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ANALYTICAL APPROXIMATIONS FOR SOME FUNCTIONS IN THE ROCHE MODEL

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Analytical approximations and tables are presented for the following functions in the Roche model: the 'barotropic' radius of the secondary companion re, the characterising angles of the Roche lobe seen from the center of the other star and from the inner Lagrangian point, the coordinates of the points at the border line separating the illuminated part of the Roche lobe and the dark one. By using Eggleton's (1983) form for the effective radius, 
\[ r_e = A q^{2/3} (B q^{2/3} + \log(1 + q^{1/3})) \]
we derived the values 
\[ A = 0.4990, 0.4394 \] and \[ B = 0.5053, 0.5333 \] for the function \( \sin \theta \) in the y- and z-directions, respectively (where \( \theta \) is the angular radius of the Roche lobe seen from the other star). For the mass ratio \( q = M_2/M_1 \leq 4 \), the maximal relative errors do not exceed 0.2 and 0.5 per cent, respectively. A more precise approximation for the 'inclination–eclipse duration' relation is derived.

KEY WORDS Stars, binaries–stars, cataclysmic.

Even though properties of the Roche lobe are well known (e.g. Kopal, 1978; Paczynski, 1971), recently some attempts were done to compute detailed tables of different functions in the Roche model (Chanan et al., 1976; Mochnacki, 1984; Pennington, 1985; Todoran, 1990), especially of the 'effective' Roche radius \( r_e \), which is usually defined as the radius \( r_e \) of the sphere whose volume is equal to the Roche lobe volume (Eggleton, 1983 and references therein). Andronov (1982) argued for a different definition of \( r_e \), which may be assumed from the baroothropic model, and is equal to the parameter \( r_0 \) of Kopal (1978). Here we tabulate also the angular dimensions of the star filling its Roche lobe as seen from its companion in the orbital plane \( \theta(0^\circ) \) and in the orthogonal plane \( \theta(90^\circ) \), and the coordinates \( x_0 \) and \( x_90 \) at the Roche lobe corresponding to the boundary of the stellar surface illuminated by its compact companion. The best fit approximations for \( r_e, \theta(0^\circ), \theta(90^\circ) \) and the 'effective' angular dimension \( \theta_e \) of the Roche lobe, are presented as well. Consider the reference frame with the origin at the secondary companion (the star filling its Roche lobe) of mass \( M_2 \). The mass of the compact primary is \( M_1 \). Using \( a \) (the distance between the centers of the stars), \( M = M_1 + M_2 \) and \( U_0 = GM/\alpha \) as unit distance, mass and potential, respectively, the Jacobi integral can be written as

\[ -\nu/r_1 - \mu/r_2 - ((x - \nu)^2 + y^2)/2 = C, \]  
where \( \mu = M_2/(M_1 + M_2) \), \( \nu = 1 - \mu \), \( r_1^2 = (1 - x)^2 + y^2 + z^2 \) and \( r_2^2 = x^2 + y^2 + z^2 \). Here \( C = -c/2 \) (where the constant \( c \) is the one used by Kopal (1978)) and thus \( C \) has the meaning of the potential as usually defined in mechanics (Landau and Lifshitz, 1973).

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The 'barothropic' radius \( r_0 \) can be defined as \( r_0 = -\mu/u \), where \( u = v + v^2 - \mu/x_1 - v/(1 - x_1) - (x_1 - v)^2/2 \), and \( x_1 \) is the coordinate of the inner Lagrangian point (cf. Andronov, 1982).

To obtain the boundary of the region at secondary's surface illuminated by the primary, we introduce the following spherical coordinates centered at the position of the primary: \( \theta \), the angle between the 'line of sight' and the 'line of centers', and the angle \( \psi \) between the orbital plane and the plane defined by the 'line of sight' and the 'line of centers'. At the boundary, \( \theta \) is a function of \( \psi \). In dimensionless units, the value of \( \sin \theta \) is the distance between a point at the boundary and the center of the secondary companion. Neglecting variations of \( \sin \theta \) with \( \psi \) (our computations show that \( \sin \theta (90^\circ)/\sin \theta (0^\circ) = 0.9628 \pm 0.0009 \) for \( \mu \leq 0.5 \)), one may write the relation between \( \mu \), inclination \( i \) and the phase of the eclipse of the primary in the form:

\[
\sin^2 \theta = \cos^2 i + \sin^2 i \cdot \sin^2 \phi \\
= \sin^2 \phi + \cos^2 \phi \cos^2 i = 1 - \sin^2 i \cos^2 \phi
\]  
(2)

(cf. Horne, 1980, Shafter, 1984; Downes et al., 1986; Garnavich et al., 1990). All these authors supposed that \( \sin \theta = r_c \) and used the approximation of Eggleton (1983):

\[
r_c = 0.49q^{2/3}/[0.6q^{2/3} + \log(1 + q^{1/3})].
\]  
(3)

