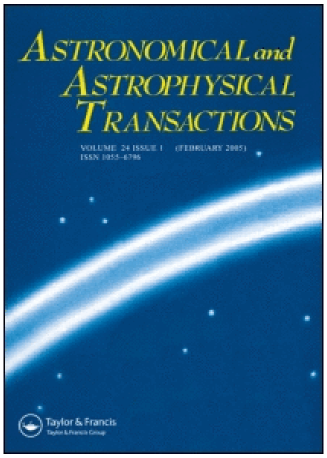

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ON SAMPLING WITH ACCOUNT OF NOISE

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(12 March, 1992)

The optimum pixel size ε of a detector under given observational conditions depends on *a priori* information about the object in focus, the *Point Spread Function* (PSF), the signal-to-noise ratio, etc. The Rayleigh criterion, according to which the theoretically attainable resolution is of the order of the PSF width Δ, leads to the generally accepted value of $\varepsilon \simeq \Delta/2$. It is shown that this conventional approach does not take into account important information concerning the image smoothing. Image restoration technique allows to reach much higher resolving power under typical conditions of observations, so that much smaller pixel size should be chosen to provide the theoretically attainable resolving power. Unmatched telescope and detector have significant losses in the resolving power and limiting magnitude. The formulae are given for approximate calculation of the corresponding parameters. As an example, the problem of the resolving power of the *Hubble Space Telescope* is discussed.

KEY WORDS Data processing, image restoration

1. INTRODUCTION

The problem under consideration can be formulated in qualitative terms as follows: what is the optimum pixel size ε (in projection to the sky if astronomical context is considered) to be used to record an image carrying both the noise due to quantum nature of light and an additive noise due to the sky background and the dark current of the detector? It is clear that an adequate choice of ε can be reached by a compromise between the necessity to ensure high spatial resolution and, at the same time, to avoid an excessively detailed representation of the data, not corresponding to available information. Of course, the final decision depends also on the field of view of the optical system employed, its permissible dimensions and other factors, but, unlike the relation between the pixel size and resolving power, all these restrictions can be easily understood, so they are not considered here.

In nontrivial case, an imaging system smooths fine details of the original image, thus the intrinsic width Δ of a point source image constructed by a given system is of primary importance. According to the accepted terminology, we interpret the point source image, normalized by unit flux, as the *Point Spread Function* (PSF).

Thus, we are dealing with three typical dimensions (Figure 1): the width of the PSF Δ, the detector pixel size ε, and the resolving power ρ_ε, which, for any correctly designed system (telescope, camera, spectrograph, etc.), should be mutually consistent and adjusted to the object under consideration and noise

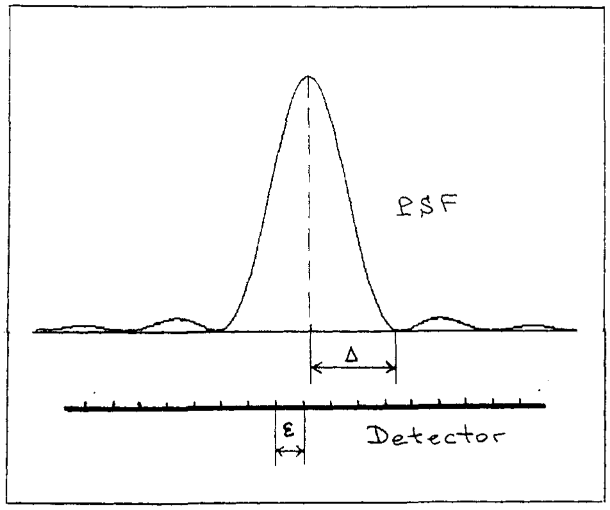

Figure 1 Schematic representation of the Point Spread Function width Δ and detector pixel size ε.

properties. The consistency problem is not of a theoretical interest only: we shall see below that intrinsic losses due to incorrect adjustment reach as much as several stellar magnitudes for the limiting brightness at fixed resolving power, or a few orders of magnitude in the resolving power at fixed flux.

These problems can be illustrated using an analogy with time series analysis (Terebizh, 1992a). Let us assume that the series counts correspond to observations with certain exposure time. Assume further that there is an advanced method to search for a periodicity of the investigated process hidden in fluctuations. Evidently, if the period is P, we must obtain the data with sufficiently short time exposures, at least not exceeding $P/2$. In the opposite case, the shortest detectable period is determined not by the physical nature of the process or the efficiency of the applied data analysis technique, but simply by the roughness of the monitoring system.

Therefore, *a reasonable sampling rate of a signal in time or space is defined by the highest corresponding temporal or spatial signal frequency which can be revealed in a noisy data.* Unlike the case of stochastic processes, searching for this cutoff frequency for deterministic functions is quite a simple problem. Just for this reason the optimum sampling in the presence of noise is to a great extent different from that following from the "standard" rule based on the well-known *sampling theorem by* Whittaker–Kotelnikov–Shannon (see, for example, Jain, 1989).

The aim of this paper is to discuss the relevant problems and to find useful approximate expressions for the optimum pixel size ε_{opt} for a wide class of observational conditions. In particular, a simple relation $\Delta/\varepsilon_{\text{opt}} \simeq (\text{Signal/Noise})^{1/2}$ follows from recent results of image restoration theory. The *Hubble Space Telescope* (HST) resolution power is discussed as an example of application of derived expressions.

2. GENERAL CONSIDERATION

2.1. *Image Space*

Since our final results are insensitive to the dimension of an image, we restrict ourselves to one-dimensional case. Let us consider, for the sake of simplicity, the resolving power in Rayleigh sense, i.e., as a problem of revealing the structure of a blurred double source with the pointlike components having about the same brightness. The detection of photo-events is a random process inevitably, so both the smoothed image and additive noise are assumed to have stochastic nature. Obviously, the smallest attainable separation ρ_ε of the components depends on the PSF form, pixel size, additive noise characteristics and the chosen significance level q of the decision. At a first approximation we may consider the resolving power $\rho_\varepsilon(\Delta, \psi, q)$ as a function of the PSF width Δ, signal-to-noise ratio ψ, and q. Our purpose is to estimate the number of pixels that cover the PSF width, that is

$$\Gamma \equiv \frac{\Delta}{\varepsilon}, \tag{1}$$

for, in some definite sense, optimum sampling.

It is noteworthy, that the "resolving power," whatever this term means, strongly depends on *a priori* information, defining, as a result, a pixel size. The role of *a priori* information has been underestimated by many authors, which leads to a number of paradoxes. Toraldo di Francia (1955) phrased that in a perfect way: "The observer is always more or less relying on his past experience of what a real object can look like. Moreover, in the great majority of particular cases, he has at his disposal a much larger amount of *a priori* information about the object than he even realizes. This information, if properly utilized, enables him to rule out some of the different objects which could correspond to the image. He thus may have the illusion that he can extract from the image more information than there is actually contained."

