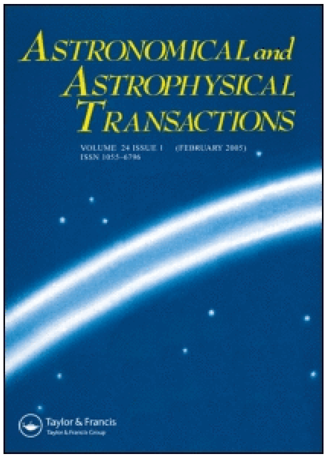

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THE ROTOR: A NEW METHOD TO DERIVE ROTATION BETWEEN TWO REFERENCE FRAMES

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The paper presents a study of two techniques to derive the relative orientation of two reference frames: the Standard Method, based on the least squares procedure, and a new method—the ROTOR, based on the representation of systematic differences in terms of orthogonal functions. It is shown that the ROTOR is preferable over the Standard Method since it (a) takes into account only the harmonics which are responsible for rotation, (b) tests them for pure rotation, (c) discovers the existence of quasi-rotational terms which may smear the rotation, (d) in the absence of quasi-rotational terms, reconstructs the rotation exactly. Numerical experiments with the 1535 fundamental stars of the FK5 show that the ROTOR may be used to obtain rotation of any reference frame (including the HIPPARCOS catalogue) with respect to the FK5.

KEY WORDS fundamental catalogues, connection of frames.

1. INTRODUCTION

At present, astrometry deals with three types of reference frames. The first one, based on meridian observations of stars, is represented by the FK5 at the accuracy level of $0.02 \div 0.03$ arcsec. The second one, resulting from VLBI observations of radio sources, has reached the accuracy of the order 0.0005 arsec. The third type of reference frames may be produced by observations from outer space. The only realization, the HIPPARCOS mission, is now near its end, and the accuracy of the resulting catalogue is expected to be $0.001 \div 0.002$ arcsec. According to the inner logics of astrometry, all the reference frames are to be connected. A lot of observational programs aiming at the connection is now in progress: observations of radio stars and extra-galactic radio sources in optical and radio regions of the spectrum, the Hubble Space Telescope measurements of the HIPPARCOS stars with respect to quasars, etc.

Very often the problem of connection between two catalogues is reduced to the determination of their mutual rotation. In this paper a theoretical study of the determination of the rotation is made. Two approaches are considered: the standard one, and a new method, based on the orthogonal representation of systematic differences. The basic principles of the new method are similar to those which were used in our previous papers on kinematic analysis of proper motions (Vityazev and Tsvetkov, 1989, Vityazev, 1990a,b).

2. EQUATIONS OF RIGID ROTATION OF ONE FRAME WITH RESPECT TO ANOTHER

Consider two rectangular systems of coordinates (X, Y, Z) and (X', Y', Z') and the associated spherical coordinates (α, δ) and (α', δ'). Let us suppose that the dashed coordinates are obtained by rotating the initial system about the axes X, Y, Z by the angles $\omega_1, \omega_2, \omega_3$ respectively. The angles ω_i, $i = 1, 2, 3$ are said to be positive if the rotation is performed in the anti-clockwise direction if viewed from the end of the axis. The result of the rotation can be written as follows:

$$\begin{bmatrix} \cos\delta'\cos\alpha' \\ \cos\delta'\sin\alpha' \\ \sin\delta' \end{bmatrix} = M(\omega_1)M(\omega_2)M(\omega_3)\begin{bmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{bmatrix}, \tag{2.1}$$

where the rotation matrices are

$$M(\omega_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & -\sin\omega_1 & \cos\omega_1 \end{bmatrix}, \tag{2.2}$$

$$M(\omega_2) = \begin{bmatrix} \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 1 & 0 \\ \sin\omega_2 & 0 & \cos\omega_2 \end{bmatrix}, \tag{2.3}$$

$$M(\omega_3) = \begin{bmatrix} \cos\omega_3 & \sin\omega_3 & 0 \\ -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.4}$$

If the angles $\omega_1, \omega_2, \omega_3$ are small, Eq. (2.1) can be rewritten in the form

$$\begin{bmatrix} \cos\delta'\cos\alpha' \\ \cos\delta'\sin\alpha' \\ \sin\delta' \end{bmatrix} = \begin{bmatrix} 1 & \omega_3 & -\omega_2 \\ -\omega_3 & 1 & \omega_1 \\ \omega_2 & -\omega_1 & 1 \end{bmatrix}\begin{bmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{bmatrix}. \tag{2.5}$$

Setting

$$\begin{aligned} \Delta\alpha &= \alpha' - \alpha, \\ \Delta\delta &= \delta' - \delta, \end{aligned} \tag{2.6}$$

we obtain from Eq. (2.5)

$$\Delta\alpha\cos\delta = \omega_1\sin\delta\cos\alpha + \omega_2\sin\delta\sin\alpha - \omega_3\cos\delta, \tag{2.7}$$

$$\Delta\delta = -\omega_1\sin\alpha + \omega_2\cos\alpha. \tag{2.8}$$

Equations (2.7)–(2.8) are the fundamental equations of our problem, since they define the differences $\Delta\alpha\cos\delta$ and $\Delta\delta$ due to a small rotation. Gavrilov and Kisliuk (1970) and Mironov (1971) were probably the first who used matrix calculus to describe relative rotation of spherical frames. The separate and combined least squares solutions of Eqs. (2.7)–(2.8) are now a common practice to derive rotational parameters $\omega_1, \omega_2, \omega_3$ (see Duma *et al.* 1980, Gubanov and Kumkova 1981, Froeschle and Kovalevsky 1982, Röser 1986, Brosche and

Sinachopoulus 1987). For the sake of reference we shall call this approach the SM (the Standard Method).

It is necessary to underline that a correct use of the least squares procedure requires that the differences $\Delta\alpha \cos\delta$ and $\Delta\delta$ are composed of no other components but rotation and noise. Meanwhile, the structure of systematic differences between astrometric catalogues is very complicated and contains not only rotation but many other components. For this reason it is necessary to study how these non-rotational components disturb the results of the SM.

3. THE REPRESENTATION OF SYSTEMATIC DIFFERENCES BETWEEN TWO REFERENCE FRAMES IN TERMS OF ORTHOGONAL FUNCTIONS

To proceed further, we are to have a model of systematic differences which comprises more details than the model defined by Eqs. (2.7)–(2.8). Such a model may be taken from a method proposed by Brosche (1966) and developed by Schwan (1977). In this paper we shall call this technique the ORM (Orthogonal Representation Method). In the ORM, the differences $\Delta\alpha \cos\delta$ and $\Delta\delta$ between two catalogues are described by equations

$$\Delta\alpha \cos\delta = \sum_{j=0}^{n} C_j Z_j(\alpha, \delta, m) + \varepsilon, \tag{3.1}$$

$$\Delta\delta = \sum_{j=\alpha}^{n'} C_j' Z_j(\alpha, \delta, m) + \varepsilon', \tag{3.2}$$

where Z_j represent a set of complete orthogonal functions over a three-dimensional space, formed by the variables α, δ, μ, C_j and C_j' are the coefficients of expansions, ε and ε' are the random parts of the differences. The separation of random and systematic parts in the ORM is made using some statistical criteria in such a way that only significant terms are allowed to enter into the systematic part of the initial data. For a given set of individual differences $\Delta\alpha_i \cos\delta_i$, $\Delta\delta_i$ ($i = 1, 2, \ldots, N$, with N the number of the stars), a chosen set of orthogonal functions Z_j and a chosen significance level the ORM derives the numbers n and n', the coefficients C_j and C_j' and the estimations of their errors with respect to the random parts ε and ε'.

Those are the main features of the method; for details the reader is referred to the papers cited above. Here we expose the sets of orthogonal functions, which are in general use in astrometry. Since the orthogonal functions form a basis in the corresponding multi-dimensional space, for the sake of brevity we shall call the sets B_1 and B_2.

