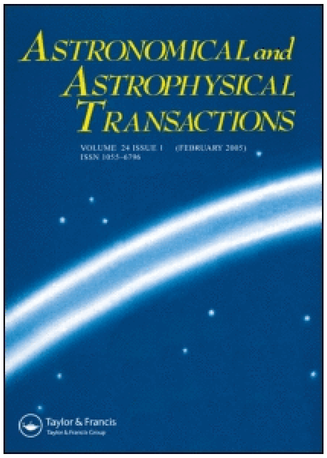


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# PARAMETRIC ADJUSTMENT OF ASTROMETRIC DATA

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The parametric adjustment method (PAM) applied to astrometrical problems of the least-squares collocation (LSC) principles is considered. In the first instance the PAM is intended for adjustment of absolute observations in radio/optical and satellite astrometry, when the coordinates of the observable selestial objects (OSO) are being determined together with other parameters of the data mathematical model. Special attention is paid to obtaining and using a priori information on the accuracy and internal correlations in both unknown parameters and observational data. It is shown that the best conditions of astrometrical data adjustment are obtained if the observations are carried out by group programs.

KEY WORDS Astrometry, data adjustment.

## INTRODUCTION

Lately the amount of astrometrical data, their diversity and accuracy greatly increased due to the application of new time-coordinate measuring facilities and the modernization of the classical ones. At the same time the demands on the quality of mathematical analysis of astrometrical data has also increased.

The principal aims of improvement of astrometrical data analysis consist in increasing the validity of the results and their accuracy estimates. The following are proposed as solutions of this problem:

a) elaboration of a more precise and complete mathematical model of observations by increasing the reduction accuracy and a more careful study of the physical conditions of the observation process;

b) a more complete use of available a priori information;

c) optimization of observational programs.

The LSC method developed by Moritz (1980) may be used as a theoretical basis for the development of a new technique of astrometric data adjustment. As applied to physical geodesy, the LSC allows to optimize the algorithm of a joint analysis of various ground and space based measurements, where gravity exists. This method also allows to determine the best estimation of the geopotential parameters if its a priori covariance function is known. This is also possible in astrometry, for example, in the determination of the Earth rotation parameters or in the compilation of fundamental catalogues.

In this paper the ideas of the LSC are used for the solution of the simpler problem that consists in the optimal estimation of astrometric parameters in the process of global adjustment of observations. It was thus necessary to introduce only some generalization of the simple least-squares (SLS) method. This fact allows the use of the data covariances and a priori information about the accuracy of the estimated parameters in the adjustment process. Furthermore, complicated problems concerning the use of this method for filtration, interpolation, approximation, forecasting and joining up of time series and random fields on the celestial sphere will be considered.

## 1. MATHEMATICAL MODELS OF ASTROMETRIC DATA

### 1.1. *Observational Data*

Consider small differences $(O - C)$,

$$l_i = h_i(L_0 - L_c)_i, \quad (i = 1, 2, \ldots, N), \tag{1}$$

between the two values, one of which $(L_0)$ is obtained from immediate positional observation of celestial objects and the other $(L_c)$ calculated on the theoretical basis. The coefficients $h_i$ are usually used for the purpose of scaling these differences. The values $L_0$ and $L_c$ must be compared with each other in the same time-coordinate reference frame $S = (C, t)$. It is convenient to use the instrument registration system $S_r$ for this purpose. In this system the immediate measured value $L_0$ is the true value $L$ $(L_0 = L + \tau)$ to within the observation *errors* $\tau$. The calculated value $L_c$ is equal to the same true value $L = L_c + \rho$ to within the reduction *corrections* $\rho$ due to inaccuracies in the initial data and underlying theories. The main uncertainties of the reduction theory are as follows:

—coordinates or orientation of the astrometric instrument;

—physical theory of the instrument and observation process;

—coordinates of the observed celestial objects (OSO),

—unsteady atmospheric effects;

—irregularity in the Earth's rotation;

—tidal and tectonic deformation of the Earth.

By substituting the values $L_0 = L + \tau$ and $L_c = L - \rho$ in (1) we obtain the data vector

$$\mathbf{l} = (l_i) = h_i(\tau + \rho)_i. \tag{2}$$

Let us separate the systematic component $\mathbf{c} = (c_i)$ from the data vector $\mathbf{l} = (l_i)$. Introducing a linear parametric model

$$c_i = \sum_{j=1}^{m} \left(\frac{\partial l_i}{\partial p_j}\right) \Delta p_j, \tag{3}$$

we can represent the data vector in the form

$$\mathbf{l} = \mathbf{G}\mathbf{z} + \mathbf{w}, \tag{4}$$

where $\mathbf{z} = (\Delta p_j)$ is the vector of unknown parameters of the linear model (3) $(j = 1, 2, \ldots, m)$, $\mathbf{G}$ is the matrix of derivatives of dimension $N \times m$ and $\mathbf{w} = (w_i)$ is a quasi-random residual vector

$$\mathbf{w} = \mathbf{u} + \mathbf{v} + \mathbf{t}. \tag{5}$$

Here $\mathbf{u}$ is the vector of random components of observational errors and the errors of modeling their systematic components, $\mathbf{v}$ is the vector of unknown individual components of the OSO coordinate corrections and the errors of modelling their systematic components, $\mathbf{t}$ is the vector of errors of that part of the data model which is not included in the model (3) and whose parameters are not included into the vector $\mathbf{z}$ and not specified.

### 1.2. *A Priori Accuracy Estimation of Observational Data*

Quasi-random vectors $\mathbf{u}$ and $\mathbf{v}$ are distributed in different fields and thus are not correlated. In addition, the covariances $\mathbf{Q}_{ut}$ and $\mathbf{Q}_{vt}$ are negligible as $\mathbf{t}$ is usually small. Therefore, the variance-covariance matrix of the residuals $w$, according to (5), is

$$\mathbf{Q}_w = \mathbf{Q}_u + \mathbf{Q}_v + \mathbf{Q}_t. \tag{6}$$

At the initial stage of the treatment of the data only preliminary estimates of the observational variances $\sigma_i^2$ are usually known, and so, we have according to (2)

$$\mathbf{Q}_u = \mathrm{diag}(h_i^2 \sigma_i^2).$$

In the general case, $\mathbf{v}$ is a linear combination of individual corrections to the OSO coordinates

$$\mathbf{v} = \mathbf{E}\mathbf{v}_\alpha + \mathbf{F}\mathbf{v}_\delta, \tag{7}$$

where

$$(\mathbf{v}_\alpha, \mathbf{v}_\delta) = (\Delta\alpha \cos\delta, \Delta\delta), \tag{8}$$

$\mathbf{E} = \mathrm{diag}(e_i)$ and $\mathbf{F} = \mathrm{diag}(f_i)$ are diagonal matrices of known coefficients. Then

$$\mathbf{Q}_v = \mathbf{E}\mathbf{Q}_{\alpha\alpha}\mathbf{E}^T + \mathbf{F}\mathbf{Q}_{\delta\delta}\mathbf{F}^T + 2(\mathbf{E}\mathbf{Q}_{\alpha\delta}\mathbf{F}^T). \tag{9}$$

In particular, we have $e_i = \pm 1$, $f_i \equiv 0$ or $e_i \equiv 0$, $f_i = \pm 1$ for meridional and $e_i \neq f_i \neq 0$ for non-meridional instruments.

Some of the modelling parameters are usually adopted as precise. They are not improved and do not enter the model (3). However, the accuracy of such parameters is often known and the estimates of their variances can be used in the adjustment process (Klenitzkij, 1982). If the vector of non-specified reductions is represented in the form (3),

$$t_i = \sum_{j=1}^{n} \left(\frac{\partial l_i}{\partial q_j}\right) \Delta q_j,$$

and the known variances of parameters $q_j$ are denoted as $(\sigma_j^2)$, then

$$\mathbf{Q}_t = \mathbf{V}\mathbf{S}\mathbf{V}^T, \tag{10}$$

where $\mathbf{S} = \mathrm{diag}(\sigma_j^2)$, $\mathbf{V}$ is the matrix of derivatives of dimension $N \times n$. As follows from Eqs. (6), (9) and (10), in the general case matrix $\mathbf{Q}_w$ is not diagonal and its off-diagonal elements are a consequence of a non-complete model (3) and the fact that the same OSO may be observed more than once.

### 1.3. *A Priori Accuracy Estimation of Unknown Parameters*

In practice some a priori information about the accuracy of unknown parameters $p_j$, that we are going to improve with the help of new observational data, is often known. It is more convenient to obtain this information from the adjustment of previous observations, when it can be expressed in the most convenient and complete form as the variance-covariance matrix of all or some parameters $p_j$ ($j = 1, 2, \ldots, m$). However, the usual form of such information, especially at the first steps of data processing, are the probability estimations:

$$P(a_j \leq p_j < b_j) = F(a_j) - F(b_j) = P_0, \tag{11}$$

where $P$ is the probability that the true values of parameters $p_j$ are in the intervals $[a_j, b_j)$; $F(a_j)$ and $F(b_j)$ are the values of the normal distribution function at the ends of this intervals. Taking the probability $P_0 = 0.997$ and assuming that the intervals $[a_j, b_j)$ are symmetric ($a_j = p_j - \Delta p_j$, $b_j = p_j + \Delta p_j$), we can express, according to the well-known rule of "three sigma," the half of the interval lengths $[a_j, b_j)$ in terms of the root-mean-square (RMS) errors $\sigma_j$, i.e., $\Delta p_j = z_j = 3\sigma_j$. This permits to obtain the diagonal elements of the matrix $\mathbf{Q}_z = \mathrm{diag}(\sigma_j^2)$.

Sometimes the accuracy estimations are available for function $c_i$, but not for the parameters $z_j$. Gubanov (1988, 1991) showed that in this case the RMS a priori estimations of the improved parameters $p_j$ can be obtained if the linear model $c = Gz$ can be represented as an expansion in ortho-normalized basic functions, as in this case an obvious equality is fulfilled:

$$\sigma_c^2 = \sigma_z^2\, \|\mathbf{G}\|^2/N = \mathrm{const},$$

where $\sigma_c$ is a known RMS of $c_i$, $\|\mathbf{G}\|$ is the spherical norm of matrix $\mathbf{G}$, and $N$ is the observation number. Thus, in this case not all the elements of the diagonal matrix $\mathbf{Q}_z = \mathrm{diag}(\sigma_j^2)$, but only the mean variances $\sigma_z^2$, can be reconstructed. This is why $\mathbf{Q}_z = \sigma_z^2\mathbf{I}$, where $\mathbf{I}$ is a $m \times m$ unit matrix.