Thus, it is extremely important to identify *a priori* information and to include it into an image restoration procedure. Consider, as the simplest example, determination of the *position* of a single blurred object. The object's shape and brightness S, as well as the total background flux B are assumed to be known. For the Poisson statistics of events (Mehta, 1970; Loudon, 1973) we can define the signal-to-noise ratio as

$$\psi = \frac{S}{\sqrt{S+B}}. \tag{2}$$

It can be easily shown (see Appendix) that for such a large amount of *a priori* information even two rough pixels are enough to determine the position of the object with high accuracy:

$$S\,\mathrm{dev}\,(\hat{\rho}/\Delta) \simeq \frac{1}{2\psi}, \tag{3}$$

where S dev denotes the root-mean-square deviation of the *estimate* $\hat{\rho}/\Delta$ of the relative shift from the true position of the object. It follows from (3) that the position of the bright object can be determined with an infinite accuracy independently of its width and number of pixels, and this is, of course, a consequence of the almost complete *a priori* information.

Consider a more complicated situation, when two different objects, a single pointlike source and a double one with pointlike components at the angular separation ρ, are the only possibilities. We have to decide, on the basis of the randomly smoothed and noised image, about the type of the initial object. Just this case was considered to formulate the known Rayleigh (1964, p. 420) rule: the smallest detectable separation ρ_t of the components is approximately equal to the PSF width Δ (we avoid some "pathological" shapes of the PSF with a few components like that for the *Hubble Space Telescope*). On the other hand, in order to distinguish two pointlike components we must have at least one empty pixel between their positions. Therefore, the pixel size ε should satisfy the following inequality:

$$\varepsilon < \rho_t/2. \tag{4}$$

Substituting here $\rho_t^{(R)} \simeq \Delta$, we obtain:

$$\varepsilon_R \simeq \Delta/2, \qquad \Gamma_R \simeq 2. \tag{5}$$

Therefore, according to the conventional approach, only two pixels are sufficient to cover the PSF width in order to transfer all the information that the image contains. Sometimes, when particularly accurate measurements are to be obtained, the Γ-factor is several times as high.

We stress again that (5) follows from the estimate $\rho_t \simeq \Delta$ for the theoretical limit of resolution. Meanwhile, it has been known since the 1940-ies that the attainable values of ρ_t are considerably smaller than the PSF width. The first investigations of this *superresolution phenomenon* have been made by Schelkunoff (1943), Bouwkamp and De Bruijn (1946), Toraldo di Francia (1952, 1953, 1955), Wolter (1961), J. Harris (1964a, b), Frieden (1967), Rushforth and R. Harris (1968); recently, the *superresolution natural limit* has been established and its strict form has been given (Terebizh, 1991, 1992b; Terebizh and Biryukov, 1991).

Let us explain the nature of the superresolution phenomenon using an abstract example, when neither additive (which is a rough approximation to reality) nor quantum (which is definitely true) components of noise are present. Figure 2 shows two *absolutely smooth* diffraction images, shifted one versus another by some distance ρ. Evidently, even if the shift ρ is extremely small relative to Δ, it is always possible to find out whether the overall picture corresponds to a binary or to a single source. Moreover, we could know nothing about the shape of the compared objects, but it turns out that if the noise were absolutely absent, the only necessary condition for exact image restoration is the finite size of the object.

In real conditions, when both types of noise are present, the resolving power is finite, though the values of ρ_t significantly lower than the PSF width are still attainable. It seems reasonable to expect that the limiting resolving power is determined not only by the PSF width but also by the signal-to-noise ratio and the required significance level of image classification. Let α be the probability to classify erroneously a single source as a double one, and β the probability to do

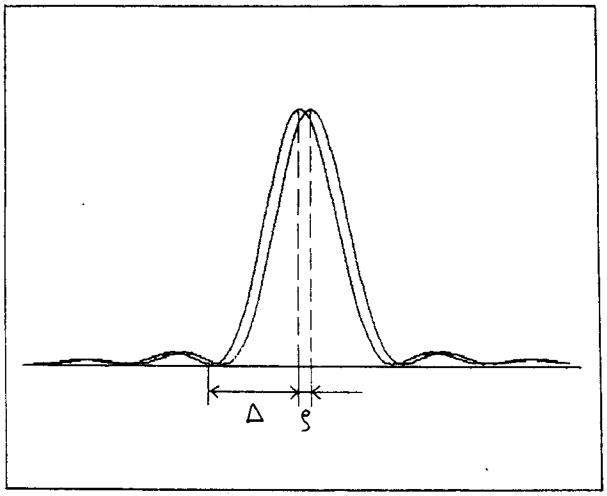

Figure 2 Detection of the shift $\rho \ll \Delta$ for absolutely smooth images.

an error of the opposite sense (Cramer, 1946; Kendall and Stuart, 1969). As it has been shown recently (Terebizh, 1990c, 1992b), the closest binary source which can be distinguished from a single source at a given significance level has the following separation ρ_t of the components:

$$\rho_t/\Delta \simeq \text{const} \cdot \left(\frac{z_\alpha + z_\beta}{\psi}\right)^{1/2}, \qquad \psi \gg 1, \tag{6}$$

where the signal-to-noise ratio is

$$\psi = \frac{S}{\sqrt{S + 2\Delta b}}, \tag{7}$$

b (events/pixel) is the density of background events, z_θ is a root of equation $\Phi(z_\theta) = 1 - \theta$, function $\Phi(z)$ is the Gaussian probability function (A.12), and the constant of the order of unity depends on the particular form of the PSF. For α and β within the range $0.02 - 0.10$ (see Table 1), we may approximately write

$$\rho_t/\Delta \simeq \frac{2}{\sqrt{\psi}}, \qquad \psi \gg 1. \tag{8}$$

The $[\rho_t - \psi]$ relation is shown schematically in Figure 3. Again $\rho_t \to 0$ when $\psi \to \infty$, but now the resolution is inversely proportional to the square root of the signal-to-noise ratio. Of course, to calculate accurately the theoretical resolution ρ_t, an exact form of the corresponding expression should be used (Terebizh, 1992b), but our purpose here is only to outline the arguments.