3a. *The basis* B_1. This basis is formed by the functions

$$Z_j(\alpha, \delta, m) = R_{nk} R_p H_p(\tilde{m}) K_j(\alpha, \delta), \tag{3.3}$$

where the spherical functions are defined as

$$K_j(\alpha, \delta) = \begin{cases} P_{n0}(\delta), & k = 0, \quad l = 1 \\ P_{nk}(\delta) \sin k\alpha, & k \neq 0, \quad l = 0 \\ P_{nk}(\delta) \cos k\alpha, & k \neq 0, \quad l = 1 \end{cases} \tag{3.4}$$

		n = 0	1	2	3	4	5	6
k=1	l=1			ω_1		ω_1		ω_1
k=1	l=0			ω_2		ω_2		ω_2
k=0	l=1	ω_3		ω_3		ω_3		ω_3

Figure 1 The spherical functions (B_1-basis) which represent rotation in the systematic differences $\Delta\alpha \cos \delta$.

Here P_{n0} and P_{nk} are the Legendre polynomials, for which one has

$$P_{nk}(\delta) = \frac{(2n)!}{2^n n!\,(n-k)!} \cos^k \delta \times \left[\sin^{n-k}\delta + \sum_{\mu=1}^{\left[\frac{n-k}{2}\right]} (-1)^\mu \frac{\prod_{\nu=0}^{2\mu-1}(n-k-\nu)}{\prod_{\nu=1}^{\mu} 2\nu(2n-2\nu+1)} \sin^{(n-k-2\mu)}\delta \right], \quad (3.5)$$

where $[x]$ denotes the integer part of x.

The spherical functions describe the dependence of the systematic differences on the right ascension and declination, while their dependence on brightness is specified by the Hermite polynomials $H_p(\bar{m})$:

$$H_{p+1}(\bar{m}) = \bar{m}H_p(\bar{m}) - pH_{p-1}(\bar{m}), \qquad p = 1, 2, \ldots,$$
$$H_0 = 1, \qquad H_1 = \bar{m}, \quad (3.6)$$

where

$$\bar{m} = (m - m_0)/\sigma_m, \quad (3.7)$$

m_0 is the average magnitude of stars common to both catalogues, σ_m^2 is the dispersion of magnitudes.

To normalize the functions Z_j, the coefficients R_{nk} and R_p are defined as:

$$R_p = (p!)^{-1/2} \quad (3.8)$$

$$R_{nk} = (2n+1)^{1/2} \begin{cases} \left(2\dfrac{(n-k)!}{(n+k)!}\right)^{1/2} & k \neq 0, \\ 1 & k = 0, \end{cases} \quad (3.9)$$

		n = 0	1	2	3	4	5	6
k=1	l=1		ω_2		ω_2		ω_2	
k=1	l=0		ω_1		ω_1		ω_1	
k=0	l=1							

Figure 2 The spherical functions (B_l-basis) which represent rotation in the systematic differences $\Delta\delta$.

	n = 0	1	2	3	4	5	6
k=1 l=1		ω_1					
k=1 l=−1		ω_2					
k=0 l=−1	ω_3		ω_3		ω_3		ω_3

Figure 3 The "Legendre-Fourier" functions (B_2-basis) which represent rotation in the systematic differences $\Delta\alpha\cos\delta$.

so the norms of the functions Z_j are

$$\|Z_j\|^2 = \int_{-\infty}^{+\infty} \exp(-\bar{m}^2/2)\,d\bar{m} \int_0^{2\pi} d\alpha \int_{-\pi/2}^{\pi/2} Z_j^2(\alpha, \delta, m)\cos\delta\,d\delta = 4\pi. \quad (3.10)$$

The indices n, k, l and j are linked by

$$j = n^2 + 2k + l - 1. \quad (3.11)$$

The correspondence of spherical functions to the indices n, k, l may be represented by a diagram of indices (Figures 1 and 2).

The basis B_2. In this case the set of basic functions is given by

$$Z_j(\alpha, \delta, m) = R_p H_p(\bar{m}) R_{nk} L_n(\delta) F_{kl}(\alpha), \quad (3.12)$$

where

$$L_n(\delta) = P_{n0}(\delta), \quad (3.13)$$

$$F_{kl}(\alpha) = \begin{cases} 1 & l=-1, \quad k=0 \\ \cos kl\alpha & l=1, \quad k\neq 0 \\ \sin(-kl\alpha) & l=-1, \quad k\neq 0, \end{cases} \quad (3.14)$$

$$R_{nk} = \begin{cases} (2n+1)^{1/2} & k=0, \\ (2(2n+1))^{1/2} & k\neq 0. \end{cases} \quad (3.15)$$

As we see, this set of functions uses the products of Legendre polynomials and Fourier terms instead of spherical functions. The normalizing condition (3.10) is valid for the basis B_2 too. The correspondence of the functions $L_n(\delta)F_{kl}(\alpha)$ to the indices n, k, l is shown in Figures 3 and 4. Later we shall show, that for deriving the rotational parameters the spherical functions are preferable over the products of Legendre polynomials and Fourier terms.

	n = 0	1	2	3	4	5	6
k=1 l=1	ω_2						
k=1 l=−1	ω_1						
k=0 l=−1							

Figure 4 The "Legendre-Fourier" functions (B_2-basis) which represent rotation in the systematic differences $\Delta\delta$.

4. PHYSICAL AND FORMAL MODELS OF SYSTEMATIC DIFFERENCES

Let us introduce two additional basis sets:

$$\Phi: \quad \begin{aligned} \varphi_1(\alpha, \delta) &= \sin\delta\cos\alpha, \\ \varphi_2(\alpha, \delta) &= \sin\delta\sin\alpha, \\ \varphi_3(\alpha, \delta) &= -\cos\delta, \end{aligned} \tag{4.1}$$

$$\Psi: \quad \begin{aligned} \psi_1(\alpha, \delta) &= -\sin\alpha, \\ \psi_2(\alpha, \delta) &= \cos\alpha. \end{aligned} \tag{4.2}$$

With this notation, Eqs. (2.7–2.8) can be written as follows:

$$\Delta\alpha\cos\delta = \sum_{i=1}^{3} \omega_i\varphi_i(\alpha, \delta), \tag{4.3}$$

$$\Delta\delta = \sum_{i=1}^{2} \omega_i\psi_i(\alpha, \delta). \tag{4.4}$$

Compare now Eqs. (3.1) and (3.2) with Eqs. (4.3) and (4.4). If the left-hand sides of all the equations are obtained by comparing two astrometric catalogues, then both pairs of equations may be regarded as models of systematic differences. The model (4.3)–(4.4) is physical since it was derived from the consideration of a certain physical process: relative rotation of two coordinate systems. At the same time, this model is not complete because the real differences may contain not only the rotational effects. On the contrary, the model (3.1)–(3.2) is complete since it is based on a complete orthogonal system of functions but it is not physical, for in general one cannot say what physics stands behind every term of Eqs. (3.1) and (3.2).

Taking into consideration the completeness of bases B_1 and B_2 and the ability of the ORM to separate noise from systematic components, we may replace the left-hand sides of Eqs. (4.3) and (4.4) by their formal models to obtain the following equations:

$$\sum_j C_jZ_j(\alpha, \delta, m) = \sum_{i=1}^{3} \omega_i\varphi_i(\alpha, \delta), \tag{4.5}$$

$$\sum_j C_j'Z_j(\alpha, \delta, m) = \sum_{i=1}^{2} \omega_i\psi_i(\alpha, \delta). \tag{4.6}$$

Two solutions of these equations represent a special interest. The first solution, "from left to right", yields the rotational parameters ω_i as functions of the given coefficients C_j and C_j'. The second solution, "from right to left", defines the coefficients of the orthogonal representations as functions of the rotational parameters ω_i. Both solutions can be obtained from the conditions:

$$I_\alpha = \int_0^{2\pi} d\alpha \int_{-\pi/2}^{\pi/2} \left[\sum_j C_jZ_j(\alpha, \delta, \mu) - \sum_{i=1}^{3} \omega_i\varphi_i(\alpha, \delta)\right]^2 \cos\delta\, d\delta = \min, \tag{4.7}$$

$$I_\delta = \int_0^{2\pi} d\alpha \int_{-\pi/2}^{\pi/2} \left[\sum_j C_j'Z_j(\alpha, \delta, m) - \sum_{i=1}^{2} \omega_i\psi_i(\alpha, \delta)\right]^2 \cos\delta\, d\delta = \min. \tag{4.8}$$

5. ROTATIONAL PARAMETERS AS FUNCTIONS OF THE COEFFICIENTS OF THE ORTHOGONAL REPRESENTATION OF THE SYSTEMATIC DIFFERENCES

In this section the separate and combined solutions of Eqs. (4.5)–(4.6) with respect to ω_1, ω_2, ω_3 are performed. The results obtained in analytical form allow us to investigate how the Standard Method works in the presence of non-rotational components.