### 1.4. *The Completeness and Orthogonality of the Data Models*

Consider a simple two-parametric model of data

$$\mathbf{l}_s = \mathbf{a}x_s + \mathbf{b}y_s,$$

where $\mathbf{l}_s = (l_i)_s$, $\mathbf{a} = (a_i)$, $\mathbf{b} = (b_i)$ are column-vectors ($i = 1, 2, \ldots, N$); $x_s$, $y_s$ are the scalar parameters determined, and $s$ is the number of the observation series ($s = 1, 2, \ldots, n$).

Let us assume that the true vectors $\mathbf{l}_s$ contain random errors $\mathbf{w}_s$ and hidden systematic errors $\mathbf{c}_s = \mathbf{g}z_s$ characterized by a stable influence vector-function $\mathbf{g}$ and variable, from series to series, parameter $z_s$. Thus, by analogy with (4) we have the following parametric equation in the vector form

$$\mathbf{l}_s = \mathbf{c}_s + \mathbf{w}_s = \mathbf{g}z_s + \mathbf{w}_s = \mathbf{a}x_s + \mathbf{b}y_s. \tag{12}$$

The SLS estimates of unknown parameters are given by

$$x_s = (x_c)_s + (x_w)_s, \qquad y_s = (y_c)_s + (y_w)_s, \tag{13}$$

where

$$(x_c)_s = [(\mathbf{ag}) - (\mathbf{bg})(\mathbf{ab})]z_s/D, \qquad (x_w)_s = [(\mathbf{aw}_s) - (\mathbf{bw}_s)(\mathbf{ab})]/D,$$
$$(y_c)_s = [(\mathbf{bg}) - (\mathbf{ag})(\mathbf{ab})]z_s/D, \qquad (y_w)_s = [(\mathbf{bw}_s) - (\mathbf{aw}_s)(\mathbf{ab})]/D.$$

Here $D = 1 - (\mathbf{ab})^2$ is the determinant of the normal equation matrix. Parenthesis denote normalized scalar products of the corresponding vectors, and we have $(\mathbf{aa}) = (\mathbf{bb}) = (\mathbf{gg}) = 1$.

The first and second terms in (13) are the effects of systematic and random errors in the observational data, respectively. Both effects are enhanced if the determinant $D$ decreases.

If the parameters $z_s$ of the hidden systematic errors $\mathbf{c}_s$ are random and their mean value equals zero, i.e. $M[z_s] = 0$, then the SLS estimations of the unknown parameters will be also random with the mutual regression

$$(y_c)_s = k_c(x_c)_s + [(\mathbf{bg})^2 - (\mathbf{ag})^2]z_s/(\mathbf{bg})D,$$

where $k_c = (\mathbf{ag})/(\mathbf{bg})$ is the coefficient of regression. However, in practice the most typical and dangerous situation is the stable effect of systematic errors, when $M[z_s] = z \neq 0$. In this case the errors of the determined parameters also become stable and the regression given above turns into a functional connection.

As to the random errors $\mathbf{w}_s$, it is obvious from (13) that their influence on the estimates of the determined parameters is minimal if $(\mathbf{ab}) = 0$ and increase rapidly if $(\mathbf{ab}) \to \pm 1$. Besides, there is the following statistical connection between these estimates:

$$(y_w)_s = k_w(x_w)_s + [(\mathbf{bw}_s)^2 - (\mathbf{aw}_s)^2]/(\mathbf{bw}_s)D,$$

where $k_w = (\mathbf{aw}_s)/(\mathbf{bw}_s)$ is a random coefficient that differs from $k_c$.

It is easy to show that, if the hidden error $\mathbf{c}_s$ is included in the data model, the mean values of all the SLS parameter estimates will be statistically true values. The random distribution of these estimates from series to series will be minimum if the basic functions of the data model are mutually orthogonal.

## 2. GENERAL LEAST-SQUARES TECHNIQUE

### 2.1. *The Parametric Adjustment Algorithm*

We see from the above discussion that it is necessary to include in the generalized concept of astrometric data not only the data vector $\mathbf{l}$, but also the a priori information on the variance-covariance matrices $\mathbf{Q}_w$ and $\mathbf{Q}_z$ of both the quasi-random residuals $\mathbf{w}$ and parameters $\mathbf{z}$, respectively. In this case the solution of the system (4) must be found under the condition (Moritz, 1980)

$$S = \mathbf{w}^T\mathbf{P}_w\mathbf{w} + \mathbf{z}^T\mathbf{P}_z\mathbf{z} = \text{min.}, \tag{14}$$

where

$$\mathbf{P}_w = \sigma_0^2\mathbf{Q}_w^{-1}, \qquad \mathbf{P}_z = \sigma_0^2\mathbf{Q}_z^{-1} \tag{15}$$

are the given weight matrices. Taking into account the extremum condition (14) and the formula (4), we obtain the system of normal equations as

$$\mathbf{Dz} = \mathbf{f}, \tag{16}$$

whose solution is

$$\mathbf{z} = \mathbf{D}^{-1}\mathbf{f} = \mathbf{Cf}, \tag{17}$$

where

$$\mathbf{D} = \mathbf{G}^T\mathbf{P}_w\mathbf{G} + \mathbf{P}_z, \qquad \mathbf{C} = \mathbf{D}^{-1}, \qquad \mathbf{f} = \mathbf{G}^T\mathbf{P}_w\mathbf{l}.$$

The parameter accuracy estimation is given by the a posteriori variance-covariance matrix

$$\mathbf{Q}_z = \sigma_0^2 \mathbf{C},$$

where $\sigma_0^2$ is the variance of unit weight (Klenitzkij, 1982)

$$\sigma_0^2 = S/(N - m + m_r). \tag{18}$$

Here

$$S = \mathbf{l}^T \mathbf{P}_w \mathbf{l} - \mathbf{f}^T \mathbf{z},$$

and $m$ is the total number of unknown parameters of vector $\mathbf{z}$, and $m_r$ is the number of regularized parameters ($m_r \leqslant m$).

If some parameter $z_j$ is not regularized, the $j$'th line and column of matrix $\mathbf{P}_z$ vanish simultaneously. If $\mathbf{P}_z = [0]$, we have $m_r = 0$. In this case the general least-squares (GLS) algorithm (14)–(18) turns into the SLS technique, which additionally assumes that the weight matrix $\mathbf{P}_w$ is of a diagonal type. Thus, the SLS supposes that the improved parameters $p_j$ have infinite variances $\sigma_j^2$ and zero weight. Therefore, the SLS estimates of the corrections $z_j = \Delta p_j$ can be arbitrary whereas due to presence of a "stabilizer" $\Omega = \mathbf{z}^T \mathbf{P}_z \mathbf{z}$ in (14), the GLS estimates $z_j$ are limited. The more precise are the initial values $p_j$ and the poorer is the accuracy of observations, the nearer to zero are the corrections $z_j = \Delta p_j$ obtained by the GLS solution (17).

Otherwise, thanks to the "stabilizer" $\Omega$, the GLS algorithm does not have chance to worsen a precise knowledge of initial parameters due to rough observations, and vice versa, the GLS estimates of the parameters having large a priori errors obtained by using precise observations are close to the SLS solution.

Comparing the above regularization procedure with that proposed by Tikhonov and Arsenin (1986), we see that the latter allows the use the regularization matrix $\mathbf{P}_z$ in the simplest form as $\mathbf{P}_z = \alpha \mathbf{I}$, where $\alpha$ is the so-called regularization parameter. Therefore, the regularization parameters $\alpha$ are equal to the mean a priori weight $p_z$ of the estimated parameters with respect to the variance of observations $\sigma_0^2$, i.e. $\alpha = \sigma_0^2/\sigma_z^2 = p_z = \text{const}$. A similar case is considered in Sect. 1.3, but usually we have the possibility to estimate all the diagonal elements of matrix $\mathbf{P}_z$ and even the whole matrix. Thus, the regularization of the system of normal equations proposed by Tikhonov and Arsenin (1986) is a partial use of a priori information on the accuracy of the estimated parameters.

### 2.2. *Orthogonalization of Estimated Parameters*

If the correlations between parameters $z_j$ are close, it is possible to enter a new system of non-correlated parameters $z'_j$ by means of the linear transform (Jenkins and Watts, 1969)

$$\mathbf{z} = \mathbf{R}^T \mathbf{z}', \tag{19}$$

where $\mathbf{R} = (r_{ij})$ is the orthogonal matrix of eigenvectors $\mathbf{r}_j$ of matrix $\mathbf{C}$. The right-hand side eigenvectors $\mathbf{r}_j$ are the columns of the matrix $\mathbf{R}$. The linear transformation

$$\mathbf{R}^T \mathbf{C} \mathbf{R} = \mathbf{L} \tag{20}$$

converts matrix $\mathbf{C}$ into a diagonal one $\mathbf{L} = \text{diag}(\lambda_j)$, which consists of eigenvalues $\lambda_j$ ($j = 1, 2, \ldots, m$).

Substituting (19) into the system of normal equations (16) and multiplying the result by $\mathbf{R}$, we obtain a new normal system

$$\mathbf{RDR}^T\mathbf{z}' = \mathbf{Rf}$$

and its solution is

$$\mathbf{z}' = (\mathbf{RDR}^T)^{-1}\mathbf{Rf}. \tag{21}$$

Using Eq. (20) and the orthogonality of matrix $\mathbf{R}$ ($\mathbf{R}^T = \mathbf{R}^{-1}$), it is easy to show that the covariance matrix $\mathbf{C}'$ of new parameters $\mathbf{z}'$ is a diagonal one:

$$\mathbf{C}' = (\mathbf{RDR}^T)^{-1} = (\mathbf{R}^T)^{-1}\mathbf{D}^{-1}\mathbf{R}^{-1} = \mathbf{RCR}^T = \mathbf{L}.$$

This is why the solution (21) can be written in a simpler form

$$\mathbf{z}' = \mathbf{LRf}.$$

Multiplying (19) by $\mathbf{R}$, we obtain the inverse transformation

$$\mathbf{z}' = \mathbf{Rz},$$

or

$$\left.\begin{array}{l} z_1' = r_{11}z_1 + r_{12}z_2 + \cdots + r_{1m}z_m, \\ z_2' = r_{21}z_1 + r_{22}z_2 + \cdots + r_{2m}z_m, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ z_2' = r_{m1}z_1 + r_{m2}z_2 + \cdots + r_{mm}z_m. \end{array}\right\} \tag{22}$$

The system (22) can be used for replacing strongly correlated parameters by their linear combinations. The calculation of the eigenvalues and vectors of matrix $\mathbf{C}$ is carried out by the Jacobi method (Press *et al.*, 1992)

## 2.3. *Parametric Adjustment Under a Constraint*

Sometimes in astrometrical practice it is necessary to take into account some connections between unknown parameters. Let these restrictions be given in the form of a linear system of equations,

$$\mathbf{Ez} = \mathbf{h}, \tag{23}$$

where $\mathbf{z} = (\mathbf{x}, \mathbf{y})$ is the $m \times 1$ vector of unknown parameters, $\mathbf{E}$ is the $M \times m$ matrix of the linear constraints and $\mathbf{h}$ is the vector of $M$ independent terms of the linear constraints.

