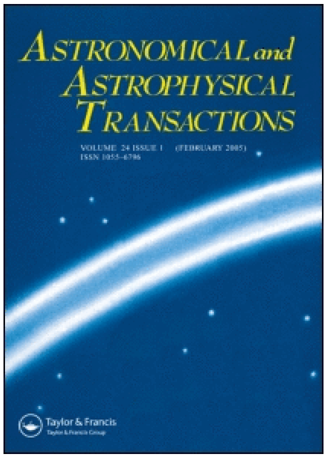

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CLASSIFICATION OF BOUND AND UNBOUND GEODESICS IN THE KERR METRIC AND THE EFFECTIVE PARTICLE CROSS-SECTION OF A REISSNER–NORDSTRÖM BLACK HOLE

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Sets of constants of motion of a particle that correspond to different types of r-motion in the Kerr metric are considered. The topology of these sets is determined and a number of constants characterizing these sets are found. The expressions are obtained for the photon and uncharged particle capture cross-sections for the Reissner–Nordström black hole.

1. TYPES OF UNBOUND GEODESICS IN THE KERR METRIC

An important problem in the study of unbound motion of particles in the Kerr metric is the description of the set of constants of motion for which a particle travelling from infinity goes below the horizon of a black hole. We shall give a qualitative description of this set and also of the set of constants of motion for which the particle asymptotically approaches a surface r = const surrounding the black hole, and the sets of constants of motion for which the particle departs to infinity. The equation of motion for the radial variable in the Kerr metric is (Chanrasekhar, 1983) $\rho^4(dr/dr)^2 = R(r)$, where

$$R(r) = r^4 + (a^2 - \xi^2 - \eta)r^2 + 2M(\eta + (\xi - a)^2]r - a\eta \text{ (Photons)},$$
$$R(r) = r^4 + (a^2 - \xi^2 - \eta) + 2M[\eta + (\xi - a)^2]r - a\eta - r^2\Delta/E \text{ (Particles)},$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$, $a = S/M$. The constants S and M refer to the black hole, namely S is the angular momentum and M is the mass of the black hole. The constants E, ξ and η refer to the particle, namely E is its energy at infinity, $\xi = L_z/E$ (L_z is the particle angular momentum with respect to the black hole rotation axis, and $\eta = Q/E^2$ (Q is given by $Q = p_\theta^2 + \cos^2\theta[a^2(\mu^2 - E_2) + L_z^2\sin^{-2}\theta]$ and μ is the mass of the particle). It is readily verified that the radial motion of the particle is controlled by the following constants: $\hat{a} = a/M$, $\hat{E} = E/\mu$, $\hat{\xi} = \xi/M$ and $\hat{\eta} = \eta/M$. The radial motion of the photon is independent of the constant E. Instead of the coordinate r, we introduce the variable $\hat{r} = r/M$. (The caret will be omitted henceforth). Thus, the character of motion in the r-coordinate for a given value of a is determined by the three constants E, ξ, η in the case of a moving particle, and by the two constants

ξ and η in the case of photons. Depending on the multiplicity of the roots of the polynomial $R(r)$ (for $r > r_+ = 1 + \sqrt{1 - a^2}$), there are three types of motion in the r-coordinate (Zakharov, 1983, 1986), namely: (1) The polynomial $R(r)$ has no roots for $r \geq r_+$. The particle then falls into the black hole. (2) The polynomial $R(r)$ has roots and $r_{\max} > r_+ (r_{\max}$ is the maximal root); for $(\partial R/\partial r)(r_{\max}) \neq 0$ we then have $(\partial R/\partial r)(r_{\max}) > 0$, and the particle departs to infinity after approaching the black hole. (3) The polynomial $R(r)$ has a root and $R(r_{\max}) = (\partial R/\partial r)(r_{\max}) = 0$; the particle now takes an infinite proper time to approach the surface $r = \text{const}$. Let us cut the space parameter (E, ξ, η) with the plane $E = \text{const} \geq 1$ and describe, in this slice, the set of constants corresponding to different types of motion. The boundary of the set of constants corresponding to the second type of motion for $\eta \geq 0$ is the set of constants for which the motion belongs to the third type. The latter set can be described in terms of the graph of the function $\eta = \eta(\xi)$. We note that the set of these constants, as functions $\xi(r)$ and $\eta(r)$, was examined by Chandrasekhar (1983). Some of the properties of the function $\eta(\xi)$ is described by Zakharov (1986). The effective capture cross-section in the field of Schwarzschild black hole for particles possessing an arbitrary velocity at infinity can be easily expressed in terms of the quantity (Zakharov 1985, 1988, 1991a).

$$L_{cr}^2 = \frac{-(\alpha^2 - 18\alpha + 27) + (\alpha + 9)\sqrt{(\alpha + 9)(\alpha + 1)}}{(2\alpha)},$$

where $\alpha = (E^2 - 1)^{-1}$. The cross-section is given by $\sigma = \pi L_{cr}^2/(E^2 - 1)$ (in units of the squared mass of the black hole). Consider a moving particle of arbitrary energy at infinity ($E > 1$). It can be verified that for $\eta_{\max} = L_{cr}^2/E^2$ we have

$$r_{\max} = [8\alpha^3/27 + \eta_{\max} \mathrm{E}^2 \alpha(\alpha/3 + 1)]^{1/3} - 2\alpha/3,$$

$\xi_{\max} = 2\alpha/(r_{\max} - 2)$, so that $R(r)$ and $\partial R/\partial r$ vanish. We also note that the values chosen according to (4) correspond to a maximal value of $\eta(\xi)$. The values $\eta_{\max}$ and $r_{\max}$ turn out to be equal to the corresponding values of these quantities for $a = 0$ (the Schwarzschild metric) (Zakharov 1986, 1991a).

2. CLASSIFICATION OF FINITE PARTICLE MOTION IN THE KERR METRIC

We consider only finite orbits for which the particle energy satisfies the inequality $E < 1$. In the present section, we classify the various types of the finite motion of test particles in the Kerr metric by investigating the roots of $R(r)$, as Synge (1960) did for the Schwarzschild metric and as we did for an unbounded particle motion in the Kerr metric. There are five possible types of motion (Zakharov, 1989a). Note that if the black hole is undergoing an extreme rotation ($a = 1$), the possible types of motion are the types 6–9, but just like the case of infinite particle motion, the orbits of the types 6–9 disappear when the rotation parameter no longer has an extreme value (the structural instability Zakharov, 1989a).

3. ON THE PHOTON AND SLOW-PARTICLE CAPTURE CROSS-SECTIONS IN THE REISSNER-NORDSTRÖM METRIC

It is well known that the radial motion of the photon is determined by the equation $(dr/d\lambda)^2 = R(r)$, where $R(r) = r^4 - \xi^2 r^2 + 2\xi^2 r - Q^2\xi^2$, Q is the charge of black hole divided by the mass of the particle, $\xi = L/E$, L is the angular momentum of the photon, and r is expressed in units of the black hole mass. The photon cross section of the black hole is given by (Zakharov, 1989b, 1991a, 1991b)

$$\xi^2 = \frac{8q^2 - 36q + 27 + \sqrt{(8q^2 - 36q + 27)^2 + 64q^3(1-q)}}{2(1-q)},$$

where $q = Q^2$.

We have also a simple derivation for the capture cross section in the case of a particle slow motion (Zakharov, 1990, 1991a, 1991b).

The autor is pleased to note that this paper stems from an outstanding paper of Kaplan (1949).

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