5a. *Solution of Eq.* (4.5). From condition (4.7) we obtain the normal system,

$$N\bar{\omega} = \bar{r}, \tag{5.1}$$

where

$$N = [n_{ij}] = [(\varphi_i, \varphi_j)], \qquad i,j = 1,2,3, \tag{5.2}$$

$$\bar{r} = \left[\sum_j C_j(Z_j, \varphi_i)\right], \qquad \bar{\omega} = [\omega_i], \qquad i = 1,2,3, \tag{5.3}$$

$$(p, q) = \int_0^{2\pi} d\alpha \int_{-\pi/2}^{\pi/2} p(\alpha, \delta) q(\alpha, \delta) \cos\delta \, d\delta. \tag{5.4}$$

It is easy to show that the functions φ_i, $i = 1,2,3$ are orthogonal with respect to the scalar product (5.4). For this reason, N is a diagonal matrix with

$$\begin{aligned} n_{11} &= \|\varphi_1\|^2 = 2\pi/3, \\ n_{22} &= \|\varphi_2\|^2 = 2\pi/3, \\ n_{33} &= \|\varphi_3\|^2 = 8\pi/3, \end{aligned} \tag{5.5}$$

and solution of Eq. (4.5) can be immediately written in the form

$$\omega_1 = (3/2\pi) \sum_j C_j(Z_j, \varphi_1), \tag{5.6}$$

$$\omega_2 = (3/2\pi) \sum_j C_j(Z_j, \varphi_2), \tag{5.7}$$

$$\omega_3 = (3/8\pi) \sum_j C_j(Z_j, \varphi_3). \tag{5.8}$$

5b. *Solution of Eq.* (4.6). In the same way, we have from condition (4.8):

$$N'\bar{\omega} = \bar{r}', \tag{5.9}$$

where

$$N' = [n'_{ij}] = [(\psi_i, \psi_j)], \qquad i,j = 1,2, \tag{5.10}$$

$$\bar{r}' = \left[\sum_j C'_j(Z_j, \psi_i)\right], \qquad \bar{\omega}' = [\omega'_i], \qquad i = 1,2. \tag{5.11}$$

Again, the functions ψ_i, $i=1,2$ are orthogonal, so

$$n'_{11} = \|\psi_1\|^2 = 2\pi, \qquad n'_{22} = \|\psi_2\|^2 = 2\pi, \tag{5.12}$$

and solution of Eq. (4.6) looks as follows:

$$\omega_1 = (1/2\pi) \sum_j C'_j(Z_j, \psi_i), \tag{5.13}$$

$$\omega_2 = (1/2\pi) \sum_j C'_j(Z_j, \psi_2). \tag{5.14}$$

5c. *Combined solution of Eqs.* (4.5) *and* (4.6). This solution is based on the condition

$$I_\alpha + I_\delta = \min, \tag{5.15}$$

which yields

$$(N + N'')\bar{\omega} = \bar{r} + \bar{r}'', \tag{5.16}$$

where

$$N'' = \begin{bmatrix} N' & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{r}'' = \begin{bmatrix} \bar{r}' \\ 0 \end{bmatrix}, \qquad \bar{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}. \tag{5.17}$$

Using symbols "α", "δ" and "c" to denote the corresponding results of separate and combined solutions, we find from Eq. (5.16)

$$\begin{aligned} \omega_1^c &= \frac{\|\varphi_1\|^2}{\|\varphi_1\|^2 + \|\psi_1\|^2}\,\omega_1^\alpha + \frac{\|\psi_1\|^2}{\|\varphi_1\|^2 + \|\psi_1\|^2}\,\omega_1^\delta, \\ \omega_2^c &= \frac{\|\varphi_2\|^2}{\|\varphi_2\|^2 + \|\psi_2\|^2}\,\omega_2^\alpha + \frac{\|\psi_2\|^2}{\|\varphi_2\|^2 + \|\psi_2\|^2}\,\omega_2^\delta, \\ \omega_3^c &= \omega_3^\alpha. \end{aligned} \tag{5.18}$$

Now, using Eqs. (5.5) and (5.12), we get finally,

$$\begin{aligned} \omega_1^c &= \tfrac{1}{4}(\omega_1^\alpha + 3\omega_1^\delta), \\ \omega_2^c &= \tfrac{1}{4}(\omega_2^\alpha + 3\omega_2^\delta), \\ \omega_3^c &= \omega_3^\alpha. \end{aligned} \tag{5.19}$$

Thus, we see that the results of the combined solution are the weighted averages of the results obtained from the separate solutions. From Eq. (5.19) one can see that the combined solution assigns to the δ-solution the weight three times larger than to the α-solution. This explains why the c-solution often turns out to be very close to the δ-solution (Fricke 1977, Brosche and Sinachopoulos 1986). For further discussion of the combined solution the reader is referred to Vityazev (1989).

The solutions obtained above are valid for any set of the functions Z_j. Now we are going to present them in the bases B_1 and B_2.

5d. *Solutions in the B_1-basis.* Taking into consideration the explicit form of the Z_j-functions in the basis B_1, we find from Eqs. (5.6)–(5.8):

Solution of Eq. (4.5) in the B_1-basis

$$\omega_1(m) = \frac{3}{2}\sum_p R_p H_p(\bar{m}) \sum_n C_{pn11}\chi_n, \tag{5.20}$$

$$\omega_2(m) = \frac{3}{2}\sum_p R_p H_p(\bar{m}) \sum_n C_{pn10}\chi_n, \tag{5.21}$$

$$\omega_3(m) = -\frac{3}{4}\sum_p R_p H_p(\bar{m}) \sum_n C_{pn01}\lambda_n, \tag{5.22}$$

where

$$\chi_n = R_{n1}\int_{-\pi/2}^{\pi/2} P_{n1}(\delta)\sin\delta\cos\delta\, d\delta, \tag{5.23}$$

$$\lambda_n = R_{n0}\int_{-\pi/2}^{\pi/2} P_{n0}(\delta)\cos^2\delta\, d\delta. \tag{5.24}$$

Solution of Eq. (4.6) in the B_1-basis

$$\omega_1(m) = -\frac{1}{2}\sum_p R_p H_p(\bar{m}) \sum_n C_{pn10}\mu_n, \tag{5.25}$$

$$\omega_2(m) = \frac{1}{2}\sum_p R_p H_p(\bar{m}) \sum_n C_{pn11}\mu_n, \tag{5.26}$$

where

$$\mu_n = R_{n1}\int_{-\pi/2}^{\pi/2} P_{n1}(\delta)\cos\delta\, d\delta. \tag{5.27}$$

The numerical values of χ_n, λ_n, μ_n are given in Table 1.

Table 1. Numerical values of χ_n, μ_n, λ_n, following from Eqs. (5.23), (5.24), (5.27). The values in brackets correspond to 1535 stars of FK5.

n	χ_n	μ_n	λ_n
0	—	—	1.5708
			[1.5746]
1	—	2.7207	—
		[2.7172]	−0.4391
2	1.5209	—	[−0.4331]
	[1.5167]		—
3	—	0.6362	
		[0.6188]	−0.0736
4	0.4657		[−0.0669]
	[0.4486]	—	—
5	—		
		0.3153	−0.0277
6	0.2535	[0.2593]	[−0.0197]
	[0.2033]	—	

5e. *Solutions in the B_2-basis.* Quite analogously to Sect. 5d, we find:

Solution of Eq. (4.5) in the B_2-basis

$$\omega_1(m) = \sqrt{6}\sum_p R_p H_p(\bar{m}) C_{p111}, \tag{5.28}$$

$$\omega_2(m) = \sqrt{6}\sum_p R_p H_p(\bar{m}) C_{p11-1}, \tag{5.29}$$

$$\omega_3(m) = -\frac{3}{4}\sum_p R_p H_p(\bar{m}) \sum_n C_{pn0-1}\lambda_n. \tag{5.30}$$

Solution of Eq. (4.6) in the B_2-basis

$$\omega_1(m) = -\sqrt{2}\sum_p R_p H_p(\bar{m}) C'_{p01-1}, \tag{5.31}$$

$$\omega_2(m) = \sqrt{2}\sum_p R_p H_p(\bar{m}) C'_{p011}, \tag{5.32}$$

Combined solutions in the bases B_1 and B_2 can be obtained from Eq. (5.19). As far as we have no further interest in them, we do not expose here the combined solutions in explicit form.