A joint solution of the systems (4) and (23) can be obtained by the Lagrange method under a standard condition (Moritz (1980))

$$S = \mathbf{w}^T\mathbf{P}_w\mathbf{w} + \mathbf{z}^T\mathbf{P}_w\mathbf{z} + 2\mathbf{k}^T(\mathbf{Ez} - \mathbf{h}) = \min.,$$

where $\mathbf{k}$ is the vector of $M$ Lagrange's multipliers. This condition brings the normal equations to the form

$$\begin{vmatrix} \mathbf{D} & \mathbf{E}^T \\ \mathbf{E} & \mathbf{0} \end{vmatrix} \cdot \begin{vmatrix} \mathbf{z} \\ \mathbf{k} \end{vmatrix} = \begin{vmatrix} \mathbf{f} \\ \mathbf{h} \end{vmatrix},$$

where matrix $\mathbf{D}$ and vector $\mathbf{f}$ are given by (16), and $[\mathbf{0}]$ is a zero-matrix of dimension $M \times M$.

The FORTRAN-subroutine for parametric adjustment of separate series by the algorithms described in Sects. 2.1 and 2.3 is given in the APPENDIX.

### 2.4. *Two-Group Parametric Adjustment*

Let us divide the systematic component $\mathbf{c}$ of the data vector $\mathbf{l}$ and also of its model (4) into two parts, $\mathbf{c} = \mathbf{a} + \mathbf{b}$. Denoting $\mathbf{G} = [\mathbf{A} \vdots \mathbf{B}]$ and $\mathbf{z} = (\mathbf{x}, \mathbf{y})$, we can write

$$\mathbf{l} = \mathbf{Ax} + \mathbf{By} + \mathbf{w} \tag{24}$$

instead of (5). In (24), we have $\mathbf{x} = (x_j)$ $(j = 1, 2, \ldots, m_x)$ and $\mathbf{y} = (y_k)$ $(k = 1, 2, \ldots, m_y)$.

This type of parametric equations is most convenient for the adjustment of the so-called "absolute" observations, when the main aim is the determination of the OSO coordinates. Their systematic corrections will be included into a separate model $\mathbf{b} = \mathbf{By}$ and all the other errors, into the model $\mathbf{a} = \mathbf{Ax}$ of the generalized instrumental system (GIS).

Let us consider some series of astrometrical observations. The corresponding system of parametric equations (24) can be solved by the GLS technique under the condition

$$S = \mathbf{w}^T\mathbf{P}_w\mathbf{w} + \mathbf{x}^T\mathbf{P}_x\mathbf{x} + \mathbf{y}^T\mathbf{P}_y\mathbf{y} = \min.,$$

which gives a two-group system of normal equations

$$\begin{vmatrix} \mathbf{D}_{xx} & \mathbf{D}_{xy} \\ \mathbf{D}_{yx} & \mathbf{D}_{yy} \end{vmatrix} \cdot \begin{vmatrix} \mathbf{x} \\ \mathbf{y} \end{vmatrix} = \begin{vmatrix} \mathbf{f}_x \\ \mathbf{f}_y \end{vmatrix}, \tag{25}$$

where

$$\mathbf{f}_x = \mathbf{A}^T\mathbf{P}_w\mathbf{l}, \qquad \mathbf{f}_y = \mathbf{B}^T\mathbf{P}_w\mathbf{l},$$

$$\mathbf{D}_{xx} = \mathbf{A}^T\mathbf{P}_w\mathbf{A} + \mathbf{P}_x, \qquad \mathbf{D}_{xy} = \mathbf{A}^T\mathbf{P}_w\mathbf{B} = \mathbf{D}_{yx}^T, \qquad \mathbf{D}_{yy} = \mathbf{B}^T\mathbf{P}_w\mathbf{B} + \mathbf{P}_y.$$

The symmetric matrix $\mathbf{D}$ on the left-hand side of (25) has a block structure, therefore we have (Korn and Korn, 1968)

$$\mathbf{D}^{-1} = \mathbf{C} = \begin{vmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xy} \\ \mathbf{C}_{yx} & \mathbf{C}_{yy} \end{vmatrix},$$

where

$$\mathbf{C}_{xx} = (\mathbf{D}_{xx} - \mathbf{D}_{xy}\mathbf{D}_{yy}^{-1}\mathbf{D}_{yx})^{-1}, \qquad \mathbf{C}_{yx} = -\mathbf{D}_{yy}^{-1}\mathbf{D}_{yx}\mathbf{C}_{xx},$$
$$\mathbf{C}_{yy} = (\mathbf{D}_{yy} - \mathbf{D}_{yx}\mathbf{D}_{xx}^{-1}\mathbf{D}_{xy})^{-1}, \qquad \mathbf{C}_{xy} = -\mathbf{D}_{xx}^{-1}\mathbf{D}_{xy}\mathbf{C}_{yy},$$

i.e., only the $m_x \times m_x$ and $m_y \times m_y$ matrices need to be inverted. Solution of the normal system (25) is given by

$$\begin{vmatrix} \mathbf{x} \\ \mathbf{y} \end{vmatrix} = \begin{vmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xy} \\ \mathbf{C}_{yx} & \mathbf{C}_{yy} \end{vmatrix} \cdot \begin{vmatrix} \mathbf{f}_x \\ \mathbf{f}_y \end{vmatrix}. \tag{26}$$

The variances and covariances of the unknown parameters are given by

$$\mathbf{Q}_x = \sigma_0^2\mathbf{C}_{xx}, \qquad \mathbf{Q}_y = \sigma_0^2\mathbf{C}_{yy}, \qquad \mathbf{Q}_{xy} = \sigma_0^2\mathbf{C}_{xy}, \tag{27}$$

where $\sigma_0^2$ is the external variance of unit weight calculated using (18) with $S = \mathbf{l}^T\mathbf{P}_w\mathbf{l} - \mathbf{f}_x^T\mathbf{x} - \mathbf{f}_y^T\mathbf{y}$, $m = m_x + m_y$, $m_r = (m_r)_x + (m_r)_y$. In the case of absolute determinations of the OSO coordinates, it is very important that vectors $\mathbf{x}$ and $\mathbf{y}$ should be uncorrelated, otherwise the instrumental system would distort the new catalogue system. The correlation between these vectors is described by the matrix $\mathbf{K}_{xy} = (k_{ij})_{xy}$ whose elements are

$$(k_{ij})_{xy} = (c_{ij})_{xy}/\sqrt{(c_{ii})_x(c_{jj})_y},$$

where $c_{ij}$ are the elements of the block matrix $\mathbf{C}$.

The cross-correlation matrix $\mathbf{K}_{xy}$ can be considered as a numerical criterion for a new catalogue system being independent of the GIS. The closer its elements are to zero, the more mutually independent are the GIS and the new catalogue system. This means that the new catalogue will be more accurate in a systematic respect. If large correlations are present in the matrix $\mathbf{K}_{xy}$, it is necessary to revise the GIS model $\mathbf{a} = \mathbf{Ax}$. If this is not possible, there are essential limitations to improving the initial catalogue using available observations.

### 2.5. *Adjustment of Independent Series*

In this section we consider the observations of several series. Let all the parameters of each series depend on its number $s$. Then the two-group model of these data can be represented as systems of $n$ equations

$$\mathbf{l}_s = \mathbf{A}_s\mathbf{x}_s + \mathbf{B}_s\mathbf{y}_s + \mathbf{w}_s, \quad (s = 1, 2, \ldots, n). \tag{28}$$

Each of these systems is independent and can be solved separately according to the formulas (25)–(27) only if the weight matrix of the data set has a block-diagonal type

$$\mathbf{P}_w = \mathrm{diag}(\mathbf{P}_s)_w, \tag{29}$$

where $(\mathbf{P}_s)_w$ are the $N_s \times N_s$ weight matrices of separate series.

Condition (29) means that there is no mutual covariations between the series of data, but strictly speaking, it is very seldom fulfilled. In the general case we have a complete $n \times n$ block-diagonal variance-covariance matrix of the data set

$$\mathbf{Q}_w = \begin{vmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \cdots & \mathbf{Q}_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{Q}_{n1} & \mathbf{Q}_{n2} & \cdots & \mathbf{Q}_{nn} \end{vmatrix}, \tag{30}$$

whose elements can be represented according to (7) as

$$(\mathbf{Q}_{sr})_w = (\mathbf{Q}_{sr})_u + (\mathbf{Q}_{sr})_v + (\mathbf{Q}_{sr})_t, \tag{31}$$

where $s,\ r = 1, 2, \ldots, n$.

If all the GIS models $\mathbf{a}_s = \mathbf{A}_s\mathbf{x}_s$ were sufficiently complete, the quasi-random errors of observation $\mathbf{u}_s$ in different series would be independent of each other, and we should have $(\mathbf{Q}_{sr})_u = [\mathbf{0}]$ with $s \neq r$ and

$$(\mathbf{Q}_{ss})_u = \mathrm{diag}(\sigma_s^2)_u. \tag{32}$$

According to (9) the mutual covariations of the OSO individual errors are

$$(\mathbf{Q}_{sr})_v = \mathbf{E}_s(\mathbf{Q}_{sr})_{\alpha\alpha}\mathbf{E}_r^T + 2\mathbf{F}_s(\mathbf{Q}_{sr})_{\alpha\delta}\mathbf{E}_r^T + \mathbf{F}_s(\mathbf{Q}_{sr})_{\delta\delta}\mathbf{F}_r^T \neq [\mathbf{0}]. \tag{33}$$

Otherwise, the mutual covariations $(\mathbf{Q}_{sr})_v$ of the OSO coordinate errors will be equal to zero for any pair of the series only if these series do not have common objects. In the general case, according to (31) and (33), the weight matrix of the data set is a $N \times N$ matrix,

$$\mathbf{P}_w = \sigma_0^2 \mathbf{Q}_w^{-1}. \tag{34}$$

It requires the inversion of the complete $N \times N$ covariance matrix of the order $N = N_1 + N_2 + \cdots + N_n$, and all the series must be adjusted together by the algorithm described in Sect. 2.2–2.3. For this purpose it is necessary to join equations (28) into the system

$$\mathbf{l} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} + \mathbf{w}, \tag{35}$$

where $\mathbf{l} = (\mathbf{l}_s)$, $\mathbf{x} = (\mathbf{x}_s)$, $\mathbf{y} = (\mathbf{y}_s)$, $\mathbf{w} = (\mathbf{w}_s)$ are the combined vectors; and $\mathbf{A}$, $\mathbf{B}$ are block-diagonal matrices, whose elements are $N_s \times m_x$ and $N_s \times m_y$ matrices, respectively.

