3}\right)\cos x+\tfrac{3}{2}T(\eta)\theta(\eta)\xi\sin\chi\cos\varphi\right]. \qquad (30)$$

The solution of Eq. (30) at $(\eta-1)\ll 1$ which satisfies the boundary conditions is

$$\begin{aligned}\phi(\eta,\xi,\varphi)&\\ &=12\phi_0\sqrt{1-\varepsilon}\,\kappa\theta_0^3\left\{\cos\chi\sum_{i=1}^{\infty}\left[\exp\left(-\frac{k_i(\eta-1)}{\theta_0\sqrt{1-\varepsilon}}\right)-1+\frac{k_i(\eta-1)}{\theta_0\sqrt{1-\varepsilon}}\right]\frac{J_0(k_i\xi)}{k_i^4J_1(k_i)}\right\}\\ &\quad+6\phi_0\sqrt{1-\varepsilon}\,\theta_0^4T(1)\delta(1)\\ &\quad\times\left\{\sin\chi\cos\varphi\sum_{i=1}^{\infty}\left[\exp\left(-\frac{\tilde{k}_i(\eta-1)}{\theta_0\sqrt{1-\varepsilon}}\right)-1+\frac{\tilde{k}_i(\eta-1)}{\theta_0\sqrt{1-\varepsilon}}\right]\frac{J_1(\tilde{k}_i\xi)}{\tilde{k}_i^4J_2(\tilde{k}_i)}\right\}.\end{aligned} \qquad (31)$$

For $(\eta-1)\gg\theta_0$, one gets

$$\begin{aligned}\phi(\eta,\xi,\varphi)&=\tfrac{1}{2}\phi_0\kappa\theta_0^2\left(1-\frac{1}{\eta^3}\right)(1-\xi^2)\cos\chi\\ &\quad+\tfrac{3}{8}\phi_0\theta_0^3T(1)\left[\frac{\theta(\eta)T(\eta)}{\theta_0T(1)}-1\right]\xi(1-\xi^2)\sin\xi\cos\varphi,\\ E_\parallel&=-\frac{3}{2}\frac{\phi_0}{a}\kappa\theta_0^2\frac{1}{\eta^4}(1-\xi^2)\cos\chi\\ &\quad-\frac{3}{8}\frac{\phi_0}{a}\theta_0^2\theta(\eta)T(\eta)\delta(\eta)\xi(1-\xi^2)\sin\chi\cos\varphi,\end{aligned} \qquad (32)$$

$$\begin{aligned}\delta(\eta)&=\frac{d}{d\eta}\ln[\theta(\eta)T(\eta)]=\frac{1}{T(\eta)}\left\{-\frac{1}{\eta^2}\left(2\varepsilon-4\frac{\kappa}{\eta^2}\right)\right.\\ &\quad+\frac{3}{f(\eta)(\eta-\varepsilon)}\left[\frac{1}{\eta}\left(\varepsilon-\frac{\kappa}{\eta^2}\right)-\left(1-\frac{3\varepsilon}{2\eta}+\frac{\kappa}{2\eta^3}\right)\right.\\ &\quad\left.\left.\times\left(\frac{4}{3}-\frac{\varepsilon}{\eta}-\frac{3}{2f}\right)\left(1-\frac{\varepsilon}{\eta}\right)^{-1}\right]\right\};\qquad \delta(\eta)\approx\frac{1}{2\eta}\quad\text{at}\quad\eta\gg1.\end{aligned}$$

Solutions (31) and (32) coincide at $\theta_0\ll(\eta-1)\ll1$. The first terms in (31) and (32), which are proportional to $\kappa=(r_g/a)(I/Ma^2)$, are associated with the dragging of inertial reference frames. The effect of the Schwarzschild spacetime is included in $T(\eta)$ and $\theta(\eta)$. The second terms in (31) and (32) transform into Eqs. (14) and (13) (which describe the effect of unipolar induction in the presence of plasma) at $\varepsilon=(r_g/a)\to0$ ($T(\eta)=1$; $\delta(\eta)=1/2\eta$ when $\varepsilon\to0$). Near the NS surface, the electric field generated by the drag effect,

$$E_\parallel^{\rm GR}\approx-\frac{3\phi_0}{2a}\kappa\theta_0^2\frac{1}{\eta^4}(1-\xi^2)\cos\chi, \qquad (33)$$

appears to be a factor of

$$\left(\frac{8\kappa}{\theta_0}\frac{1}{\xi}\frac{1}{\eta^{3.5}}\frac{\cos\chi}{\sin\chi\cos\varphi}\right)$$

larger than the field (15) produced by the unipolar induction. For $\chi = 0.16$, $\theta_0 = 0.014$ ($P = 1$ s) and $\xi \times 0.5$, the enhancement term is ~ 160. The electric field (33) is high enough to generate an electron-positron avalanche for pulsars with $P \le P_c(B_c/10^{12}G)^{4/7}$, where $P_c \approx 0.5$ s (Muslimov and Tsygan, 1990b). Let us consider the solution for the case when the electron-positron avalanche occurs and the discharge height z_0 is small, $z_0 = \eta_0 - 1 \ll \theta_0$, in comparison with the polar cap radius. This case has been considered by Beskin (1990). Then the boundary conditions $\phi|_{\eta=1} = \phi|_{\xi=1} = 0$ should be supplemented by the condition $\partial\phi/\partial\eta|_{\eta=\eta_0} = 0$ at the discharge top. In Eq. (30), one can keep the derivatives with respect to η only. For $z = \eta - 1 \ll \theta_0$, we obtain the following equation:

$$\frac{d^2\phi}{dz^2} = -\phi_0\frac{2}{1-\varepsilon}\Big\{[1-\kappa+3\kappa z + A(\xi)]\cos\chi + \tfrac{3}{2}[\xi\theta_0 T(1)(1+\delta(1)z) + D(\xi)]\sin\chi\cos\varphi\Big\}. \tag{34}$$

For $(1-\xi) \gg z_0$, its solution is

$$\phi = \phi_0\frac{z^2}{1-\varepsilon}\left(\frac{z_0}{2}-\frac{z}{3}\right)\{3\kappa\cos\chi + \tfrac{3}{2}\xi\theta_0 T(1)\delta(1)\sin\chi\cos\varphi\},$$

$$E_\parallel = -\phi_0\frac{1}{1-\varepsilon}\frac{z}{a}(z_0 - z)\{3\kappa\cos\chi + \tfrac{3}{2}\xi\theta_0 T(1)\delta(1)\sin\chi\cos\varphi\}, \tag{35}$$

$$A(\xi) = -(1-\kappa+\tfrac{3}{2}\kappa z_0), \qquad D(\xi) = -\xi\theta_0 T(1)\left[1+\frac{z_0}{2}\delta(1)\right].$$

The first term in Eq. (35), which is proportional to the relativistic parameter κ, is again essential.

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