The next step is to introduce the finite pixel size. It can be suggested that for any given object, noise, PSF and pixel size the limiting resolution is

$$\rho_\varepsilon \simeq \max(\rho_t, 2\varepsilon). \tag{9}$$

Table 1 The z_θ-function

θ	z_θ	θ	z_θ
0.01	2.326	0.30	0.524
0.05	1.645	0.35	0.385
0.10	1.282	0.40	0.253
0.15	1.036	0.45	0.126
0.20	0.842	0.50	0.000
0.25	0.674		

Indeed, let us imagine first that we have very fine pixel structure with $\varepsilon \ll \rho_t$. In this case the resolution does not depend on ε and is equal to the theoretically attainable value ρ_t. In the opposite case, when $\varepsilon \gg \rho_t$, the resolution is determined simply by the pixel size. Therefore, if we are interested only in obtaining the highest resolving power for given object and imaging system, we should choose the optimum pixel size according to the following relation:

$$\varepsilon_{\text{opt}} \simeq \rho_t/2. \tag{10}$$

From Eqs (8) and (10) we obtain:

$$\varepsilon_{\text{opt}} \simeq \frac{\Delta}{\sqrt{\psi}}, \qquad \Gamma_{\text{opt}} \simeq \sqrt{\psi}. \tag{11}$$

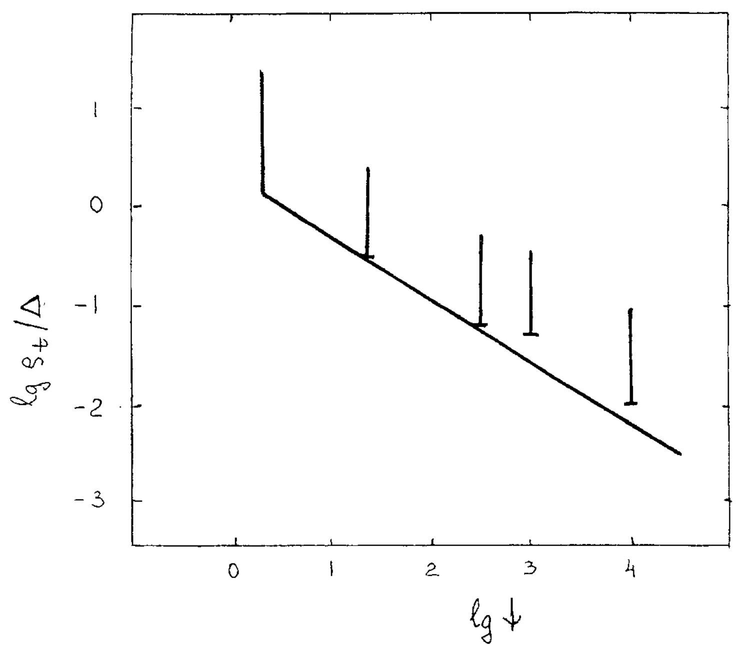

Figure 3 Theoretically minimum resolution ρ_t as a function of signal-to-noise ratio for the choice of objects (solid line) and results of numerical simulations for non-negativity as the only *a priori* information (vertical bars).

Thus, for the approximate approach under consideration, the optimum Γ-factor depends only on the signal-to-noise ratio. The latter is given by Eq. (7) for one-dimensional images; in a general case, it should be written as

$$\psi = \frac{S}{\sqrt{S + S_0}}, \tag{12}$$

where S is the mean value of integrated number of counts during exposure time caused by the investigated source, and S_0 is the similar value in the absence of the object.

Strictly speaking, Eq. (11) is valid only when the probabilities of the first and second type errors α and β are equal to 0.05 and 0.10, respectively. For arbitrary significance level, it follows from (6) and (10):

$$\varepsilon_{\text{opt}} \simeq \Delta \cdot \left(\frac{z_\alpha + z_\beta}{\psi}\right)^{1/2}, \qquad \Gamma_{\text{opt}} \simeq \left(\frac{\psi}{z_\alpha + z_\beta}\right)^{1/2}. \tag{13}$$

The values of the z_θ-function are given in Table 1. Note again that exact equations with allowance for the PSF form are to be used for accurate calculations.

According to Eq. (11), rather large number of pixels, of the order of $\Gamma \simeq 10$–100, should cover the PSF width for the conditions typical of astronomical observations if we want to reach the highest theoretical resolving power. More extensive discussion of this case is given in Section 3.

We have discussed above two imaginary situations when *a priori* information was quite extensive. Such cases are not rare in astronomical practice, but investigations in situations, when the only known information about the object is its non-negativity (since any image is an energy distribution) are much more frequent. As an example, we can consider the problem of star/galaxy separation in faint object surveys, on the one hand, and investigations of the structure of active galactic nuclei, on the other hand. How the resolving power would degrade in the general case, when the non-negativity is the only additional available information about the object? Using numerical simulations, Terebizh (1992b) has shown that, when the most efficient (theoretically) method for image restoration is used, the resolving power does not affect strongly the threshold given above (it was suggested that the sampling does not introduce restrictions). The resolution for a binary pointlike source the absence of any *a priori* information, except the non-negativity, is shown by vertical lines in Figure 3.

Thus, in the case of scarce *a priori* information one can assume that the Γ-factor is smaller than implied by Eq. (11), although the difference is not very large. Taking into account that advanced, expensive devices are usually employed to solve such problems through obtaining extensive *a priori* information, it is worth to consider the sampling rate basing on the relations discussed above.

2.2. *Frequency Space*

An equivalent and generally complementary approach to sampling is associated with the analysis of the frequency space (for the sake of simplicity, we shall consider angular frequencies with dimension 1/radian).

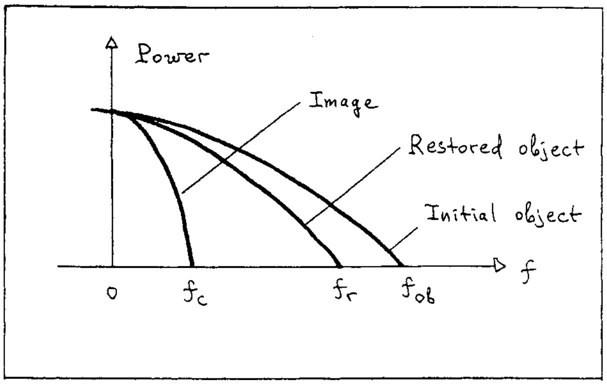

Figure 4 Schematic representation of power spectra.

Let f_{ob} be the largest frequency in the spectrum of the object (so that f_{ob}^{-1} is the angular size of the smallest detail). Schematically, the power spectrum is shown in Figure 4. Smoothing by a linear imaging system is described by the multiplication of the spectrum by the *Modulation Transfer Function* (MTF). It is known, (see, e.g., Born and Wolf, 1964) that the MTF of any optical system is strictly equal to zero for frequencies exceeding some critical value f_c (for a telescope with the aperture diameter D, in an ideal case we have $f_c \simeq D/\lambda$). Thus, the observed image of the object contains only frequencies below f_c. A similar result is valid for non-linear systems as well.