## 3. JOINT ADJUSTMENT OF AN ASTROMETRIC DATA SET

### 3.1. *The General Concept of a Joint Data Set*

Several series of observations in which data models have some common parameters will be called joint data set. Combining these common parameters into vector $\mathbf{y}$, we obtain the following model of the joint data

$$\mathbf{l}_s = \mathbf{A}_s\mathbf{x}_s + \mathbf{B}_s\mathbf{y} + \mathbf{w}_s, \quad (s = 1, 2, \ldots, n). \tag{36}$$

In optical astrometry, the components of the vector $\mathbf{y}$ are the unknown coefficients of a linear decomposition in ortho-normalized two-dimensional functions of the systematic errors of the improved reference catalogue in an available declination zone.

In radio astrometry, the separation of the errors of the reference catalogue into systematic and random components is not used as there are not many enough observable radio sources. Neglecting small reduction errors $\mathbf{t}$ in Eq. (5), we obtain a VLBI data model

$$\mathbf{l}_s = \mathbf{A}_s\mathbf{x}_s + \mathbf{B}_s\mathbf{y} + \mathbf{u}_s, \quad (s = 1, 2, \ldots, n), \tag{37}$$

where, according to (7), we have

$$\mathbf{B}_s = [\mathbf{E}_s \,\vdots\, \mathbf{F}_s], \qquad \mathbf{y} = (\mathbf{v}_\alpha, \mathbf{v}_\delta).$$

In satellite astrometry, the joint data set is formed from observations of the same satellite at several ground stations ($s = 1, 2, \ldots, n$) during a limited period of time $\Delta T \cong 5$ days. In this case the elements of the satellite orbital arc forming vector $\mathbf{y}$ are constants or simple functions of time. Neglecting small vectors $\mathbf{t}$ and $\mathbf{v}$, we obtain the satellite observation model in the form (37).

### 3.2. *Adjustment of Non-Correlated Series*

Consider now a joint data of series whose weight matrix has a block-diagonal form (29). These series are independent in the sense that their mutual correlations are absent. Is such a case possible in the practice of astrometry? As mentioned in Sect. 2.5, the following conditions are necessary:

(1) the reduction errors $\mathbf{t}$ are negligible;

(2) the models of systematic errors $\mathbf{a}$ and $\mathbf{b}$ are sufficiently complete;

(3) the random errors $\mathbf{v}$ of the OSO coordinates are zero or the adjusted series do not contain common OSO.

From the practical point of view, conditions (1) and (2) are acceptable at least at the first step of the iteration process. The third condition is realized in the case of satellite and the VLBI observations (see 3.1).

Let us consider the adjustment process for the joint series data described in the parametric model (37) with a block-diagonal weight matrix

$$\mathbf{P}_u = \sigma_0^2 \mathbf{Q}_u^{-1} = \sigma_0^2 \,\mathrm{diag}(\mathbf{Q}_s)_u^{-1} = \mathrm{diag}(\mathbf{P}_s)_u. \tag{38}$$

To first approximation, preliminary estimates (32) can be used instead of (38).

Using the weight matrices (38) or (32), we can construct a system of normal equations of the type (25) for all the data series of a given group described by parametric equations (37). Then we have

$$\left.\begin{aligned} \mathbf{F}_s\mathbf{x}_s + \mathbf{H}_s\mathbf{y} &= \mathbf{f}_s, \\ \mathbf{H}_s^T\mathbf{x}_s + \mathbf{G}_s\mathbf{y} &= \mathbf{g}_s, \end{aligned}\right\} \tag{39}$$

where

$$\mathbf{F}_s = (\mathbf{D}_{xx})_s, \quad \mathbf{H}_s = (\mathbf{D}_{xy})_s, \quad \mathbf{G}_s = (\mathbf{D}_{yy})_s, \quad \mathbf{f}_s = (\mathbf{f}_x)_s, \quad \mathbf{g}_s = (\mathbf{f}_y)_s.$$

Eliminating $\mathbf{x}_s$ from (39), we obtain a system of equations for estimating $\mathbf{y}$ from all the data series

$$\mathbf{V}_s\mathbf{y}_s = \mathbf{h}_s, \tag{40}$$

where

$$\mathbf{V}_s = (\mathbf{G}_s - \mathbf{H}_s^T\mathbf{F}_s^{-1}\mathbf{H}_s), \qquad \mathbf{h}_s = \mathbf{g}_s - \mathbf{H}_s^T\mathbf{F}_s^{-1}\mathbf{f}_s.$$

As $\mathbf{V}_s$ can be a degenerate matrix we cannot solve the system (40). We shall accumulate such systems for all the series of this group by simple summation

$$(\mathbf{V}, \mathbf{h}) = \sum_{s=1}^{n} (\mathbf{V}_s, \mathbf{h}_s).$$

After that we obtain the final estimation of vector $\mathbf{y}$:

$$\mathbf{y} = \mathbf{V}^{-1}\mathbf{h} \tag{41}$$

with the weight $\mathbf{P}_y = \mathbf{V}$. It is obvious that matrix $\mathbf{V}$ is much better conditioned than individual matrices $\mathbf{V}_s$.

It is easy to show that, if the weight matrix $\mathbf{P}_u$ is of a block-diagonal type (38), the estimate (41) coincides exactly with that obtained from the global adjustment of all the series by the GLS solution of the system (35). We can obtain the same estimation by weighting independent solutions $\mathbf{y}_s$ of systems (37) as

$$\mathbf{y} = \left[\sum_{s=1}^{n} (\mathbf{P}_s)_y\right]^{-1} \sum_{s=1}^{n} (\mathbf{P}_s)_y\mathbf{y}_s = \mathbf{V}^{-1}\sum_{s=1}^{n} \mathbf{V}_s\mathbf{y}_s = \mathbf{V}^{-1}\sum_{s=1}^{n} \mathbf{h}_s = \mathbf{V}^{-1}\mathbf{h} = \mathbf{y}.$$

Substituting the estimate (41) into all the equations (37) and solving the separate systems

$$\mathbf{l}'_s = \mathbf{l}_s - \mathbf{B}_s\mathbf{y} = \mathbf{A}_s\mathbf{x}_s + \mathbf{u}_s, \quad (s = 1, 2, \ldots, n),$$

by the GLS technique, we obtain the GIS parameters $\mathbf{x}_s$ for all the series related to a common vector $\mathbf{y}$

$$\mathbf{x}_s = \mathbf{F}_s^{-1}(\mathbf{f}_s - \mathbf{H}_s\mathbf{y}) \tag{42}$$

with the weight matrix $(\mathbf{P}_s)_x = \mathbf{F}_s$. It is easy to show that, in this case, a variance of unit weight should be calculated as

$$\sigma_0^2 = S/(N - m_x n - m_y),$$

where

$$S = \sum_{s=1}^{n} \mathbf{u}_s^T\mathbf{P}_s\mathbf{u}_s = \sum_{s=1}^{n} (\mathbf{l}_s^T\mathbf{P}_s\mathbf{l}_s - \mathbf{f}_s^T\mathbf{F}_s^{-1}\mathbf{f}_s) - \mathbf{h}^T\mathbf{y}.$$

The residuals

$$\mathbf{u}_s = \mathbf{l}_s - \mathbf{A}_s\mathbf{x}_s - \mathbf{B}_s\mathbf{y}, \qquad (s = 1, 2, \ldots, n), \tag{43}$$

can be used for further improvement of the weight matrices $(\mathbf{P}_s)_u$ (see Sect. 3.5). Then one should repeat the whole adjustment procedure from the very beginning.

The main aim of the satellite data adjustment is to determine most precisely the coordinates of the observation point. Therefore, observations on only one arc not sufficient and it is necessary to use some groups of series $(k = 1, 2, \ldots, m)$, corresponding to several arcs of a satellite orbit. The residuals $\mathbf{u}_s$, defined by Eq. (43), are the original data of subsequent adjustment. If we separate these residuals according to the series index $s$ and choose only those which correspond to the observations at basic stations, we obtain a group data set whose model can be presented as

$$\mathbf{l}_k = \mathbf{A}_k\mathbf{x} + \mathbf{B}_k\mathbf{y}_k + \mathbf{u}_k, \qquad (k = 1, 2, \ldots, m), \tag{44}$$

where $\mathbf{l}_k = (\mathbf{u}_s)_k$; $\mathbf{A}_k = \mathrm{diag}(\mathbf{A}_s)_k$, $\mathbf{B}_k = \mathrm{diag}(\mathbf{B}_s)_k$ are compound influence matrices, $\mathbf{x} = (\mathbf{x}_s)$ is the vector of corrections to the coordinates of the basic stations obtained by Eq. (42) and $\mathbf{y}_k$ is the vector of corrections to the elements of new orbital arcs $(k = 1, 2, \ldots, m)$.

In radioastrometry, the inter-group adjustment can be applied to diurnal series of VLBI observations of a given radio source at different bases, labelled with index $k$. In this case the GIS parameter vector $\mathbf{x}_k$ depends on the base index but the vector $\mathbf{y}$ does not, therefore we have

$$\mathbf{l}_k = \mathbf{A}\mathbf{x}_k + \mathbf{B}_k\mathbf{y} + \mathbf{u}_k. \tag{45}$$

As long as the group data models (44) and (45) completely coincide with the series data model (37), the joint adjustment of groups can be carried out in the same as the above series have been adjusted into groups.

### 3.3. *Adjustment of Synchronous Group Observations*

The above adjustment process is not used for the treatment of absolute observations in optical astrometry as the coordinates of stars, even of the best fundamental catalogues, have significant individual errors. A data model for such observations cannot be described by Eqs (37) and (38). This model has the general form (36) with the weight matrix (34). In the case of joint adjustment of such data, it is necessary to use the general algorithm (see Sect. 2.2–2.3) that requires operations with very large matrices. This can be avoided only in the case of observations carried out by group programs.