6. DISCUSSION OF THE ANALYTICAL SOLUTIONS

The results obtained in Sect. 5 show some interesting features, which we are going to discuss here. First of all, we note that, due to the dependence of Z_j on magnitude, the rotational parameters ω_i, $i = 1,2,3$ turn out to be m-dependent. This fact can be easily explained, for the magnitude equation causes the stars belonging to various groups of brightness to produce different rotation of frames. In general, when the angles ω_i are dependent on some parameters, the rotation is no longer rigid. In modern astrometric catalogues, the magnitude equation is rather small. For example, the systematic differences FK5-FK4 show no magnitude equation in $\Delta\delta$, and only 4 of 39 terms yielding $\Delta\alpha\cos\delta$ are m-dependent (Fricke *et al.* 1988). For this reason, in our further study we shall restrict ourselves to the rigid rotation and neglect the dependence of the basic functions Z_j on magnitude. In this simplified case the analytical solutions of Eqs. (4.5) and (4.6) can be represented in the following form:

The B_1-basis

solution of Eq. (4.5)

$$\omega_1 = \frac{3}{2}\sum_n C_{0n11}\chi_n, \tag{6.1}$$

$$\omega_2 = \frac{3}{2}\sum_n C_{0n10}\chi_n, \tag{6.2}$$

$$\omega_3 = -\frac{3}{4}\sum_n C_{0n01}\lambda_n, \tag{6.3}$$

solution of Eq. (4.6)

$$\omega_1 = -\frac{1}{2}\sum_n C'_{0n10}\mu_n, \tag{6.4}$$

$$\omega_2 = \frac{1}{2}\sum_n C'_{0n11}\mu_n, \tag{6.5}$$

The B_2-basis

solution of Eq. (4.5)		solution of Eq. (4.6)	
$\omega_1 = \sqrt{6}\, C_{0111}$,	(6.6)	$\omega_1 = -\sqrt{2}\, C'_{001-1}$,	(6.9)
$\omega_2 = \sqrt{6} C_{011-1}$,	(6.7)	$\omega_2 = \sqrt{2} C'_{0011}$.	(6.10)
$\omega_3 = -\frac{3}{4} \sum_n C_{0n0-1} \lambda_n$,	(6.8)		

The solutions (6.6)–(6.10) show that the rotational parameters ω_1 and ω_2 in the basis B_2 are proportional to the coefficients C_{0111}, C_{011-1}, and C'_{001-1}, C'_{0111}. This result is not unexpected, for the functions φ_1, φ_2, ψ_1, ψ_2, except for the normalization convention, coincide with the functions Z_{0111}, Z_{011-1}, Z_{001-1}, Z_{0011}. On the contrary, the function φ_3 does not belong to the basis B_2, whence the parameter ω_3 is composed of the combination of the zonal coefficients C_{0n0-1}. Analogously, no one of the functions φ_i, $i = 1,2,3$ and ψ_i, $i = 1,2$ belongs to the B_1-basis. For this reason, all the rotational parameters ω_1, ω_2, ω_3 in the B_1-basis are dependent on a series of the corresponding coefficients C_j and C'_j.

7. WHAT DO THE COEFFICIENTS C_j AND C'_j MEAN IN CASE OF ROTATION?

To answer this question we are to perform the "right to left" solutions of Eqs. (4.5) and (4.6). Having in mind that the basic functions Z_j are orthogonal on sphere, and setting $p = 0$, we get

$$C_j = \sum_{i=1}^{3} \omega_i \frac{(\varphi_i, Z_j)}{(Z_j, Z_j)}, \tag{7.1}$$

$$C'_j = \sum_{i=1}^{2} \omega_i \frac{(\psi_i, Z_j)}{(Z_j, Z_j)}, \tag{7.2}$$

where the scalar products are defined by Eq. (5.4). Evaluating the integrals corresponding to various values of the indices n, k, l, we find the solutions in both bases as follows:

Solution of Eq. (4.5) in the B_1-basis

$$C_{0n11} = \frac{\chi_n}{4} \omega_1, \qquad n = 2, 4, 6, \ldots, \qquad k = 1, \tag{7.3}$$

$$C_{0n10} = \frac{\chi_n}{4} \omega_2, \qquad n = 2, 4, 6, \ldots, \qquad k = 1, \tag{7.4}$$

$$C_{0n01} = -\frac{\lambda_n}{2} \omega_3, \qquad n = 0, 2, 4, \ldots, \qquad k = 0, \tag{7.5}$$

$$C_{0nkl} = 0 \qquad \text{for all the rest } n, k, l. \tag{7.6}$$

Solution of Eq. (4.6) in the B_1-basis

$$C'_{0n10} = -\frac{\mu_n}{4} \omega_1, \qquad n = 1, 3, 5, \ldots, \qquad k = 1, \tag{7.7}$$

$$C'_{0n11} = \frac{\mu_n}{4} \omega_2, \qquad n = 1, 3, 5, \ldots, \qquad k = 1, \tag{7.8}$$

$$C'_{0nkl} = 0 \qquad \text{for all the rest } n, k, l. \tag{7.9}$$

Solution of Eq. (4.5) in the B_2-basis

$$C_{0111} = \omega_1/\sqrt{6}, \tag{7.10}$$

$$C_{011-1} = \omega_2/\sqrt{6}, \tag{7.11}$$

$$C_{0n0-1} = -\frac{\lambda_n}{2}\omega_3, \quad n = 0, 2, 4, \ldots, \quad k = 0, \tag{7.12}$$

$$C_{0nkl} = 0 \quad \text{for all the rest } n, k, l. \tag{7.13}$$

Solution of Eq. (4.6) in the B_2-basis

$$C'_{001-1} = -\omega_1/\sqrt{2}, \tag{7.14}$$

$$C'_{0011} = \omega_2/\sqrt{2}, \tag{7.15}$$

$$C'_{0nkl} = 0 \quad \text{for all the rest } n, k, l. \tag{7.16}$$

The results obtained in this section are represented graphically in Figs 1–4, where the corresponding diagrams of indices show those orthogonal functions which are responsible for rigid rotation. It is interesting to note that the coefficients C_j and C'_j, which are proportional to the angles ω_1, ω_2, ω_3 are the same that enter Eqs. (6.1)–(6.10) to define the rotational parameters in the "left to right" solutions of Eqs. (4.5)–(4.6). This enables us to substitute the solutions (7.3)–(7.9) into Eqs. (6.1)–(6.5) to obtain the following identities:

$$\frac{3}{8}\sum_{n=1}^{\infty} \chi_{2n}^2 = 1, \tag{7.17}$$

$$\frac{3}{8}\sum_{n=0}^{\infty} \lambda_{2n}^2 = 1, \tag{7.18}$$

$$\frac{1}{8}\sum_{n=1}^{\infty} \mu_{2n-1}^2 = 1, \tag{7.19}$$

which will be used later.