Further, according to a widely known theorem by Whittaker–Kotelnikov–Shannon (see Shannon, 1948), to obtain an exact representation of a continuous function whose spectrum vanishes at frequencies above some f_{max}, it is sufficient to have sampling at the step

$$\varepsilon = \frac{1}{2f_{max}}. \tag{14}$$

This means simply that the smallest detail should be covered by at least two counts. How to define f_{max}? If we accepted that $f_{max} = f_{ob}$, the step would be very small, being adequate for the original object but not its blurred image. Following a conventional approach, we should adopt $f_{max} = f_c \simeq \Delta^{-1}$, which yields the Rayleigh solution (5).

The conclusion that the observed image spectrum does not contain frequencies exceeding the cutoff frequency f_c is correct in the most strict sense, but in fact *we have at our disposal not only the image spectrum but also the information about the way how the degradation of the spectrum was obtained,* i.e., we know the PSF and its Fourier transform (MTF) (of course, some other *a priori* information may be also available). If noise were absent completely, this information would be sufficient to restore the whole spectrum up to f_{ob}. However noise is inevitable (at least the *photon noise*), so the spectrum restoration is feasible up to certain frequency f_r ($<f_{ob}$) determined by the available information, signal-to-noise ratio

and other parameters (see Figure 4). In fact $f_r \simeq \rho_t^{-1}$ and $f_c \simeq \Delta^{-1}$, so that

$$f_r \simeq \frac{\Delta}{\rho_t} \cdot f_c. \tag{15}$$

Since usually $\rho_t \ll \Delta$ (Figure 3), we obtain $f_r \gg f_c$, i.e., the details considerably smaller than the PSF width can be restored. The corresponding value of the optimum sampling step can be derived from (14) for $f_{max} \simeq f_r \simeq \rho_t^{-1}$ to be coinciding with the above expressions.

2.3. *Some Additional Remarks*

In order not to complicate the problem but at the same time to investigate the role of individual parameters, we neglected some effects of secondary importance. Particularly, the upper limit for f_{ob} was not considered. Meanwhile, a possibility to study the superresolution phenomenon depends also on the shape of the object as well. In addition, advanced methods of flux measurement by CCD detectors assume flux integration within pixels which are separated by significantly smaller space than their proper dimension. This averaging effect represents a problem different from the sampling procedure of the functions at widely separated moments of time. Moreover, in a consistent theory both mentioned effects are to be considered simultaneously.

A more general analysis of the sampling problem, including the factors mentioned above, will be published elsewhere. We only outline here the main features of the corresponding effects.

Consider object's power spectrum before its degradation by an imaging system. If the object had deterministic nature and an additive noise were absolutely absent, the typical power spectrum would be decreasing gradually from low spatial frequencies up to very high ones. Introducing an additive white noise not only adds a constant power density, but inevitably introduces large fluctuations of the power density at close frequencies. It was shown almost a century ago by Arthur Schuster (1898) that the probability density distribution of the sampling white noise power spectrum is an *exponential* function (that decreases slower than the Gaussian distribution), and the neighbouring values of the sampling spectrum at the separation of the order of the reciprocal of the image width are uncorrelated. This implies that very large fluctuations of the power density are expected to be typical, and this phenomenon is familiar to anybody who deals with Fourier representation of observational data.

Another source of the power spectrum fluctuations is the quantum nature of light. This inevitable *photon noise* has the mean level equal to the total flux of the object, and the same exponential distribution as the additive noise. A combined action of the two types of noise "hides" those parts of the original spectrum which have low power, usually those in the high-frequency domain. Thus emerges the concept of *noise-limited spectra*.

An example of this phenomenon is shown in Figure 5, where the power spectra of a randomized Gaussian object and the Poisson background are given. The integral flux, standard deviation and the total width of the object have been chosen to be $S = 10^3$ counts, $\sigma = 20$ pixels and $N = 256$ pixels, respectively; the background level is $b = 5$ counts per pixel. The frequency is $j = f/\Delta f$, where the

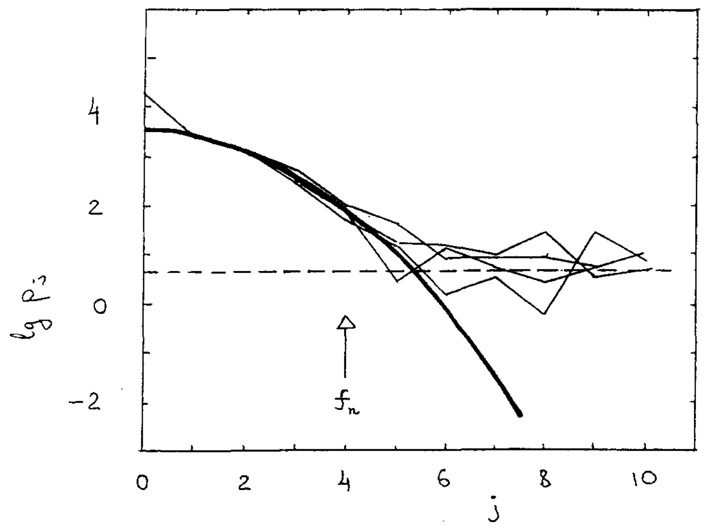

Figure 5 The power spectra for deterministic Gaussian object (thick line), background (dotted line), and four random simulations of the noisy Gaussian object.

frequency step is $\Delta f = (N \cdot \Delta x)^{-1}$. We see that at some critical frequency $f_n^{(0)}$, dependent on the chosen significance level, fluctuations are so high that it is impossible to decide whether or not a high-frequency domain is present in the object's spectrum.

This assertion allows to consider quantitatively the problem of the noise limit in power spectra. Indeed, we can compare two objects: the real one and the same object but with a high-frequency power spectrum cut-off for $f > f_*$ (that is, smoothed by sinc-filter). If these objects are statistically different at some adopted significance level (for the probabilities of the first and second type errors α and β), the frequency f_* should be considered as accessible in the spectrum of the object, and the corresponding spatial details can be revealed. In the opposite case, when the compared objects are indistinguishable, we should consider f_* (ψ, α, β) as the limiting frequency due to noise. The corresponding theory of the recognition of stochastic objects has been recently developed by Terebizh (1990c, 1992b). Application of this theory to the problem considered gives strict but rather complicated results that we will describe elsewhere. Let us consider here only a numerical example.

Figure 6 shows the relation between the noise-limit frequency $f_n^{(0)}$ (in units of σ^{-1}) and the signal-to-noise ratio for the random Gaussian discussed above object in the presence of an additive noise. We see, as it can be expected, that the critical frequency slowly increases with the signal-to-noise ratio.