The group program in optical astrometry is a list of transits of the stars across the instrument sighting device according to sideral time. This list is divided into several fragments of equal duration $\Delta T = 2^h$. Let us call these the "groups" of the program. During one night, two or more consequent groups of stars ("link" of the program, Fig. 1) are observed as usual.

Let consider some aggregate of series observation of one and the same link having the number $k$. As the same stars are observed in each series, their data model will obviously have two common parameter vectors $\mathbf{y}$ and $\mathbf{v}$, which define both a systematic and individual correction to the positions of the stars in the initial catalogue. Moreover, as each star is observed in al the series at the same hour angle, i.e., synchronously from series to series relative to the sideral time scale, the matrices $\mathbf{A}$ and $\mathbf{B}$ do not depend on the number of the series $s$. In these conditions the group data model turns into the form

$$\mathbf{l}_s = \mathbf{A}\mathbf{x}_s + \mathbf{B}\mathbf{y} + \mathbf{w}_s, \quad (s = 1, 2, \ldots, n), \tag{46}$$

where $\mathbf{w}_s = \mathbf{u}_s + \mathbf{v}$.

As every next link is displaced by one group relative to the subsequent one, there is a 50% overlapping of the neighbouring links. In this common group they have the same stars. However, as can be seen from Fig. 1, only even or only odd sequences of links do not overlap.

Though equations (46) are coupled by the complete weight matrix (34), nothing prevents us from forming their differences for common stars in different series. The vector of stellar coordinate corrections $\mathbf{v}$ cancels in this differences. Thus, for each pair of series with the numbers $s$ and $r$ we can obtain a system of equations

$$\Delta\mathbf{l}_{s,r} = \mathbf{A}\,\Delta\mathbf{x}_{s,r} + \Delta\mathbf{u}_{s,r}, \tag{47}$$

where

$$\Delta\mathbf{l}_{s,r} = \mathbf{l}_s - \mathbf{l}_r, \qquad \Delta\mathbf{x}_{s,r} = \mathbf{x}_s - \mathbf{x}_r, \qquad \Delta\mathbf{u}_{s,r} = \mathbf{u}_s - \mathbf{u}_r.$$

At the first step of adjustment we do not have the necessary information for the creation of the complete variance-covariance matrix $(\mathbf{Q}_s)_u$, and we can only reconstruct their diagonal elements according to Eq. (32) by using the intrinsic measurement accuracy. Therefore, we can accept the following preliminary form

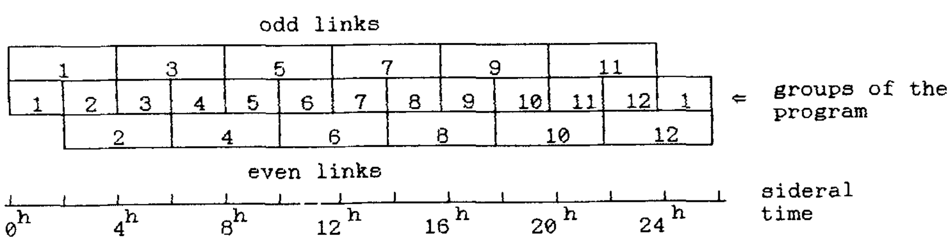


**Figure 1** A diagram of the traditional group program.

of this matrix:

$$(\mathbf{Q}_s)_u = \mathrm{diag}(\sigma_s^2)_u. \tag{48}$$

It is obvious that

$$(\mathbf{Q}_{s,r})_{\Delta u} = (\mathbf{Q}_s)_u + (\mathbf{Q}_r)_u.$$

Taking into account (32) and (48), we have

$$\mathbf{P}_{s,r} = \sigma_0^2(\mathbf{Q}_{s,r})_{\Delta u}^{-1} = \mathrm{diag}(\sigma_0^2/(\sigma_s^2 + \sigma_r^2)_u) \tag{49}$$

where $(\sigma_s^2)_u$, $(\sigma_r^2)_u$ are the variances of the errors $\mathbf{u}_s$ and $\mathbf{u}_r$, respectively.

The GLS adjustment of the system (47) with the data weight matrix (49) gives the normal equation system

$$\mathbf{D}_{s,r}\,\Delta\mathbf{x}_{s,r} = \Delta\mathbf{f}_{s,r},$$

where

$$\mathbf{D}_{s,r} = \mathbf{A}^T\mathbf{P}_{s,r}\mathbf{A}, \qquad \Delta\mathbf{f}_{s,r} = \mathbf{A}^T\mathbf{P}_{s,r}\,\Delta\mathbf{l}_{s,r}.$$

Its solution is

$$\Delta\mathbf{x}_{s,r} = \mathbf{x}_s - \mathbf{x}_r = \mathbf{D}_{s,r}^{-1}\,\Delta\mathbf{f}_{s,r}$$

with the weight matrix being $\mathbf{D}_{s,r}$.

Calculating all the reductions $\Delta\mathbf{x}_{s,r}$ of the series with the number $s$ to the rest of the series of a given link and averaging them with the weight matrix $\mathbf{D}_{s,r}$, we obtain the reduction

$$\Delta\mathbf{x}_s = \mathbf{x}_s - \mathbf{x} = \mathbf{D}_s^{-1}\sum_{r=1}^{n}\mathbf{D}_{s,r}\,\Delta\mathbf{x}_{s,r} \tag{50}$$

of this series to the mean GIS of a given link determined by weighted parameters

$$\mathbf{x} = \mathbf{D}^{-1}\sum_{s=1}^{n}\mathbf{D}_s\mathbf{x}_s. \tag{51}$$

The weight matrices of the vectors $\Delta\mathbf{x}_s$ and $\mathbf{x}$ are

$$\mathbf{D}_s = \sum_{r=1}^{n}\mathbf{D}_{s,r}, \qquad \mathbf{D} = \sum_{s=1}^{n}\mathbf{D}_s.$$

Substituting the reductions (50) in all the equations of the system (46), we obtain the data set reduced to the mean GIS, as follows:

$$\mathbf{l}'_s = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} + \mathbf{w}_s, \qquad (s = 1, 2, \ldots, n),$$

where

$$\mathbf{l}'_s = \mathbf{l}_s - \mathbf{A}\,\Delta\mathbf{x}_s. \tag{52}$$

Averaging the values (52) that correspond to the same star observed in different series, and using the preliminary weights

$$(\mathbf{P}_s)_u = \sigma_0^2(\mathbf{Q}_s)_u^{-1} = \mathrm{diag}(\sigma_0^2/\sigma_s^2)_u, \tag{53}$$

we obtain a shortened data set

$$\mathbf{l} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} + \mathbf{w}, \tag{54}$$

where

$$(\mathbf{l}, \mathbf{w}) = (\mathbf{P}_u)^{-1} \sum_{s=1}^{n} (\mathbf{P}_s)_u (\mathbf{l}', \mathbf{w})_s, \qquad \mathbf{P}_u = \sum_{s=1}^{n} (\mathbf{P}_s)_u. \tag{55}$$

At this stage the first iteration of the series adjustment ends. The following step is a joint adjustment of all the links. Their data model according to (54) has the form analogous to (36):

$$\mathbf{l}_k = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{y} + \mathbf{w}_k, \qquad (k = 1, 2, \ldots, m), \tag{56}$$

where $\mathbf{x}_k$ is the GIS parameter vector of the mean-link system with the number $k$, defined by Eq. (49), $\mathbf{y}$ is the common parameter vector of the initial catalogue systematic errors and $\mathbf{l}_k$ and $\mathbf{w}_k$ are the vectors of data and quasi-random residuals, respectively, averaged over all the series according to Eq. (55).

The model (56) differs from (36) in that its residuals are the sums $\mathbf{w}_k = \mathbf{u}_k + \mathbf{v}_k$. The variance-covariance matrix of these residuals is

$$\mathbf{Q}_w = (\mathbf{Q}_{kh})_w = (\mathbf{Q}_{kh})_u + (\mathbf{Q}_{kh})_v, \qquad (k, h = 1, 2, \ldots, m), \tag{57}$$

and the weight matrix is defined by (34).

A preliminary estimate of the matrix $\mathbf{Q}_u$ is a block-diagonal matrix

$$\mathbf{Q}_u = \mathrm{diag}(\mathbf{Q}_{kk})_u,$$

whose blocks follow from Eqs. (48) and (55) as simple diagonal matrices of the form

$$(\mathbf{Q}_{kk})_u = \mathrm{diag}((\sigma_s^2)_u / n). \tag{58}$$

The a priori variance-covariance matrix $\mathbf{Q}_v$ of the individual errors of the OSO coordinates can be constructed according to Eq. (33) if we replace the indices $(s, r)$ by $(g, h)$ and use the accuracy estimations of the OSO adopted coordinates in random relation. As one and the same star can be observed in different links, matrix $\mathbf{Q}_v$ is not diagonal and looks like a full rarefied matrix. According to (57), the total matrix $\mathbf{Q}_w$ will have the same form:

$$\mathbf{Q}_w = (\mathbf{Q}_{kh})_w = \begin{vmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \cdots & \mathbf{Q}_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{Q}_{m1} & \mathbf{Q}_{m2} & \cdots & \mathbf{Q}_{mm} \end{vmatrix}.$$

For a joint adjustment of all the equations (56) with the weight matrix

$$\mathbf{P}_w = \sigma_0^2 \mathbf{Q}_w^{-1},$$

an inversion of the full matrix $\mathbf{Q}_w$ is required. However, this matrix can have a very large dimension equal to $mN$ and can be degenerate. This is mostly dangerous for observations by traditional group programs, when neighbouring links have common group of stars in overlapping zones. Actually, suppose that matrix $\mathbf{Q}_u$ is of a strictly diagonal type (58) with equal elements given by the variance of observations $\sigma_u^2/n$. Let us take into account the fact that each star of the program is observed only twice in neighbouring overlapping links with the numbers $k$ and $k+1$. Assume also that all the stars have the same accuracy of individual coordinates in the form of variance $\sigma_v^2$. Denoting $a = \sigma_u^2 + \sigma_v^2$ and $b = \sigma_v^2$, we obtain the covariance matrix of residuals $\mathbf{w}$ of a three-diagonal type:

$$\mathbf{Q}_w = \begin{vmatrix} a & 0 & 0 & 0 & 0 & \cdots & b \\ 0 & a & b & 0 & 0 & \cdots & 0 \\ 0 & b & a & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b & 0 & 0 & 0 & 0 & \cdots & a \end{vmatrix}.$$

The determinant in the case of an $m$-group program with 50% overlapping is equal to $\det(\mathbf{Q}_w) = (a^2 - b^2)^m$. This determinant is approximately zero if $a \to b$, which means $\sigma_u \to 0$. In other words, the more precise are the observational data, the more degenerative is matrix $\mathbf{Q}_w$! If $\sigma_u = 0$, matrix $\mathbf{Q}_w$ cannot be inverted which means that the weighting matrix $\mathbf{P}_w$ also cannot be calculated.