8. TESTS FOR ROTATION

The results obtained in Sect. 7 lead us to a very important conclusion: if $\Delta\alpha\cos\delta$ and $\Delta\delta$ are composed of rotational terms defined by Eqs. (2.7)–(2.8), then the coefficients C_j and C'_j in the B_1-basis should satisfy the following relations:

(a) for the angles ω_1 and ω_2 from $\Delta\alpha\cos\delta$

$$T_1(n, m) = \frac{C_{0n11}\chi_m}{C_{0m11}\chi_n} = 1; \qquad T_2(n, m) = \frac{C_{0n10}\chi_m}{C_{0m10}\chi_n} = 1;$$
$$n, m = 2, 4, 6, \ldots, n \neq m; \tag{8.1}$$

(b) for the angle ω_3 from $\Delta\alpha\cos\delta$

$$T_3(n, m) = \frac{C_{0n01}\lambda_m}{C_{0m01}\lambda_n} = 1, \qquad n, m = 0, 2, 4, \ldots, n \neq m; \tag{8.2}$$

(c) for the angles ω_1 and ω_2 from $\Delta\delta$

$$T_1'(n,m)=\frac{C'_{0n10}\mu_m}{C'_{0m10}\mu_n}=1;\qquad T_2'(n,m)=\frac{C'_{0n11}\mu_m}{C'_{0m11}\mu_n}=1;$$

$$n,m=1,3,5,\ldots,n\neq m. \tag{8.3}$$

The inversion statement can be proved quite easily. We shall do this for the angle ω_3 (the others are treated analogously). From Eq. (8.2) it follows that we may set

$$C_{0n01}=-\frac{1}{2}\lambda_n. \tag{8.4}$$

Consider now the function

$$F(\delta)=\sum_n C_{0n01}R_{n0}P_{n0}(\delta). \tag{8.5}$$

It is clear that, if C_{0n01} are determined by Eq. (8.4), then due to Eq. (5.24) we get

$$F(\delta)=-\frac{1}{2}\sum_{n=0}^{\infty}\lambda_n R_{n0}P_{n0}(\delta)=-\cos\delta=\varphi_3(\delta). \tag{8.6}$$

We have just proved that Eqs. (8.1)–(8.3) afford the necessary and sufficient conditions for the systematic differences $\Delta\alpha\cos\delta$ and $\Delta\delta$ be composed of the rotational terms and of *nothing else*. Thus Eqs. (8.1)–(8.3) may be regarded as tests for rotation. They permit us to discover rotation even before the angles ω_1, ω_2, ω_3 have been derived. Eqs. (8.1)–(8.3) are valid in the B_1-basis. It is necessary to say that in the B_2-basis only the angle ω_3 can be tested, since solutions (7.10), (7.11) and (7.14), (7.15) are independent of n. Now we see, that to test all the three angles the spherical functions are preferable over the products of Legendre polynomials and Fourier terms.

In practice there are two reasons why conditions (8.1)–(8.3) can be violated. These are: an uneven distribution of the stars in the catalogues and the noise. The quantities χ_n, λ_n and μ_n, defined by Eqs. (5.23), (5.24) and (5.27), correspond to continuous or even distributions. Obviously, the evaluation of the integrals (5.23), (5.24) and (5.27) on every specific set of stars produces its own values of χ_n, λ_n and μ_n. As an example, Table 1 shows in brackets the values of these quantities, derived for 1535 stars of the FK5. These values should be used in the tests (8.1)–(8.3) if rotation of any catalogue with respect to the FK5 is studied provided all the 1535 stars are employed.

When noise is present, the coefficients C_j and C'_j are random quantities. They are derived by the ORM together with the mean square root errors σ and σ' which are independent of j due to Eq. (3.10). In this case the strict tests (8.1)–(8.3) should be replaced by the following unequalities:

(a) for the angles ω_1, ω_2, ω_3 from $\Delta\alpha\cos\delta$

$$1-\sigma_i\leq T_i(n,m)\leq 1+\sigma_i,\qquad i=1,2,3, \tag{8.7}$$

where

$$\sigma_1=\sigma T_1(n,m)(C_{0n11}^{-2}+C_{0m11}^{-2})^{1/2}, \tag{8.8}$$

$$\sigma_2=\sigma T_2(n,m)(C_{0n10}^{-2}+C_{0m10}^{-2})^{1/2},\qquad n,m=2,4,6,\ldots,n\neq m, \tag{8.9}$$

$$\sigma_3=\sigma T_3(n,m)(C_{0n01}^{-2}+C_{0m01}^{-2})^{1/2}\qquad n,m=0,2,4,\ldots,n\neq m, \tag{8.10}$$

(b) for the angles ω_1, ω_2 from $\Delta\delta$:

$$1-\sigma_i' \leq T_i'(n,m) \leq 1+\sigma_i', \qquad i=1,2, \tag{8.11}$$

where

$$\sigma_1' = \sigma' T_1'(n,m)(C_{0n10}'^{-2} + C_{0m10}'^{-2})^{1/2}, \tag{8.12}$$

$$\sigma_2' = \sigma' T_2'(n,m)(C_{0n11}'^{-2} + C_{0m11}'^{-2})^{1/2}, \qquad n,m = 1, 3, 5, \ldots, n \neq m. \tag{8.13}$$

9. THE DEFICIENCIES OF THE STANDARD METHOD

We are now in position to return to the Standard Method and to see how it derives the rotational parameters when the systematic differences contain non-rotational components. From the analytical study of the SM, performed in Sect. 5, we can point out two essential faults.

(a) The statistical significance of the SM-solution is diminished by non-rotational components. Indeed, in the least squares procedure the root mean square errors are defined by

$$\sigma_{\omega_i} = \sigma_0 / n_{ii}, \qquad i = 1, 2, 3, \tag{9.1}$$

where, for the N stars employed, the r.m.s.e. of a unit weight is

$$\sigma_0 = \left[\frac{\sum_i \varepsilon_i^2}{N-3}\right]^{1/2}. \tag{9.2}$$

Here

$$\varepsilon_i = \Delta\alpha_i \cos\delta_i - \sum_{k=1}^{3} \omega_k \varphi_k(\alpha_i, \delta_i). \tag{9.3}$$

In a classical form of the least squares procedure, the root mean square errors describe the level of the random part in the observed quantities. In our case, as it is seen from Eqs. (9.1)–(9.3), even in the absence of noise the quantities σ_{ω_i} may happen to become very large, since the SM does not discriminate between the real noise and any other components which are beyond the rotation. In this sense, the non-rotational terms may be regarded as a "systematic noise".

(b) The SM may yield a solution which would be adopted as a rotation even if the systematic differences $\Delta\alpha\cos\delta$ and $\Delta\delta$ contain no rotational components. Indeed, the orthogonal functions, denoted as ω_1, ω_2, ω_3 in Figures 1–4, may represent either rotational terms (R-terms), given by Eqs. (2.7)–(2.8), or other functions, which are not orthogonal to the basic functions. Henceforth we shall call such functions the quasi-rotational terms (the Q-terms). Let us suppose now, that the systematic differences $\Delta\alpha\cos\delta$ and $\Delta\delta$ are composed of the Q-terms and of nothing else. In this case the coefficients C_j, C_j' will not be equal to zero, and, after summation according to Eqs. (6.1)–(6.5), they will produce non-zero values of the rotational parameters. Since the summation is necessarily finite, it may happen that the residuals of the corresponding series give but small contribution to the systematic differences $\Delta\alpha\cos\delta$ and $\Delta\delta$. In this case the formal errors of the rotational parameters may be found too small to reject the spurious solution. Numerical examples demonstrating these shortcomings of the Standard Method are shown in Sect. 11.

10. THE ROTOR: A NEW METHOD TO DERIVE THE PARAMETERS OF ROTATION

A new procedure free of the SM′ deficiencies is based on the results obtained in Sects. 7 and 8. For the sake of reference we shall call this method the ROTOR (Rotation by Orthogonal Representation). To derive the rotational parameters by the ROTOR, the following steps are to be performed:

1. Perform the orthogonal representation of the initial individual differences $\Delta\alpha\cos\delta$ and $\Delta\delta$ in B_1- or in B_2-basis either by the ORM, or by any other suitable technique, which, with a chosen significance level, separates noise from the systematic components and yields the coefficients C_j and C_j' together with the estimates of their root mean square errors σ and σ' (the r.m.s.e. are independent of j due to the normalization of the basic functions).

2. Test C_j and C_j' for rotation with the help of Eqs. (8.1)–(8.3) and unequalities (8.7) and (8.12). If the B_2-basis has been used, the only test of ω_3 is possible. In the following points it is supposed that the ROTOR is based on the spherical functions.