To allow for the smoothing effect due to the finite pixel size ε, the power density should be multiplied by $\text{sinc}^2(\varepsilon f)$. This has zeros at $f = \varepsilon^{-1}, 2\varepsilon^{-1}, \ldots$, so the *noise-limit frequency* $f_n(\varepsilon)$ is somewhat smaller than the corresponding frequency $f_n^{(0)}$ for $\varepsilon = 0$. On the other hand, the averaging over a finite pixel affects the statistics of the counts (the sampling variance decreases), and the resulting dependence $f_n(\varepsilon)$ looks like shown in Figure 7.

These questions deserve a detailed numerical investigation.

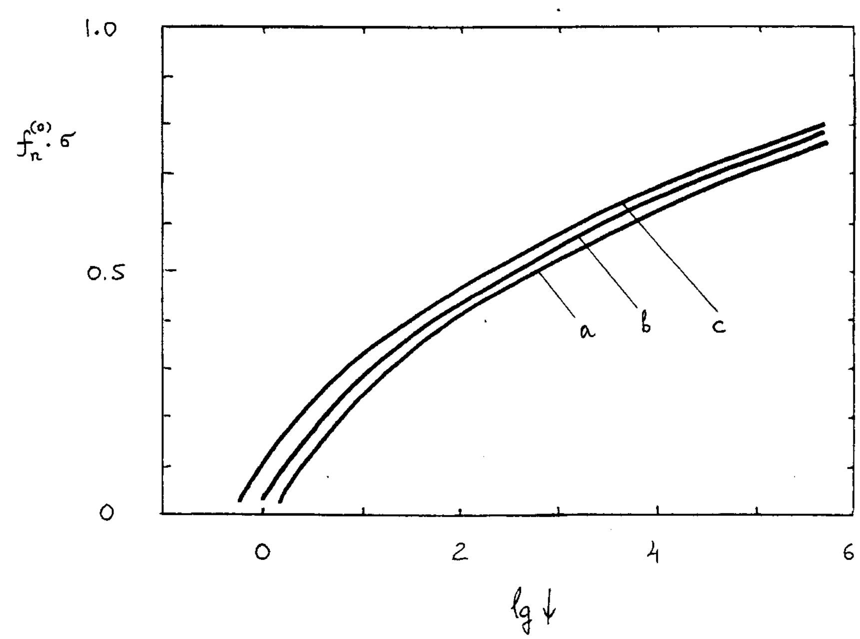

Figure 6 Relation between the noise-limit frequency $f_n^{(0)}$ and signal-to-noise ratio for Gaussian stochastic object; $\alpha = 0.10$, $\beta = 0.05$ (a), $\alpha = \beta = 0.20$ (b), $\alpha = \beta = 0.30$ (c).

3. SAMPLING FOR THE HUBBLE SPACE TELESCOPE

The discussion of this section is restricted to those aspects of the problem which illustrate the above approach in a most clear way, so much more extensive calculations including many additional factors (like stray light, the exact values of the bandwidth and dark current, etc.) are to be performed to obtain accurate results.

Let us consider the *Faint Object Camera* (FOC) of the HST. Assume Δ to be the radius of the first dark diffraction circle in the pointlike source image (Figure 1). For the monochromatic flux at wavelength λ, we have $\Delta = 1.22\lambda/D$, where D is the aperture diameter. For λ given in Angstroms, D, in mm and Δ, in arcsec,

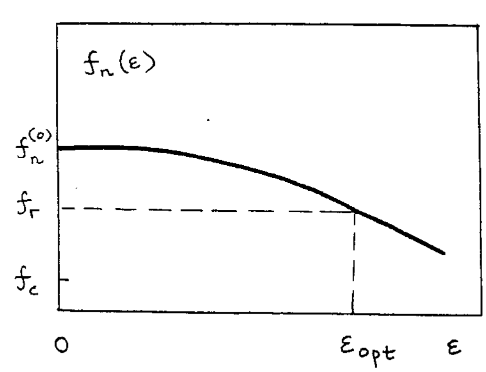

Figure 7 Qualitative form of the relation between the noise-limit frequency f_n and pixel size ε.

we have:

$$\Delta'' \simeq \frac{\lambda_A}{40 \cdot D_{mm}}, \tag{16}$$

and $\Delta = 0.058''$ for the HST ($D = 2400$ mm) at $\lambda = 5556$ Å. Let $\varepsilon_{\mu m}$ to be the pixel size in micrometers. Then we obtain, in arcsec:

$$\varepsilon'' = 206.265 \frac{\varepsilon_{\mu m}}{F_{mm}}, \tag{17}$$

where F_{mm} is the equivalent focal length. Then the Γ-factor (1) for an image diffraction-limited at wavelength λ is

$$\Gamma_d = 1.22 \times 10^{-4} \cdot (F/D)\lambda_A/\varepsilon_{\mu m}. \tag{18}$$

Here F/D is the relative focal length of the telescope or the complete optical system including the telescope and spectrograph. Relation (18) allows us to estimate some parameters after adopting the other, usually the F/D ratio given the detector, wavelength and resolving power.

Five columns of Table 2 give, respectively, the F/D ratio, focal length F, field of view 2 w, pixel size ε and diffraction Γ-factor for the three modes of the FOC at 512 × 512 pixels format. We have adopted $\lambda = 5556$ Å and $\varepsilon = 25$ μm (Macchetto, 1982).

As one can see from Table 2, the FOC was designed according to the Rayleigh point of view, so it was supposed to cover the diffraction PSF by a couple of pixels. The $f/288$ facility has been provided to exploit the conventional resolution power at shorter wavelengths ($\Gamma_d \simeq 1.8$ at $\lambda = 1250$ Å). Such a choice was quite natural at the time of designing the HST project because the image restoration techniques needs significant additional efforts and extremely powerful computers but these problems seem to be not very restrictive for the modern and, all the more, future projects.