There is one way out of this situation: the neighbouring links should not have common stars. In this case the elements $b$ of matrix $\mathbf{Q}_w$ disappear, and its determinant becomes equal to $a^{2m} \neq 0$. This condition is satisfied automatically if the even links are adjusted separately from the odd links. As both the links include all the stars of one and the same observational program, we obtain two independent parameter vectors of systematic corrections to the same initial catalogue together with their covariance and weight matrices. Then these vectors can be averaged according to Eq. (55).

Having determined parameters $\mathbf{x}_k$ and $\mathbf{y}$, we can obtain the residual vectors

$$\mathbf{w}_k = \mathbf{l}_k - \mathbf{A}_k\mathbf{x}_k - \mathbf{B}_k\mathbf{y}, \qquad (k = 1, 2, \ldots, m).$$

Selecting the residuals $\mathbf{w}_k$ which pertain to the observations of one and the same star, we form a vector of independent terms of Eqs. (9). Solving these equations, we obtain the corrections $(v_\alpha)_i = (\Delta\alpha \cos\delta)_i$ and $(v_\delta)_i = (\Delta\delta)_i$ of individual coordinates. This procedure is performed for all the stars of the observational program. We then determine the vectors of the observational errors $\mathbf{u}_k = \mathbf{w}_k - \mathbf{v}_k$ $(k = 1, 2, \ldots, m)$ averaged over all the series of each link. Reconstructing theGIS parameters $\mathbf{x}_s$ for all the series using Eq. (50), we obtain all the vectors of quasi-random observational errors:

$$\mathbf{u}_s = \mathbf{l}_s - \mathbf{A}_s\mathbf{x}_s - \mathbf{B}_s\mathbf{y} - \mathbf{v}_s.$$

### 3.4. *Improvement of the Data Variance-Covariance Matrix*

As we have seen above, a first-step adjustment of all types of astrometrical observation allows to determine parameter vectors $\mathbf{x}$ and $\mathbf{y}$, and also quasi-random residual vectors $(\mathbf{u}_s)_k$ $(s = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m)$. We show now how this can be used for the improvement of the a priori variance-covariance matrix of observational errors. This is important for the improvement of parameter estimates in the second iteration also for the agreement between the adopted mathematical models and the real physical process of observation.

Consider first of all the residuals $(\mathbf{u}_s)_k$ corresponding to satellite observations. In this case it is natural to seek for a correlation between the residuals as a function of time and the station number $s$. The current time of observation $t_i$ will be calculated within the interval $\Delta T = T_2 - T_1$ corresponding to the temporal length of the orbital arc ($\Delta T \geq 5$ days). If the step of time is $\Delta t$, then $t_i = T_1 + (i-1)\,\Delta t$, where $i = 1, 2, \ldots, N$. As several residuals $(\mathbf{u}_s)_k = (u_{is})_k$ corresponding to observations of several orbital arcs at a given station can appear in the interval $\Delta t_i = t_i \pm \Delta t/2$, we obtain $n$ vectors $\bar{\mathbf{u}}_s$ after averaging the original residuals $(u_{is})_k$ over $k$ as follows:

$$\bar{\mathbf{u}}_s = (\bar{u}_{is}) = (u'_{is}) - u'_i, \tag{60}$$

where

$$(u'_{is}) = \frac{1}{m} \sum_{k=1}^{m} (u_{is})_k, \qquad u'_i = \frac{1}{n} \sum_{s=1}^{n} (u'_{is}).$$

The changeable numbers $m = m_{is}$ may be used as the weights of the mean residuals $(u'_{is})$. In the general case considerable variations of the weights are possible. The residuals (60) can be represented in the form of a two-dimensional matrix

$$\mathbf{U} = (\bar{u}_{is}) = \begin{vmatrix} \bar{u}_{11} & \bar{u}_{12} & \cdots & \bar{u}_{1n} \\ \bar{u}_{21} & \bar{u}_{22} & \cdots & \bar{u}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{u}_{N1} & \bar{u}_{N2} & \cdots & \bar{u}_{Nn} \end{vmatrix}, \tag{61}$$

whose columns are vectors $\bar{\mathbf{u}}_s$.

If it is necessary to investigate correlations between the orbital arcs, i.e., groups of series, we shall interchange the indices $s$ and $k$ in Eq. (60). Then matrix (61) will be transformed to the form $\mathbf{U} = (\bar{u}_{ik})$ $(i = 1, 2, \ldots, N;\ k = 1, 2, \ldots, m)$.

In radio astrometry, series of VLBI observations is carried out during twenty-four hours, therefore $\Delta T = 1$ sideral day. In addition, index $k$ designates some base of the VLBI network. In optical astrometry in the case of synchronous observations by a group program, one usually has $\Delta T = 4$ hours and index $k$ designates the number of the link. In other respects, matrices $\mathbf{U}_s$ and $\mathbf{U}_k$ are constructed in the same way.

The columns of matrix (61) can be considered as separate realizations of a random process $U(t)$, and their rows, as its sections at the moments $t_i$. The covariance function $q(t_i, t_j)$ of a process $U(t)$ can be presented as a $N \times N$ matrix $\mathbf{Q}_\tau = (q_{ij})$ with the elements

$$q_{ij} = q(t_i, t_j) = \frac{1}{n-1} \sum_{s=1}^{n} u_{is} u_{js}. \tag{62}$$

Assuming that $i = j$, we obtain the variances of observations

$$\sigma_i^2 = q(t_i, t_i) = q_{ii} \qquad (i = 1, 2, \ldots, N).$$

These values can be used to control the completness of data models over the time interval $\Delta T$. In the ideal case, individual values of $\sigma_i^2$ should not differ significantly from each other. Otherwise, the GIS model should be expanded by taking into account the time-depending errors of observations.

Meanwhile, the transposed matrix (61) can be considered as $N$ realizations of the random process $U(s) = (\bar{u}_{si})$. By subtracting the mean value

$$\bar{u}_s = \frac{1}{N}\sum_{i=1}^{N}(\bar{u}_{si})$$

from each row of matrix (61), we obtain the covariance function $q(s, r)$ of the centralized process $U(s)$ in the form of a $n \times n$ matrix $\mathbf{Q}_\sigma = (q_{sr})$ with the elements

$$q_{sr} = \frac{1}{N-1}\sum_{i=1}^{N}\bar{u}_{si}\bar{u}_{ri}.$$

When $s = r$, we obtain the variances of the series

$$\sigma_s^2 = q(s, s) = q_{ss}, \qquad (s = 1, 2, \ldots, n).$$

It is obvious that the mean variance of all the data is

$$\sigma_0^2 \cong \frac{1}{nN}\sum_{i=1}^{N}\sum_{s=1}^{n}u_{is}^2 \cong \frac{1}{N}\sum_{i=1}^{N}\sigma_i^2 \cong \frac{1}{n}\sum_{s=1}^{n}\sigma_s^2. \tag{63}$$

Practical experience (Gubanov *et al.*, 1992) shows that the covariance function (62) should be smoothed in order to choose a regular part of covariances. This procedure is carried out most effectively in terms of a certain system of physical variables $(\xi, \eta)$ that are connected with indices $(i, j)$ by some linear transformation $(i, j) \Rightarrow (\xi, \eta)$. The following can be taken as the new variables: the temperature of the air, the OSO positional angle or the zenit distance, etc. After two-dimensional smoothing, the covariance function $\tilde{q}(\xi, \eta)$ is converted to a smoothed matrix $\tilde{\mathbf{Q}}_\tau = (\tilde{q}_{ij})$ by means of the inverse transformation $(\xi, \eta) \Rightarrow (i, j)$.

If the original data are sorted according to the indices $i$ and $s$, the specified covariance matrix of the united residual vectors $\bar{\mathbf{u}} = (\bar{u}_{is})$ can be defined as the following block-matrix:

$$\mathbf{Q}_u = (\mathbf{Q}_{sr})_u = \begin{vmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \cdots & \mathbf{Q}_{1n} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \cdots & \mathbf{Q}_{2n} \\ & & & \\ \mathbf{Q}_{n1} & \mathbf{Q}_{n2} & \cdots & \mathbf{Q}_{nn} \end{vmatrix}, \tag{64}$$

where $\mathbf{Q}_{sr}$ is the $N \times N$ matrix of covariances between the series determined as

$$\mathbf{Q}_{sr} = q_{sr}\tilde{\mathbf{K}}_\tau. \tag{65}$$

Here $\tilde{\mathbf{K}}_\tau = (\tilde{k}_{ij})$ is the correlation matrix, obtained by transforming the smoothed matrix $\tilde{\mathbf{Q}}_\tau$:

$$\tilde{k}_{ij} = \tilde{q}_{ij}/\sqrt{\tilde{q}_{ii}\tilde{q}_{jj}}.$$

It should be noted that the variance-covariance matrix of the errors $\bar{u}_k = (\bar{u}_{ik})$ obtained by averaging of all the series of a given link (arc, base) over $k$ can be obtained from (64) using the rule of covariance summation,

$$\mathbf{Q}_u(t_i, t_j) = \frac{1}{n^2}\sum_{s=1}^{n}\sum_{r=1}^{n}(\mathbf{Q}_{sr})_u. \tag{66}$$

We can judge from (65) that if the mutual covariations of the series vanish, i.e., $q_{sr} = 0$ if $s \neq r$, the covariance matrix of observations (64) turns into a block-diagonal matrix,

$$\mathbf{Q}_u = \mathrm{diag}(\mathbf{Q}_{ss})_u, \qquad (s = 1, 2, \ldots, n).$$

In such a case the adjustment process can be carried out using a simple version described in Sect. 3.2. However, if also $\tilde{k}_{ij} = 0$ when $i \neq j$, the matrices $\mathbf{Q}_{ss}$ become strictly diagonal,

$$\mathbf{Q}_{ss} = \mathrm{diag}(\sigma_{is}^2)_u. \tag{67}$$

In the limiting case when, according to (63), $\sigma_0^2 = \sigma_i^2 = \sigma_s^2$, we have $\mathbf{Q}_u = \sigma_0^2 \mathbf{I}$ where $\mathbf{I}$ is the $nN \times nN$ unit matrix.