3. If the coefficients C_j and C_j' satisfy the tests for rotation, calculate the rotational parameters ω_1, ω_2, ω_3 and their r.m.s.e. with the help of the following formulae:

from $\Delta\alpha\cos\delta$

$$\omega_1^{(n)}=\frac{4C_{0n11}}{\chi_n},\qquad \sigma_{\omega_1}^{(n)}=\frac{4\sigma}{\chi_n},\qquad n=2,4,6,\ldots, \tag{10.1}$$

$$\omega_2^{(n)}=\frac{4C_{0n10}}{\chi_n},\qquad \sigma_{\omega_2}^{(n)}=\frac{4\sigma}{\chi_n},\qquad n=2,4,6,\ldots, \tag{10.2}$$

$$\omega_3^{(n)}=-\frac{2C_{0n01}}{\lambda_n},\qquad \sigma_{\omega_3}^{(n)}=\frac{2\sigma}{|\lambda_n|},\qquad n=0,2,4,\ldots; \tag{10.3}$$

from $\Delta\delta$

$$\omega_1^{(n)}=-\frac{4C'_{0n10}}{\mu_n},\qquad \sigma_{\omega_1}^{(n)}=\frac{4\sigma'}{\mu_n},\qquad n=1,3,5,\ldots, \tag{10.4}$$

$$\omega_2^{(n)}=\frac{4C'_{0n11}}{\mu_n},\qquad \sigma_{\omega_2}^{(n)}=\frac{4\sigma'}{\mu_n},\qquad n=1,3,5,\ldots. \tag{10.5}$$

Obviously, the values $\omega_1^{(n)}$, $\omega_2^{(n)}$, $\omega_3^{(n)}$ are independent of n, and this is another proof of rotation. On the contrary, the quantities $\sigma_{\omega_i}^{(n)}$ are n-dependent, and it is quite understandable, since the higher the order of the harmonic used to derive the rotation, the less reliable is the result. From this point of view, the lowest values of n in Eqs. (10.1)–(10.5) are recommended to ensure the highest weights of the rotational parameters.

It is interesting to note that averaging of the values $\omega_i^{(n)}$ with the weights proportional to $[\sigma_{\omega_i}^{(n)}]^{-2}$ yields again Eqs. (6.1)–(6.5), from which, due to Eqs. (7.17)–(7.19), the strict solution for the r.m.s.e. may be derived as follows:

from $\Delta\alpha\cos\delta$ from $\Delta\delta$

$$\sigma_{\omega_1}=\sigma_{\omega_2}=\sigma\sqrt{6},\qquad \sigma_{\omega_1}=\sigma_{\omega_2}=\sigma'\sqrt{2}, \tag{10.6}$$

$$\sigma_{\omega_3}=\sigma\sqrt{3/2}. \tag{10.7}$$

It is remarkable that the estimates $\sigma^{(n)}_{\omega_i}$ corresponding to the lowest values of n are very close to the strict values, for example $\sigma^{(0)}_{\omega_3}/\sigma_{\omega_3} = 1.04$, $\sigma^{(2)}_{\omega_1}/\sigma_{\omega_1} = 1.07$, $\sigma^{(1)}_{\omega_1}/\sigma_{\omega_1} = 1.04$. Thus the following two procedures may be used to derive the rotational parameters. One is to calculate them from Eqs. (10.1)–(10.5) for the lowest values of n, another is to calculate them from Eqs. (6.1)–(6.5) and (10.6)–(10.7).

4. If the coefficients C_j and C'_j do not satisfy the tests for rotation, two conclusions can be made: either the systematic differences under consideration contain no rotation, or the rotation exists, but it is masked by quasi-rotational terms (Q-terms), which are strongly correlated with the rotational terms. For simplicity we again use the angle ω_3 to illustrate the situation (the remaining angles are treated in the same way).

When noise is absent and stars are regularly distributed over the sky, the violation of the condition (8.2) implies that the coefficients C_{0n01} can be presented in the form

$$C_{0n01} = -\frac{\omega_3}{2}\lambda_n + \gamma_{0n01}, \qquad n = 0, 2, 4, \ldots, \tag{10.8}$$

where the coefficients γ_{0n01} define the orthogonal representation of a quasi-rotational function

$$q(\delta) = \sum_n \gamma_{0n01} R_{n0} P_{n0}(\delta), \tag{10.9}$$

of which, generally speaking, we know nothing. Let us rewrite Eq. (10.8) as follows:

$$\frac{2C_{0n01}}{\lambda_n} = -\omega_3 + \frac{2\gamma_{0n01}}{\lambda_n}, \qquad n = 0, 2, 4, \ldots \tag{10.10}$$

For any finite n, Eq. (10.10) yields a system of n equations with $(n+1)$ unknowns. It is clear that to solve the system, an additional equation is needed. Thus, we come to an important conclusion: in order to make the choice between the two possibilities formulated above, an *a priori* information concerning the function $q(\delta)$ is required.

Keeping in mind that the complete knowledge of $q(\delta)$ is hardly available (otherwise the problem becomes trivial), we shall show two examples, when a partial information is sufficient.

(a) Suppose that the rate of decreasing γ_{0n01} is known to be faster than that of λ_n. In this case we find from Eq. (10.10),

$$\lim_{n\to\infty} \frac{2C_{0n01}}{\lambda_n} = -\omega_3. \tag{10.11}$$

In practice, when noise is small, the values of ω_3 can be evaluated from the upper value of the index n obtained above.

(b) Suppose that the scalar product (φ_3, q) is known. Now from Eq. (10.8) with the help of Eqs. (8.6) and (10.9) we can construct the function

$$F(\delta) = \sum_n C_{0n01} R_{n0} P_{n0}(\delta) = \omega_3 \varphi_3(\delta) + q(\delta). \tag{10.12}$$

Consider the following equations:

$$A\varphi_3(\delta) = F(\delta), \tag{10.13}$$

$$A\varphi_3(\delta) = \omega_3\varphi_3(\delta) + q(\delta). \tag{10.14}$$

The least squares solution of Eq. (10.13) yields the estimation of the rotational parameter A,

$$A = \frac{(F, \varphi_3)}{(\varphi_3, \varphi_3)} = -\frac{3}{4}\sum_n C_{0n01}\lambda_n, \tag{10.15}$$

and it is just the same result as obtained using the SM (see Sect. 6). Another estimation of A comes from Eq. (10.14) as

$$A = \omega_3 + \frac{(\varphi_3, q)}{(\varphi_3, \varphi_3)}. \tag{10.16}$$

Equating (10.15) and (10.16), we get,

$$\omega_3 = -\frac{3}{4}\sum_n C_{0n01}\lambda_n - \frac{(\varphi_3, q)}{(\varphi_3, \varphi_3)}, \tag{10.17}$$

from which it follows that if the scalar product (φ_3, q) is known, the spurious SM-solution can be corrected to become the true rotation. Here we clearly see the difference between the SM and the ROTOR. In the presence of the R- and Q-terms the least squares procedure fits the functions of the Φ- and Ψ-bases to *all the components* of which the systematic differences are comprised, and this leads to a fictitious solution. On the contrary, the ROTOR separates the rotational components from all others and, dealing with only the first ones, tries to derive the rotation exactly. As we have seen, the rigorous solution is possible if some additional information about the quasi-rotational terms is available. In the absence of any information about the Q-terms, the infringement of conditions (8.1)–(8.3) may be regarded as a warning that the solution produced by the Standard Method is not at least a pure rotation.

11. THE ROTOR IN PRACTICE

To see how the ROTOR works in practice, we made several numerical experiments with the FK5, J2000.0. In all the numerical runs the systematic differences $\Delta\alpha\cos\delta$ and $\Delta\delta$ between some artificial catalogues and the FK5 have been calculated for each star in such a way that the systematic differences were composed of rotational and/or non-rotational components. The rotational parameters were derived with the Standard Method and with the ROTOR. Below the three experiments are described and the comparison between the two methods is made.