It is known that the real radius of the point source image obtained by the HST at 84% level is $\simeq 1.6''$, that is approximately 30 times as large as the diffraction radius. The corresponding values of the Γ-factor are given in the last column of Table 2. At the same time, it should be stressed that the situation with the HST point spread function is more complicated since the PSF can be presented as a sum of two completely different components: a sharp central core of diameter 0.25″ containing 15% of the light, and very broad "wings." It was noted by White and Burrows (1990) on this occasion, that "the fundamental loss of HST imaging science as a result of the spherical aberration is not a loss of resolution; rather, it is a loss of the ability to detect faint objects, especially in crowded fields." This

Table 2 The faint object camera modes of the HST

Focal ratio	*Focal length (m)*	*Field of view (arcsec)*	*Pixel size (arcsec)*	Γ_d	Γ
48	115.2	22.5	0.044	1.3	36
96	230.4	11.2	0.022	2.6	73
288	691.2	3.8	0.0075	7.8	213

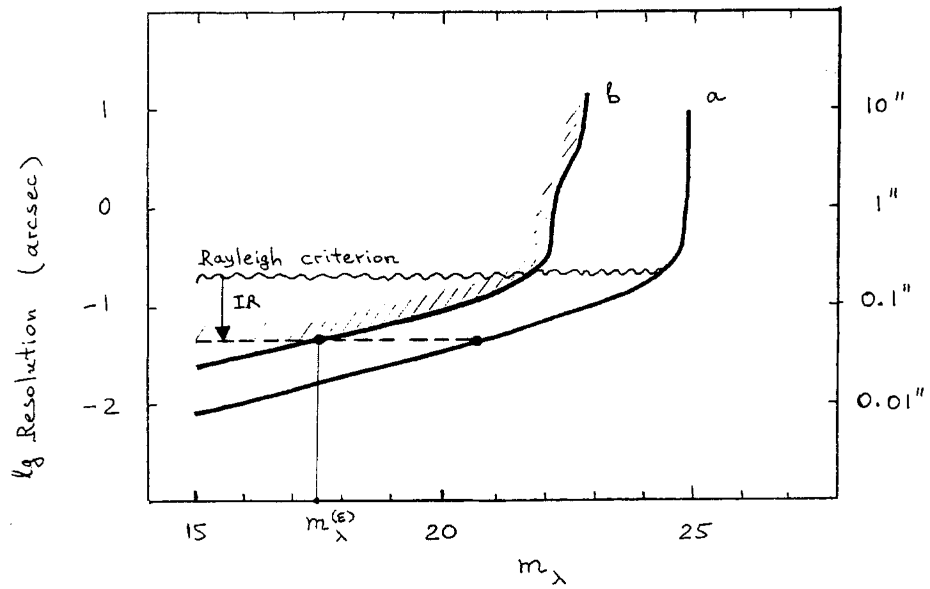

Figure 8 The resolution-magnitude relation for $f/96$ FOC HST images under the conditions specified in Table 3. Thick lines correspond to the theoretical limit for the supposed (a) and real (b) HST optics wavy line corresponds to the Rayleigh law and dashed line shows the resolution limit due to the pixel size. The shaded area is accessible for the actual HST optics. The symbols "IR" indicate the image restoration techniques abilities.

statement shows the essence of the problem in a quite correct manner, but if we want to describe the situation more accurately, some loss of resolution power should be accounted as well.

As an example, the abilities of the FOC direct imaging in the $f/96$ mode are shown in Figure 8. The parameters from Table 3 were adopted for the calculations of the resolving power (Terebizh, 1992b). The PSF characteristics refer to the one-dimensional representation of the FOC images. The monochro-

Table 3 Adopted parameters for illustrative observations with the FOC HST

Central wavelength	5000 A
Bandpass	100 A
Sky background	23 map/square arcsec
Stray light	27 mag/square arcsec
Exposure time	1000 s
PSF core component radius	0.157 arcsec
PSF diffuse component radius	1.5 arcsec
PSF core component fraction	1.0 and 0.23
Overall quantum efficiency	0.02 events/photon
Pixel size	25 micrometers
Dark current	$5 \cdot 10^{-5}$ events/s pixel
Significance levels	$\alpha = 0.10$, $\beta = 0.05$

matic stellar magnitude has been defined as

$$m_\lambda = -21.10 - 2.5 \cdot \lg f_\lambda \,(\text{erg/s cm}^2\,\text{Å}). \tag{19}$$

First of all, it can be seen from Figure 8 that the theoretical resolving power of the real HST is approximately 3 times worse than that for the supposed optics for $m_\lambda < 22^m$. For the object magnitude in the 22^m–25^m range, the losses are infinite. Rather moderate losses of the resolving power in the bright domain are caused mostly by the existence of a sharp core in the HST PSF. If there were no such a sharp detail, the losses would be more than one order of magnitude.

The second important note concerns the absolute value of the resolving power. We see that $\rho_t \simeq 0.01''$ were attainable with the supposed optics for the objects of $m_\lambda \simeq 15^m$, and higher resolving power for brighter objects. Nevertheless, the theoretical resolving power of the actual HST optics remains quite high. For the conditions considered, it follows from (9) that for the objects brighter than the *critical magnitude* $m_\lambda^{(\varepsilon)} \simeq 17.5^m$ the resolving power is restricted simply by the pixel size, $\varepsilon \simeq 25$ mcm rather than by signal-to-noise ratio (for the supposed optics, the critical magnitude is $\simeq 20.5^m$).

Let us discuss, finally, the abilities of image restoration techniques. Evidently, the gain of resolving power corresponds to the transition from the Rayleigh criterion to the ρ_ε-limit (the shaded area border). It can be seen from Figure 8 that the most efficient image restoration technique can improve the resolution only by a factor of about five for the objects brighter than $m_\lambda^{(\varepsilon)}$. If the focal ratio were greater, the gain could be more significant. For the present situation, we can shift the critical point $m_\lambda^{(\varepsilon)}$ to the faint magnitudes region by increasing the exposure time and/or spectral bandpass, and also by using other ways to increase the signal-to-noise ratio.

Similar diagrams can be calculated for various observational conditions, but it is appropriate to specify them for a special discussion of the HST abilities. We consider here only the example of restoration of the HST image.

The upper part of Figure 9 shows the *Supernova 1987A* region as seen by the FOC HST; the lower panel demonstrates the image processed using the *Maximum Likelihood Image Restoration* (MLIR) method (Terebizh, 1990a, b, c; 1991). The restoration was carried out by Terebizh and Biryukov (1992). Unfortunately, it is just the existing optics of the HST (where the PSF width is covered by dozens or hundreds of pixels) that allows to demonstrate the effect of superresolution in practice. According to the Rayleigh criterion one cannot expect to reach the resolving power drastically different from the PSF core width, $\simeq 0.20$ arcsec. However, the restored images shown in Figure 9 reveal details as small as $\simeq 0.03$–0.04 arcsec, i.e., of the order of pixel size. It is noteworthy that the nearest star to the *Supernova* (*Star 3* of Jakobsen *et al.*, 1991) appeared to be a binary, with the component separation $\simeq 0.06''$ and the intensity contrast of 32 times. The *Supernova* itself reveals the arc millisecond structure with the pointlike central source and surrounding envelope, consisting of the material ejected during the outburst at the velocity $\simeq 10^4$ km/s.

As one can see from Figure 8, the resolution power provided by the MLIR for the *Star 3* ($m_\lambda \simeq 15.6$) and *Supernova* ($m_\lambda \simeq 17.0$) attains the limit caused by the finite pixel size.