In this way, one of the problems connected with the adjustment of group observations is the reduction of the covariance matrix $\mathbf{Q}_u$ to the block-diagonal type. If the data model is complete enough, this can be achieved by iterative process specifying in each consecutive order the parameters of the data model and the covariances of the residuals. However, if the possibilities of expansion of the data model become too limited, it is necessary to make arrangements for the stabilization of the generalized instrumental system.

### 3.5. *Global Adjustment of Independent Catalogues*

Let us consider the problem of global adjustment of individual independent catalogues of stellar coordinates obtained from observations with several instruments located at different latitudes. Assuming that the declination zones of these catalogues cover the whole sky, it is necessary to obtain the general global catalogue of star coordinates using this data.

The individual catalogues $C_k$ ($k = 1, 2, \ldots, M$) represent a joint data set only if they have a common global initial catalogue $C$ (see Sect. 3.1). At present this is the FK5.

In the general case, when observations have been carried out with instruments of a non-orthogonal type (e.g. the astrolabe), the systematic errors of the initial catalogue $C$ influence the observations according to (7) and (8) as follows:

$$\mathbf{b} = \mathbf{E}\mathbf{b}_\alpha + \mathbf{F}\mathbf{b}_\delta = \mathbf{E}\mathbf{B}_\alpha \mathbf{y}_\alpha + \mathbf{F}\mathbf{B}_\delta \mathbf{y}_\delta = \mathbf{B}\mathbf{y}, \tag{68}$$

where $\mathbf{E}$ and $\mathbf{F}$ are the diagonal matrices of known coefficients determined by the type of instrument and the program of observations. Vector functions $\mathbf{b}_\alpha$ and $\mathbf{b}_\delta$ can be represented according to Schwan (1983) as follows:

$$\left.\begin{aligned} b_\alpha(\alpha, \delta) &= \sum_m \sum_n (c_{mn} \cos m\alpha + d_{mn} \sin m\alpha) P_n(\vartheta), \\ b_\delta(\alpha, \delta) &= \sum_m \sum_n (c'_{mn} \cos m\alpha + d'_{mn} \sin m\alpha) P_n(\vartheta). \end{aligned}\right\} \tag{69}$$

Here $P_n(\vartheta)$ are the Legendre polynomials with normalized arguments

$$\vartheta = 2(\delta - \delta_1)/(\delta_2 - \delta_1) - 1,$$

where $\delta \in [\delta_1, \delta_2]$; $\delta_1$ and $\delta_2$ are the declination zone boundary of the individual catalogue.

Thus, according to (68) and (69),

$$\left.\begin{aligned} \mathbf{B} &= [\mathbf{E}\mathbf{B}_\alpha \,\vdots\, \mathbf{F}\mathbf{B}_\delta], & \mathbf{y} &= (\mathbf{y}_\alpha, \mathbf{y}_\delta), \\ \mathbf{y}_\alpha &= (c_{mn}, d_{mn}), & \mathbf{y}_\delta &= (c'_{mn}, d'_{mn}). \end{aligned}\right\} \tag{70}$$

The rows of matrices $\mathbf{B}_\alpha$ and $\mathbf{B}_\delta$ are the basis functions of the expansions (69).

In particular case when orthogonal meridian instruments are used, one of the matrices $\mathbf{E}$ and $\mathbf{F}$ in (68) is a zero-matrix, and then $\mathbf{b} = \mathbf{b}_\alpha = \mathbf{B}_\alpha\mathbf{y}_\alpha$ (observations of right ascensions) or $\mathbf{b} = \mathbf{b}_\delta = \mathbf{B}_\delta\mathbf{y}_\delta$ (observations of declinations).

Let us first consider the global adjustment process for one of the equatorial coordinates of stars. It is assumed that the parameter vector $\mathbf{y}_k$ and the influence matrix $\mathbf{B}_k$ of the systematic errors of the initial catalogue $C$ are known from the internal adjustment of observations with the instrument of the number $k$, as described above (see Sect. 3.2–3.3). Representing systematic errors of this catalogue in the form of a global expansion over spherical functions,

$$\mathbf{b} = \mathbf{G}\mathbf{z} = \sum_\mu \sum_\nu (c_{\mu\nu} \cos \mu\alpha + d_{\mu\nu} \sin \mu\alpha) P^\mu_\nu(\sin \delta), \tag{71}$$

where $\mathbf{z} = (c_{\mu\nu}, d_{\mu\nu})$ is the parameter vector, $P^\mu_\nu$ is the associated Legendre polynomial, we obtain a system of parametric equations:

$$\mathbf{b}_k = \mathbf{B}_k\mathbf{y}_k = \mathbf{G}_k\mathbf{z},$$

where the left-hand side $\mathbf{b}_k(\alpha, \delta)$ can be calculated using one of equations (69) for each star of the initial catalogue $C$ in the declination zone $[\delta_1, \delta_2]_k$. Adding the residual vector $\mathbf{w}_k$ known from the internal adjustment of the catalogue $C_k$, to vector $\mathbf{b}_k$, we obtain the following parametric data model:

$$\mathbf{l}_k = \mathbf{G}_k\mathbf{z} + \mathbf{w}_k, \qquad (k = 1, 2, \dots, M), \tag{72}$$

where $\mathbf{l}_k = \mathbf{b}_k + \mathbf{w}_k$, $\mathbf{w}_k = \mathbf{u}_k + \mathbf{v}$, $\mathbf{u}_k$ is the quasi-random vector of the residual errors of observations, $\mathbf{v}$ defined by Eq. (7) is the global vector of individual corrections to the star coordinates, and $\mathbf{G}_k$ is the influence matrix defined in zone $[\delta_1, \delta_2]_k$ by Eq. (71).

The weight matrix of the joint data vector $\mathbf{l} = (\mathbf{l}_k)$ is

$$\mathbf{P}_w = \sigma_0^2\mathbf{Q}_w^{-1},$$

where

$$\mathbf{Q}_w = \mathbf{Q}_u + \mathbf{Q}_v.$$

As the residual quasi-random errors of observations $\mathbf{u} = (\mathbf{u}_k)$ are independent for different catalogues $C_k$, their variance-covariance matrix

$$\mathbf{Q}_u = \mathrm{diag}(\mathbf{Q}_{kk})_u, \qquad (k = 1, 2, \dots, M), \tag{73}$$

where the blocks $(\mathbf{Q}_{kk})_u$ are given by Eq. (66) for each individual catalogue $C_k$ and can be evaluated at the last step of its internal adjustment.

As regards the covariances of individual coordinate corrections, the form of matrix $\mathbf{Q}_v$ depends on the overlapping of the declination zones $[\delta_1, \delta_2]_k$ of the individual catalogues $C_k$. If these zones do not overlap, matrix $\mathbf{Q}_v$ is of the block-diagonal type,

$$\mathbf{Q}_v = \mathrm{diag}(\mathbf{Q}_{kk})_v,$$

and its blocks are strictly diagonal,

$$(\mathbf{Q}_{kk})_v = \text{diag}(\sigma_i^2)_k, \qquad (i = 1, 2, \ldots, N_k), \tag{74}$$

where $(\sigma_i)_k$ are known RMS values of the random errors of stellar coordinates for the initial catalogue $C_k$. In this case the solution of the system (72) can be obtained by the simplified algorithm described in Sect. 3.2. When the catalogue declination zones overlap, the matrix $\mathbf{Q}_v$ is not of a block-diagonal type and the general algorithm described Sect. 2.1 should be used for solving the system (72).

In the case of observations with non-orthogonal instruments, vectors $\mathbf{y}_\alpha$ and $\mathbf{y}_\delta$, determined by internal adjustment of the catalogues $C_k$, may be correlated, therefore the corrections to both coordinates must be adjusted simultaneously. Then it is necessary to take, by analogy with (70),

$$\mathbf{l} = (\mathbf{l}_\alpha, \mathbf{l}_\delta), \qquad \mathbf{w} = (\mathbf{w}_\alpha, \mathbf{w}_\delta), \qquad \mathbf{v} = (\mathbf{v}_\alpha, \mathbf{v}_\delta), \qquad \mathbf{b} = (\mathbf{b}_\alpha, \mathbf{b}_\delta),$$
$$\mathbf{G} = [\mathbf{G}_\alpha \,\vdots\, \mathbf{G}_\delta], \qquad \mathbf{z} = (\mathbf{z}_\alpha, \mathbf{z}_\delta), \qquad \mathbf{z}_\alpha = (c_{\mu\nu}, d_{\mu\nu}), \qquad \mathbf{z}_\delta = (c'_{\mu\nu}, d'_{\mu\nu}),$$

where elements of matrices $\mathbf{G}_\alpha$ and $\mathbf{G}_\delta$ are defined in terms of the same spherical functions (71).

Sorting the data vector $\mathbf{l}$ in the order $\mathbf{l} = (\mathbf{l}_\alpha, \mathbf{l}_\delta)$ and taking into account that the residuals $\mathbf{w}_\alpha$ and $\mathbf{w}_\delta$ are mutually independent, we obtain after simple calculations

$$\mathbf{Q}_w = \mathbf{Q}_u + \mathbf{Q}_y + \mathbf{Q}_v,$$

where

$$\mathbf{Q}_u = \begin{vmatrix} \mathbf{Q}_{\alpha\alpha} & \mathbf{Q}_{\alpha\delta} \\ \mathbf{Q}_{\delta\alpha} & \mathbf{Q}_{\delta\delta} \end{vmatrix}_u, \qquad \mathbf{Q}_y = \begin{vmatrix} \mathbf{Q}_{\alpha\alpha} & \mathbf{Q}_{\alpha\delta} \\ \mathbf{Q}_{\delta\alpha} & \mathbf{Q}_{\delta\delta} \end{vmatrix}_y, \qquad \mathbf{Q}_v = \begin{vmatrix} \mathbf{Q}_{\alpha\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{\delta\delta} \end{vmatrix}_v.$$

Matrix $\mathbf{Q}_u$ is determined by Eq. (66) in the internal adjustment process of the joint catalogue of right ascensions and declinations. If the data model used is sufficiently complete, matrix $\mathbf{Q}_u$ may be obtained in a strictly diagonal form (67) after several iterations of such processes.