The first experiment. The differences $\Delta\alpha\cos\delta$ and $\Delta\delta$ were taken to be "rotation plus noise":

$$\Delta\alpha\cos\delta = \sum_{i=1}^{3}\omega_i\varphi_i(\alpha, \delta) + \sqrt{d_0}\, r(\alpha, \delta), \tag{11.1}$$

$$\Delta\delta = \sum_{i=1}^{2}\omega_i\psi_i(\alpha, \delta) + \sqrt{d_0}\, r(\alpha, \delta), \tag{11.2}$$

		n = 0	1	2	3	4	5	6
k=1	l=1		−0.027 ±27	0.374 ±30	0.021 ±30	0.088 ±31	−0.002 ±27	0.052 ±28
k=1	l=0		0.054 ±27	0.410 ±30	−0.004 ±30	0.139 ±31	0.025 ±27	0.019 ±27
k=0	l=1	− 0.786 ±30	0.004 ±33	0.220 ±34	−0.024 ±32	0.061 ±32	−0.029 ±29	0.001 ±29

Figure 5 The orthogonal representation of the systematic differences $\Delta\alpha \cos\delta$ (11.1) in the basis B_1 (spherical functions).

where $r(\alpha, \delta)$ denotes the random variables with the zero mean, normally distributed within the interval $[-1, 1]$ with the dispersion equal to unity. The differences were computed with $\omega_i = 1''$, $i = 1, 2, 3$ and $\sqrt{d_0} = 1''$.

The Standard Method yields the following results:

solution from $\Delta\alpha \cos\delta$	solution from $\Delta\delta$
$\omega_1 = 0''.985 \pm 0''.070$,	$\omega_1 = 0''.940 \pm 0''.036$,
$\omega_2 = 1''.075 \pm 0''.070$,	$\omega_2 = 1''.074 \pm 0''.036$.
$\omega_3 = 1''.001 \pm 0''.029$,	

Now we exhibit step by step the work of the ROTOR.

1. The orthogonal representations of the systematic differences (11.1)–(11.2) are shown in Figures 5 and 6.

		n = 0	1	2	3	4	5	6
k=1	l=1		0.655 ±28	0.034 ±30	0.148 ±30	0.012 ±31	0.000 ±28	0.020 ±28
k=1	l=0		−0.667 ±28	0.004 ±30	−0.160 ±31	−0.008 ±31	−0.024 ±28	0.011 ±28
k=0	l=1	0.006 ±31	−0.028 ±34	−0.038 ±35	−0.027 ±32	0.000 ±33	−0.064 ±30	0.005 ±30

Figure 6 The orthogonal representation of the systematic differences $\Delta\delta$ (11.2) in the basis B_1 (spherical functions).

2. The tests for rotation (8.7), (8.11), having the form,

$$0.56 < (T_1(2, 4) = 1.26) < 1.44,$$
$$0.80 < (T_2(2, 4) = 0.87) < 1.20,$$
$$0.86 < (T_3(0, 2) = 0.98) < 1.14,$$
$$0.79 < (T_1'(1, 3) = 1.01) < 1.21,$$
$$0.82 < (T_2'(1, 3) = 0.95) < 1.18$$

give evidence that the differences (11.1) and (11.2) consist of rotational terms plus noise and of nothing else.

3. The rotational parameters derived from Eqs. (10.1)–(10.5) with $n = 0, 1, 2$ turn out to be

from $\Delta\alpha \cos\delta$	from $\Delta\delta$
$\omega_1 = 0''.986 \pm 0''.073,$	$\omega_1 = 0''.964 \pm 0''.042,$
$\omega_2 = 1''.081 \pm 0''.073,$	$\omega_2 = 0''.982 \pm 0''.042,$
$\omega_3 = 0''.998 \pm 0''.037.$	

Thus, in the presence of the R-terms and noise both methods are equally reliable. It is necessary to add that this is the only situation when the SM is permissible.

The second experiment. Here the differences $\Delta\alpha \cos\delta$ and $\Delta\delta$ were taken to be "rotation plus systematic noise":

$$\Delta\alpha \cos\delta = \sum_{i=1}^{3} \omega_i \varphi_i(\alpha, \delta) + \sum_{j \in G} C_j Z_j(\alpha, \delta), \tag{11.5}$$

$$\Delta\delta = \sum_{i=1}^{2} \omega_i \psi_i(\alpha, \delta) + \sum_{j \in G'} C_j' Z_j(\alpha, \delta), \tag{11.6}$$

where G and G' denote the sets of empty cells in Figs 1–4. The input differences have been calculated with $\omega_i = 0''.5$, $i = 1, 2, 3$ and $C_j = C_j' = 3''$ in the B_1-basis for $k = 0, 1$; $n = 0, 1, \ldots 6$.

The Standard Method yields the following results:

solution from $\Delta\alpha \cos\delta$	solution from $\Delta\delta$
$\omega_1 = 0''.622 \pm 0''.505,$	$\omega_1 = 0''.561 \pm 0''.291,$
$\omega_2 = 0''.632 \pm 0''.506,$	$\omega_2 = 0''.584 \pm 0''.293.$
$\omega_3 = 0''.451 \pm 0''.213,$	

It is likely that, guided by the "three sigma" rule, one would not hesitate to reject these solutions though the differences $\Delta\alpha \cos\delta$ and $\Delta\delta$ contain rotation. It is the situation when the "systematic noise" is too large and spoils the expected results (see Section 9a).

Now we are tracing the results which the ROTOR produced at each step.

1. The orthogonal representations of the systematic differences (11.5–11.6) in the B_1-basis are shown in Figs. 7 and 8.

	n = 0	1	2	3	4	5	6
k=1 l=1		3.000 ±1	0.189 ±1	3.000 ±1	0.056 ±1	3.000 ±1	0.025 ±1
k=1 l=0		3.000 ±1	0.190 ±1	3.000 ±1	0.056 ±1	3.000 ±1	0.026 ±1
k=0 l=1	− 0.394 ±1	3.000 ±1	0.108 ±1	3.001 ±1	0.017 ±1	3.000 ±1	0.005 ±1

Figure 7 The orthogonal representation of the systematic differences $\Delta\alpha \cos\delta$ (11.5) in the basis B_1 (spherical functions).

2. The tests for rotation (8.1)–(8.3) yield the quantities:

$$T_1(2,4) = 1.00, \qquad T_2(2,4) = 1.00,$$
$$T_3(0,2) = 1.00,$$
$$T'_1(1,3) = 1.00, \qquad T'_2(1,3) = 1.00.$$

This tells us that the systematic differences $\Delta\alpha \cos\delta$ and $\Delta\delta$ (11.5–11.6) contain the rotational terms.

3. The rotational parameters calculated from Eqs. (10.1)–(10.5) with $n = 0, 1, 2$ turned out to be:

from $\Delta\alpha \cos\delta$	from $\Delta\delta$
$\omega_1 = 0''.498 \pm 0''.002$,	$\omega_1 = 0''.501 \pm 0''.001$,
$\omega_2 = 0''.501 \pm 0''.002$,	$\omega_2 = 0''.501 \pm 0''.001$.
$\omega_3 = 0''.500 \pm 0''.001$,	

	n = 0	1	2	3	4	5	6
k=1 l=1		0.340 ±1	3.001 ±1	0.077 ±1	3.001 ±1	0.032 ±1	3.001 ±1
k=1 l=0		−0.340 ±1	2.999 ±1	−0.077 ±1	2.999 ±1	−0.032 ±1	2.999 ±1
k=0 l=1	3.000 ±1	3.001 ±1	3.001 ±1	3.001 ±1	3.001 ±1	3.001 ±1	3.001 ±1

Figure 8 The orthogonal representation of the systematic differences $\Delta\delta$ (11.6) in the basis B_1 (spherical functions).

We see that the ROTOR reconstructed the rotation hidden in our systematic differences practically exactly since it ignores the "systematic noise". In the absence of a "real" noise small non-zero values of the r.m.s.e. are explained by the traces of rotation in upper harmonics, corresponding to $n > 6$.