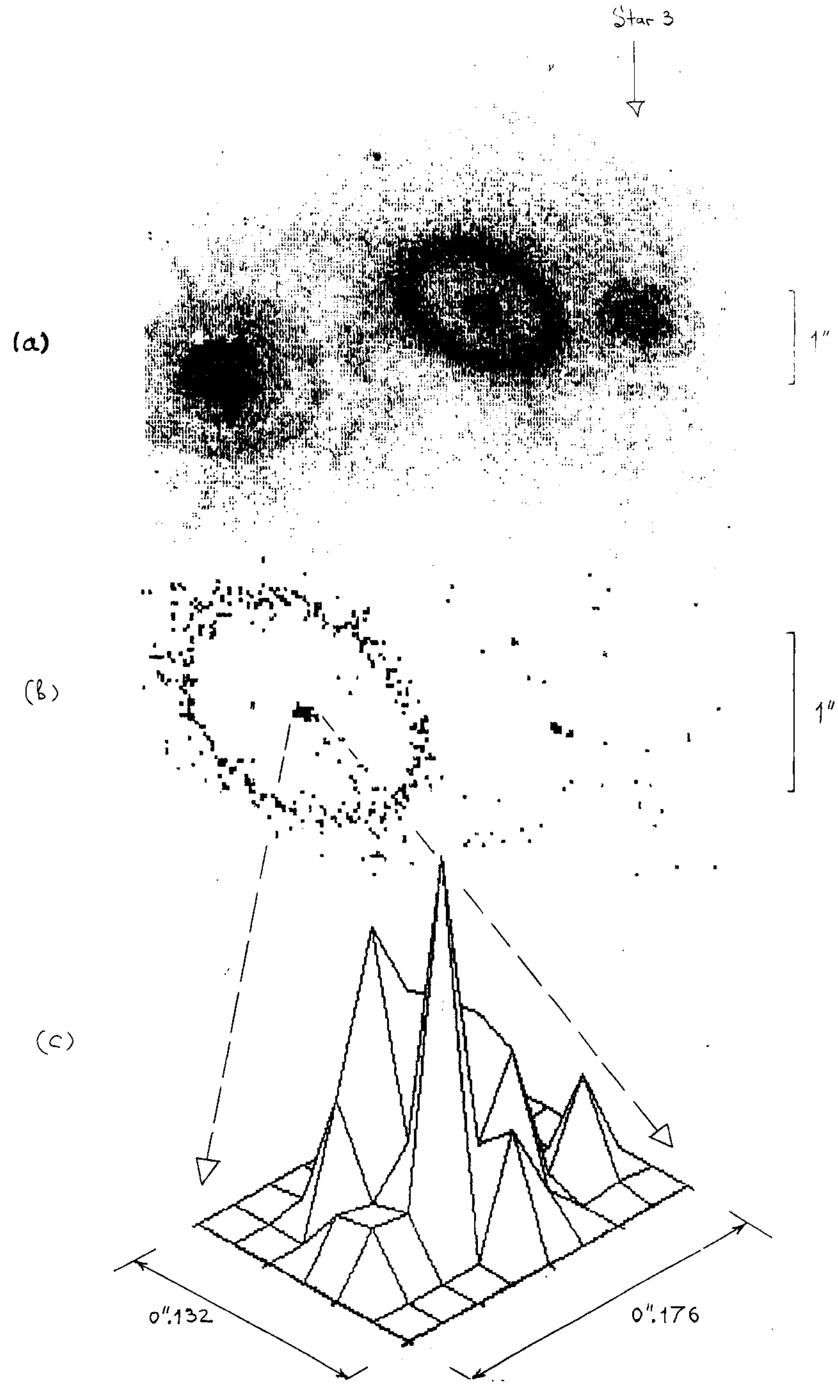

Figure 9 The *Supernova 1987A* as seen by HST (a), a part of the restored image including *Supernova* and *Star 3* (b), and the intensity distribution within the restored *Supernova* image (c).

4. CONCLUDING REMARKS

The general inference from the above discussion is that the image restoration abilities should be taken into account when astronomical and similar projects are being designed. In astronomy, the easiest way to obtain a self-consistent device is connected with the adjustment of the equivalent focal length F. At the same time, the adjustment of F to improve the resolution power should correspond, in one's turn, to the detector characteristics. For example, at very large values of F the reading noise of CCD devices becomes the dominant factor in the limiting magnitude. So, all aspects of any experiment should be considered consistently and simultaneously.

It should be stressed that almost any image, even the one obtained in a non-consistent way, contains much more information than can be anticipated by such a powerful system as human "eye + brain." A fine structure of an object can be revealed by proper restoration, and this should not introduce any subjective motives which are usual for widespread methods of restoration. The corresponding method-independent approach has been proposed by Terebizh (1990a, 1991).

For the sake of simplicity, our discussion has been restricted to the case of direct imaging. Of course, similar considerations are valid for spectral measurements, where the gain in resolving power is no less significant than for direct imaging. The spectroscopy has evident specific features, caused by limitations in the exposure time, spectral range width, etc.

As usually, it seems quite natural that the optimum solution combining high resolving power with a large field of view and faint limiting magnitude can be obtained for several modes of the optical system through introducing auxiliary optical elements. The only, but important, change to the conventional approach is connected with the allowance for the possibilities of image restoration techniques.

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APPENDIX

ESTIMATES OF POSITION AND BRIGHTNESS FOR A BLURRED SOURCE

The problem considered below is so typical for astronomy that, perhaps, it was analyzed earlier. Nevertheless, in view of importance of the result, it is worth to discuss the problem once more in the framework of a simple version of the image restoration approach.

Let us assume that a pointlike source of intensity S is located near the origin of a coordinate system. Since the final expressions are independent of the number of dimensions, consider a one-dimensional case. Let ρ be the unknown true position of the source. It is supposed further that the pointlike object is blurred stochastically, so that each photo-event S is distributed randomly and independently of other ones, with some probability $h(x-\rho)\,dx$, over the interval $[x, x+dx]$ around the initial position ρ. The detector has only two large pixels: one from $-\infty$ to 0, and the other from 0 to $+\infty$, and there is additional Poisson noise due to the detector dark current or/and sky background with the mean value of b events per pixel.

The *Point Spread Function* $h(x)$ is assumed to be known, as well as the intensity of the object S and mean intensity of the additive noise b. Thus, the observer has only two random counts N_1 and N_2 of the events for the left and right pixels correspondingly. One should obtain an estimate of the true object position, $\hat{\rho}$, on the basis of the set of numbers (N_1, N_2) and the aforementioned *a priori* information.