The blocks of matrix $\mathbf{Q}_y$ are

$$(\mathbf{Q}_{\alpha\alpha})_y = \mathbf{B}_\alpha(\mathbf{Q}_{y_\alpha y_\alpha})\mathbf{B}_\alpha^T, \qquad (\mathbf{Q}_{\alpha\delta})_y = \mathbf{B}_\alpha(\mathbf{Q}_{y_\alpha y_\delta})\mathbf{B}_\delta^T, \qquad (\mathbf{Q}_{\delta\delta})_y = \mathbf{B}_\delta(\mathbf{Q}_{y_\delta y_\delta})\mathbf{B}_\delta^T,$$

and are determined by the internal and mutual covariations of vectors $\mathbf{y}_\alpha$ and $\mathbf{y}_\delta$ obtained from the internal adjustment of the catalogue $C_k$. As the basis in expansion (69) is orthogonal, diagonal blocks $(\mathbf{Q}_{\alpha\alpha})_y$ and $(\mathbf{Q}_{\delta\delta})_y$ are practically strictly diagonal and determined by Eq. (74). Only in this case the variances $\sigma_i^2$ are the a priori accuracy estimates of the independent catalogue $C_k$ in a systematic respect.

Information about mutual correlations between random errors of the FK5 right ascension and declination values is absent. Therefore, non-diagonal blocks of matrix $\mathbf{Q}_v$ vanish and diagonal blocks $(\mathbf{Q}_{\alpha\alpha})_v$ and $(\mathbf{Q}_{\delta\delta})_v$ are strictly diagonal and determined by Eq. (74).

The parametric adjustment of VLBI observations made with a separate radio interferometer or a multi-base complex yields the vector $\mathbf{v}_k = (\mathbf{v}_\alpha, \mathbf{v}_\delta)_k$ of corrections to initial coordinates of extragalactic radio sources and the complete variance-covariance matrix,

$$(\mathbf{Q}_{kk})_v = \begin{vmatrix} \mathbf{Q}_{\alpha\alpha} & \mathbf{Q}_{\alpha\delta} \\ \mathbf{Q}_{\delta\alpha} & \mathbf{Q}_{\delta\delta} \end{vmatrix}_v.$$

Thus, the joint adjustment process for these data converges to the solution of the parametric system $\mathbf{v}_k = \mathbf{v}$ $(k = 1, 2, \ldots, M)$ with a block-diagonal weight matrix,

$$(\mathbf{P}_k)_v = \sigma_0^2 \operatorname{diag}(\mathbf{Q}_{kk})_v^{-1}.$$

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## APPENDIX

### THE PROGRAM OF PARAMETRIC ADJUSTMENT BY GLS TECHNIQUES

```
      SUBROUTINE PAM (A, B, C, D, E, F, P, R, X, SX, S, N, M, K, IB, IC,
      IP, IR, M1, M2, DET)

C-----------------------------------------------------------------------

C       PAM—PARAMETRIC ADJUSTMENT METHOD
C          A—INPUT DESIGN MATRIX OF DIMENSION NA = N * M
C             GIVEN BY COLUMNS (KEEP)
C          B—INPUT DATA VECTOR OF LENGTH N (KEEP); IF 'IB' = 0
C             IT IS NOT USED
C          C—WORK AREA OF DIMENSION NC = (M + K) * (M + K):
C             OUTPUT MATRIX OF THE ESTIMATED PARAMETERS
C             MUTUAL CORRELATIONS OF DIMENSION MC = M * M IF
C             'IC' = 0 OR THEIR VARIANCE-COVARIANCE MATRIX OF
C             THE SAME DIMENSION IF 'IC' ≠ 0
C          D—WORK AREA OF LENGTH N; OUTPUT RESIDUALS
C             VECTOR IF 'IB' ≠ 0
C          E—INPUT MATRIX OF THE LINEAR RESTRICTIONS AND
C             THEIR INDEPENDENT TERMS OF DIMENSION NE = K *
C             (M + 1) GIVEN BY ROWS (KEEP); IF 'K' = 0 IT IS NOT
C             USED
C          F—WORK AREA OF LENGTH M + K
C          P—INPUT WEIGHT MATRIX OF DATA (KEEP)
C          R—INPUT MATRIX OF REGULARIZATION (KEEP)
C          X—OUTPUT PARAMETER VECTOR OF LENGTH M; IF
C             'IB' = 0 IT IS NOT USED
C         SX—OUTPUT R.M.S. VECTOR OF ESTIMATED PARAMETERS
C             OF LENGTH M
```

```
C           S—OUTPUT R.M.S. OF UNIT WEIGHT; IF 'IB' = 0 IT IS INPUT
C              PARAMETER
C           N—INPUT NUMBER OF ROWS IN MATRIX A (N > 1)
C           M—INPUT NUMBER OF COLUMNS IN MATRIX A (M < N)
C           K—INPUT NUMBER OF LINEAR RESTRICTIONS (K =
C              0, 1, . . .)
C          IB—INPUT SYMBOL OF DATA VECTOR MODE PRESENCE:
C              IF 'IB' = 1 THE DATA ARE GIVEN,
C              IF 'IB' = 0 THE DATA ARE NOT USED: 'S' BECOME AN
C              INPUT PARAMETER,
C              R.M.S. VECTOR 'SX' AND MATRIX 'C' ARE CALCULED,
C              BUT AREAS 'B' AND 'X' ARE NOT USED
C          IC - INPUT SYMBOL OF THE OUTPUT MATRIX 'C' MODE
C              PRESENCE:
C              IF 'IC' = 0, C IS THE CORRELATION MATRIX,
C              IF 'IC' ≠ 0, C IS THE VARIANCE-COVARIANCE MATRIX
C          IP—INPUT SYMBOL OF THE INPUT MATRIX P MODE
C              PRESENCE:
C              IF 'IP' = 1, P IS SYMMETRIC MATRIX OF DIMENSION
C              NP = N * (N + 1)/2,
C              IF 'IP' = 2, P IS DIAGONAL MATRIX OF LENGTH NP = N,
C              IF 'IP' = 0, P IS NOT USED
C          IR—INPUT SYMBOL OF THE INPUT MATRIX R MODE
C              PRESENCE:
C              IF 'IR' = 1, R IS SYMMETRIC MATRIX OF DIMENSION
C              NR = M * (M + 1)/2,
C              IF 'IR' = 2, R IS DIAGONAL MATRIX OF LENGTH NR = M,
C              IF 'IR' = 0, R IS NOT USED
C      M1, M2—WORK AREAS OF LENGTH NM = M + K (INTEGER)
C         DET—OUTPUT DETERMINANT OF THE NORMAL SYSTEM
C
C
C              NOTES:
C              1. IN CASE OF IB, K, IP, IR = 0 UNUSED AREAS B, X, E,
C                 P, R ARE DESCRIBED IN MAIN PROGRAM AS B(1),
C                 X(1), E(1), P(1), R(1).
               2. OTHER ROUTINES REQUIRED: MINV, LOC.
C------------------------------------------------------------------

$LARGE: a
      real a(1), b(1), c(1), d(1), e(1), f(1), p(1), r(1), x(1), sx(1)
      mk = m + k


C     CONSTRUCTION OF THE NORMAL EQUATIONS
      do1 i = 1, m
      f(i) = 0.
      do2 j = 1, n
      d(j) = a(n * (i - 1) + j)
```

```
      if (ip.eq.2)d(j) = d(j) * p(j)
      if(ip.ne.1)goto 2
      d(j) = 0.
      do3 ij = 1,n
      CALL LOC(ij,j,ljp,n,n,1)
    3 d(j) = d(j) + a(n * (i - 1) + ij) * p(ljp)
    2 continue
      do4 ij = i,m
      il = (ij - 1) * mk + i
      li = (i - 1) * mk + ij
      c(il) = 0.
      do5 j = 1,n
      if(ij.eq.i)f(i) = f(i) + d(j) * b(j)
    5 c(il) = c(il) + d(j) * a(n * (ij - 1) + j)
    4 c(li) = c(il)
    1 continue

C     BORDERING OF THE NORMAL SYSTEM
C     WITH LINEAR RESTRICTIONS
      if (k.eq.0)goto 8
      do6 j = 1,k
      f(m + j) = e((m + 1) * j)
      do7 i = 1,m
      ji = mk * (i - 1) + j + m
      ij = mk * (m + j - 1) + i
      c(ji) = e(i + (m + 1) * (j - 1))
    7 c(ij) = c(ji)
      do6 i = 1,k
    6 c(mk * (m + j - 1) + m + i) = 0.

C     REGULARIZATION OF THE NORMAL SYSTEM
    8 if(ir.eq.0)goto 10
      do9 i = 1,m
      do9 j = i,m * (2 - ir) + i * (ir - 1)
      CALL LOC(i,j,ijr,m,m,ir)
      ij = mk * (j - 1) + i
      ji = mk * (i - 1) + j
      c(ij) = c(ij) + r(ijr)
      if(ij.ne.ji)c(ji) = c(ji) + r(ijr)
    9 continue

C     INVERSION OF THE MATRIX BY GAUSS-JORDAN WAY
   10 CALL MINV(c,mk,det,m1,m2)
      if(ib.eq.0)goto 17

C     ESTIMATION OF THE PARAMETER VECTOR
      do11 i = 1,m
      x(i) = 0.
```

```
      do11 j = 1,mk
   11 x(i) = x(i) + c(mk * (j - 1) + i) * f(j)

C     COMPUTATION OF THE RESIDUAL VECTOR
      do12 i = 1,n
      d(i) = b(i)
      do12 j = 1,m
   12 d(i) = d(i) - a(n * (j - 1) + i) * x(j)

C     COMPUTATION OF THE UNIT WEIGHT R.M.S
      s = 0.
      do13 i = 1,n
      if(ip.eq.0)sum = d(i) * d(i)
      if(ip.eq.2)sum = p(i) * d(i) * d(i)
      if(ip.ne.1)goto 13
      sum = 0.
      do14 j = 1,n
      CALL LOC(i,j,ij,n,n,1)
   14 sum = sum + p(ij) * d(j) * d(i)
   13 s = s + sum
      zn = FLOAT(n - m)
      if(ir.eq.0)goto 15
      do16 i = 1,m
      CALL LOC(i,i,ii,m,m,ir)
      if(r(ii).eq.0.)goto 16
      zn = zn + 1.
   16 continue
   15 s = SQRT(s/zn)

C     COMPUTATION OF THE ESTIMATED PARAMETERS R.M.S
   17 do18 i = 1,m
      f(i) = c(mk * (i - 1) + i)
   18 sx(i) = s * SQRT(f(i))

C     COMPUTATION OF THE CORRELATION OR COVARIANCE
C     MATRIX
      if(ic.ne.0)goto 20
      do19 i = 1,m
      do19 j = 1,m
   19 c(m * (j - 1) + i) = c(mk * (j - 1) + i)/SQRT(f(i) * f(j))
   20 return
      end
C---------------------------------------------------------------
```

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