The third experiment. Now the systematic differences $\Delta\alpha\cos\delta$ and $\Delta\delta$ were changed to contain rotational and quasi-rotational terms:

$$\Delta\alpha\cos\delta = \sum_{i=1}^{3} \omega_i \varphi_i(\alpha, \delta) + \sum_{i=1}^{3} Q_i(\alpha, \beta), \tag{11.7}$$

$$\Delta\delta = \sum_{i=1}^{2} \omega_i' \psi_i(\alpha, \beta) + \sum_{i=1}^{2} Q_i'(\alpha, \delta), \tag{11.8}$$

where $\omega_1 = \omega_2 = 1''$, $\omega_3 = 0''$; $\omega_1' = \omega_2' = 2''$. In Eqs. (11.7)–(11.8), the Q-terms were taken to be:

$$Q_1 = -\sin\delta\cos^2\delta\cos\alpha, \qquad Q_2 = -\sin\delta\cos^2\delta\sin\alpha,$$

$$Q_3 = -\cos^2\delta;$$

$$Q_1' = -2\cos^2\delta\cos\alpha, \qquad Q_2' = 2\cos^2\delta\sin\alpha.$$

The Standard Method produced the results:

solution from $\Delta\alpha\cos\delta$	solution from $\Delta\delta$
$\omega_1 = 0''.528 \pm 0''.012$,	$\omega_1' = 0''.515 \pm 0''.019$,
$\omega_2 = 0''.512 \pm 0''.012$,	$\omega_2' = 0''.525 \pm 0''.019$.
$\omega_3 = 0''.907 \pm 0''.005$,	

No-one would hesitate to adopt these solutions as a reliable evidence of rotation, though the rotational parameters turned out to be far from their real values, being blurred by quasi-rotational terms (see Section 9b).

Let us see how the ROTOR works in this situation.

1. The orthogonal representations of (11.7) and (11.8) are shown in Figs. 9 and 10.

	n = 0	1	2	3	4	5	6
k=1 l=1		0.000 ±2	0.189 ±2	0.000 ±3	0.155 ±2	0.000 ±2	0.056 ±2
k=1 l=0		0.000 ±2	0.189 ±2	0.000 ±2	0.156 ±2	0.001 ±2	0.057 ±2
k=0 l=1	− 0.668 ±2	0.000 ±2	0.296 ±2	0.000 ±2	−0.002 ±2	−0.001 ±2	−0.002 ±2

Figure 9 The orthogonal representation of the systematic differences $\Delta\alpha\cos\delta$ (11.7) in the basis B_1 (spherical functions).

	n = 0	1	2	3	4	5	6
k=1 l=1		0.338 ±4	0.000 ±4	0.468 ±4	0.000 ±4	0.148 ±4	0.001 ±4
k=1 l=0		−0.339 ±4	0.000 ±4	−0.468 ±4	0.001 ±4	−0.148 ±4	−0.003 ±4
k=0 l=1	0.000 ±4	−0.005 ±5	0.001 ±5	−0.007 ±5	0.002 ±5	−0.002 ±4	0.003 ±4

Figure 10 The orthogonal representation of the systematic differences $\Delta\delta$ (11.8) in the basis B_1 (spherical functions).

2. The tests for rotation look now as follows:

$$T_1(2, 4) = 0.36, \qquad T_2(0, 4) = 0.36,$$
$$T_3(0, 2) = 0.62;$$
$$T_1'(1, 3) = 0.16, \qquad T_2'(1, 3) = 0.16.$$

These values being far from unity tell us that the SM-solutions hardly can be trusted, and the suspicion arises that they are smeared by quasi-rotational terms. In our example we know the Q-terms, and a rigorous solution of our problem is possible. Indeed, with the corresponding values

$$(\varphi_1, q_1) = (\varphi_2, q_2) = -4\pi/15; \qquad (\varphi_3, q_3) = 3\pi^2/4;$$
$$(\varphi_1, q_1') = (\varphi_2, q_2') = -4\pi/3,$$

where q_i, $i = 1, 2, 3$; q_i', $i = 1, 2$ denote the δ-dependent parts of the functions Q_i and Q_i', the method described in Section 10.4 yields the corrected solutions:

from $\Delta\alpha\cos\delta$	from $\Delta\delta$
$\omega_1 = 0''.963 \pm 0''.012,$	$\omega_1' = 1''.966 \pm 0''.018,$
$\omega_2 = 0''.960 \pm 0''.012,$	$\omega_2' = 1''.966 \pm 0''.018.$
$\omega_3 = 0''.001 \pm 0''.005,$	

We see that at the 96 per cent level of accuracy the ROTOR has reconstructed the true values of rotational parameters. (A small loss of accuracy is explained by weak non-orthogonalities which exist between the functions of the Φ- and Ψ-bases due to uneven distribution of the FK5 stars in the α- and δ-directions).

Comparison between the SM- and the ROTOR-solutions is very instructive for it displays what we may adopt as a rotation if the quasi-rotational terms occur in the systematic differences. As we have seen, the Standard Method has no protection against the Q-terms, and it is the ROTOR which discovers the Q-terms and (if *a priori* information is available) eliminates them from the solution. In the absence of *a priori* information the beneficial properties of the ROTOR over the SM still remain, since it is more valuable to know that something is wrong than to trust it blindly.

CONCLUSIONS

It is likely that the ideas on which the ROTOR is based, may be used in other problems where parameters of a certain physical model are derived from observed data by the least squares procedure. Briefly they are summarized as follows.

1. Any physical model of the observed values is not complete, since the observations, without speaking of noise, usually contain effects beyond the adopted model.
2. The completeness of the model may be ensured only by representing the data with the help of a complete set of orthogonal functions. In the presence of noise statistical criteria should be applied to separate random and systematical parts in observed values.
3. From various sets of orthogonal functions, a set which does not contain the basic functions of the physical model is to be chosen.
4. The representation of the physical model by the chosen orthogonal functions is to be made. This representation yields two important tools: (a) the tests of the fact that the observed data contain only the effects of the adopted physical model; (b) the equations which link the parameters of the physical and formal models.
5. The representation of the observed data by means of the same set of orthogonal functions is to be made too.
6. The coefficients of the orthogonal representation derived at point 5 are to be checked by the tests obtained at point 4.
7. If the tests are successful then the parameters of the adopted physical model can be calculated over the coefficients of the orthogonal representation.
8. If the tests are unsuccessful, then two conclusions are possible: either the physical model is wrong, or it is weak, and a more realistic model is needed.

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References

Bien, R., Fricke, W., Lederle, T. and Schwan, H. (1978). Methods for the Comparison of Systems of Star Positions to be applied in the Construction of FK5. *Veroff. Astron. Rechen. Inst.*, Heidelberg N 29.

Brosche, P. (1966). Representation of Systematic Differences in Positions and Proper Motions of Stars by Spherical Harmonics. Veroff. *Astron. Rechen. Inst.*, Heidelberg, N 17.

Brosche, P. and Sinachopoulos, D. (1987). *Mon. Not. R. Astr. Soc.*, **227**, 341–346.

Fricke, W. (1977). Basic Material for the Determination of Precession and of Galactic Rotation and a Review of Methods and Results. *Veroff. Astron. Rechen. Inst.*, Heidelberg, N 28.

Fricke, W., Schwan, H. and Lederle, T. (1988). Fifth Fundamental Catalogue (FK5). Part 1. The Basic Fundamental Stars. *Veroff. Astron. Rechen. Inst.*, Heidelberg, N 32.
Duma, D. P., Kizl'uk, L. N. and Safronov, Yu. I. (1980). Orientation of the FK4 coordinate system based on meridian observations of planets. *Naukova Dumka,* Kiev.
Gavrilov, I. I. and Kisliuk, V. S. (1970). *The Moon,* **12,** 363–364.
Gubanov, V. S. and Kumkova, I. I. (1981). Astrometric Investigations. Proceedings of the 21-st Astrometric Conference. Naukova Dumka, Kiev.
Froeschle, M. and Kovalevsky, J. (1982). *Astron. and Astrophys.*, **116,** 89-94.
Mironov, N. T. (1971). *Astrometry and Astrophysics,* **13,** 13–18.
Röser, S. (1986). *Astrometric Technique,* 773–778.
Vityazev, V. V. (1989). VINITI, N 5696–b89, 1–17.
Vityazev, V. V. and Tsvetkov, A. S. (1989). *Vestn. Leningr. Univ.*, S.1. **2,** N8, 73–78.
Vityazev, V. V. (1990a). *Kinematics and Physics of Celestial Bodies,* **6,** N3, 22–29.
Vityazev, V. V. (1990b). *Kinematics and Physics of Celestial Bodies,* **6,** N4, 74–82.