In the most general formulation of the problem we know nothing about the source except its intensity $S \geq 0$. This case is considered in this Appendix, but first we discuss a simpler case, when additional *a priori* information is available concerning the source nature. Namely, it is supposed, that the total number of counts due to the emission from the source is the Poisson random variable with the known S mean value. This is just the case for usual observational conditions for a non-coherent object, and strictly valid for the one-mode laser radiation (Mehta, 1970; Loudon, 1973). As discussed by Terebizh *et al.* (1991), the numbers of counts for the left and right pixels are then independent Poisson variables with mean values $\langle \zeta_1 \rangle = S \cdot p(\rho) + b$ and $\langle \zeta_2 \rangle = S \cdot [1 - p(\rho)] + b$, respectively. Here $p(\rho)$ is the probability of occurrence of a count in the left pixel, which can be written in terms of the PSF in the following way:

$$p(\rho) = \int_{-\infty}^{0} h(x-\rho)\,dx. \tag{A1}$$

The probability of occurrence of N_1 counts in the left pixel and N_2 counts in the right one under given shift ρ is

$$f(N_1, N_2 \mid \rho) = \exp(-Sp - b)\frac{(Sp+b)^{N_1}}{N_1!} \cdot \exp[-S(1-p) - b]\frac{[S(1-p)+b)^{N_2}}{N_2!} \tag{A2}$$

The desired maximum likelihood estimate $\hat{\rho}(N_1, N_2)$ is the root of the following equation:

$$\frac{\partial}{\partial \rho} \ln f(N_1, N_2 \mid \rho) = 0, \tag{A3}$$

or

$$p(\hat{\rho}) = \frac{N_1 + (N_1 - N_2) \cdot b/S}{N_1 + N_2}. \tag{A4}$$

The solution of this equation for known PSF shape gives a concrete expression for $\hat{\rho}$.

Let us consider the simplest case when the PSF has a rectangular shape of the width Δ. Then $p(\rho) = 1/2 - \rho/\Delta$, and we have from (A4):

$$\frac{\hat{\rho}}{\Delta} = \frac{1}{2}\left(1 + \frac{2b}{S}\right)\frac{N_2 - N_1}{N_2 + N_1}. \tag{A5}$$

Taking into account the Poisson nature of random variables N_1 and N_2, it is easy to find for the mean value $\langle \hat{\rho} \rangle \simeq \rho$, i.e., the estimate $\hat{\rho}$ is unbiased, and for the standard deviation we have

$$S\,\text{dev}\,(\hat{\rho}/\Delta) \simeq \frac{1}{2\psi}, \qquad \psi \equiv \frac{S}{\sqrt{S + 2b}}. \tag{A6}$$

Here ψ should be considered as the signal-to-noise ratio, so, we come to Eqs (2) and (3). A similar result with slightly different proportionality coefficient is valid for other PSF forms.

In a general case, when only the intensity S of the pointlike source is known, the probability $f(N_1, N_2 \mid \rho)$ to obtain given numbers of counts is equal to

$$f(N_1, N_2 \mid \rho) = \sum_{k=S-N_2}^{N_1} C_S^k p^k (1-p)^{S-k} \cdot \exp(-b)\frac{b^{N_1-k}}{(N_1-k)!}\exp(-b)\frac{b^{N_2-S+k}}{(N_2-S+k)!}, \tag{A7}$$

where $N_1 + N_2 - S \geq 0$, and we consider the most interesting case when N_1 and N_2 do not exceed S. Since $\rho \ll \Delta$, the probability $p(\rho)$ is close to 1/2, and the binomial probability density including $p(\rho)$ in (A7) has a maximum near $k \simeq S/2$. On the other hand, factor the $[(N_1 - k)! \cdot (N_2 - S + k)!]^{-1}$ has very sharp maximum at $k_{\max} = (S + N_1 - N_2)/2$, which is close to $S/2$ as well, so it is permissible to retain in (A7) only the term corresponding to $k_{\max}$. Therefore, the probability f is approximately

$$f(N_1, N_2 \mid \rho) \simeq \text{const} \cdot p^{(S+N_1-N_2)/2} \cdot (1-p)^{(S-N_1+N_2)/2}. \tag{A8}$$

As follows from Eqs. (A3) and (A8), the maximum likelihood estimate $\hat{\rho}$ is the root of the equation

$$p(\hat{\rho}) = \frac{1}{2}\left(1 + \frac{N_1 - N_2}{S}\right). \tag{A9}$$

Now, for rectangular shape of PSF of the width Δ, we obtain an expression somewhat different from (A5):

$$\frac{\hat{\rho}}{\Delta} = \frac{N_2 - N_1}{2S}, \tag{A10}$$

but, according to (A10), the standard deviation of $\hat{\rho}$ is again given by (A6).

The variance of the maximum likelihood estimate of the pointlike source position is a useful characteristic of the position accuracy, but more informative is the least possible shift of the source ρ_t at the accepted significance levels α and β (see Section 2). The corresponding general analysis has been carried out by

Terebizh (1990c; 1992b). As follows from Eq. (28) in Terebizh (1990c) for $n = 2$,

$$\rho_t(\psi, \alpha, \beta)/\Delta \simeq \frac{z_\alpha + z_\beta}{\psi\sqrt{2}}, \tag{A11}$$

where z_θ is the root of Eq. $\Phi(z) = 1 - \theta$, and $\Phi(z)$ is the Gaussian distribution function:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp(-t^2/2)\, dt. \tag{A12}$$

As follows from (A11), the limiting, at the given significance level, shift of the source is inversely proportional to ψ. According to Eqs. (A6), (A11) and Table 1, the standard deviation of $\hat{\rho}$ corresponds approximately to $\alpha \simeq \beta \simeq 0.35$.

Equation (A6) is valid even when both the object intensity S and its position ρ are unknown. To $\hat{S}$ and $\hat{\rho}$, estimate one should maximize (A2) with respect to S and ρ simultaneously. The corresponding estimates are:

$$\hat{S} = N_1 + N_2 - 2b,$$

$$p(\hat{\rho}) = \frac{N_1 - b}{N_1 + N_2 - 2b}. \tag{A13}$$

The standard deviation of $\hat{S}/S$ is evidently equal to

$$S\,\mathrm{dev}\,(\hat{S}/S) = \psi^{-1}, \tag{A14}$$

while that for $\hat{\rho}/\Delta$ remains to be approximately equal to $(2\psi)^{-1}$ when signal-to-noise ratio ψ is large enough. Note that all the above estimates can be considered, in a first approximation, as unbiased.

The above estimates provide a reasonable approximation for any shape of the object and any number of pixels.

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NOTE ADDED IN PROOF

Since an initial "object" is, in our determination, the set of **mean** pixel intensities for an ideal imaging system, it should be considered as deterministic function. Therefore, strictly speaking, we should say "a random object estimate for an ideal imaging system" instead of more simple term "a stochastic object", which is used in the text.