Here \( r_c \) is supposed to be equal to \( r_0 \). Martin et al. (1987) pointed out that \( r_c \) is by 3 per cent larger than \( r_c \), and proposed a corrected value of \( r_c \), neglecting the dependense \( \theta(\psi) \). Some other relations can be obtained from spherical trigonometry:

\[
\sin \theta \sin \psi = \cos i, \\
\sin \theta \cos \psi = \sin i \sin \phi, \\
\cos \theta = \sin i \cos \phi.
\]  
(4)

The value of \( \psi \) can be obtained from the following expression:

\[
\sin^2 \psi = \cos^2 i/\left(\cos^2 i + \sin^2 i \cdot \sin^2 \phi \right),
\]  
(5)

while the values \( i \) and \( \phi \) can be determined from observations. Numerical results can be approximated by

\[
\sin \theta(\psi) = \sin \theta (0^\circ) + [\sin \theta (90^\circ) - \sin \theta (0^\circ)] \sin^2 \psi.
\]  
(6)

An exact value of \( \sin \theta(\psi) \) is smaller than that given by this expression, but the relative error does not exceed \( 5 \cdot 10^{-4} \) for intermediate values of \( \psi \). The value \( \theta (0^\circ) \) corresponds to the half-width of the eclipse of the compact companion when \( i = 90^\circ \); the eclipses occur when the inclination \( i \) exceeds the value \( 90^\circ - \theta (90^\circ) \).

To derive the \( \phi - i \) relation for a fixed value of \( q \), one may use the interpolating expression

\[
\sin^2 \theta(\psi) = \sin^2 \theta (0^\circ) + (\sin^2 \theta (90^\circ) - \sin^2 \theta (0^\circ)) \sin^2 \psi = \alpha + \beta \sin^2 \psi.
\]  
(7)

Thus one may obtain from Eqs. (2), (5) and (7) the following expression:

\[
\sin^2 i \cos^2 \phi = \left(2 - \alpha - [\alpha^2 + 4\beta(1 - \sin^2 i)]^{1/2}\right)/2,
\]  
(8)
which allows to compute \( \phi \) for each appropriate value of \( i \). In the previously published \( 'q - i' \) and \( '\psi - i' \) diagrams (Horne, 1985; Shafter et al., 1988), the approximations \( a = r^2, \beta = 0 \) were assumed. In this case the systematic error of Eq. (2) reaches a few percent, i.e. a few degrees in \( i \) or \( \phi \). Thus the more accurate Eq. (8) is recommended to be used when evaluating the \( 'i - \phi' \) diagrams. Of course, to obtain the exact dependence \( i(\phi) \) one should use precise values of \( \theta(\psi) \) and to solve Eq. (2) numerically (cf. Chanan et al., 1976). However, the good approximation provided by Eq. (7) allow to neglect the arbitrary deviations of a few \( 10^{-4} \).

The solid angle \( \Omega \) of the secondary companion can be evaluated as

\[
\Omega = \int_0^{2\pi} (1 - \cos \theta(\psi)) \, d\psi.
\]  
(9)

The total energy heating the secondary companion is \( L \cdot \Omega/4\pi \), where \( L \) is the luminosity of the primary one. The numerical values of \( \Omega/4\pi \) are given in Table 1. One may use the following approximation (cf. Andronov, 1986):

\[
\Omega/4\pi = \begin{cases} 
0.0473 \mu^{2/3} & (m \leq 0.2), \\
0.004 + 0.061\mu & (0.1 \leq \mu \leq 0.5).
\end{cases}
\]  
(10)

For \( \Omega \) itself, the corresponding best fit coefficients are 0.594, 0.051 and 0.766, respectively. The 'effective' angular radius of the Roche lobe \( \theta_e \) may be thus defined from the expression \( \Omega(\psi) = 2\pi[1 - \cos \theta_e(\psi)] \). It may be noted that the difference between \( \sin \theta_e \) and \( r_e \) does not exceed 2.5 percent for \( \mu \leq 0.7 \).

We used Eggleton's (1983) form

\[
r_e = Aq^{2/3}/[Bq^{2/3} + \log(1 + q^{1/3})],
\]  
(11)

where \( q = \mu/(1 - \mu) \), to obtain the coefficients for the abovementioned quantities: \( r_e(A = 0.4660, \, B = 0.5929, \, \delta = 0.010), \, \sin \theta(0^\circ) \quad (A = 0.4990, \, B = 0.5053, \, \delta = 0.002), \, \sin \theta(90^\circ) \quad (A = 0.4394, \, B = 0.5333, \, \delta = 0.005), \, \sin \theta_e \quad (A = 0.4441, \, B =

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\( \mu \) & \( \theta(0^\circ) \) & \( \theta(90^\circ) \) & \( \theta_e \) & \( \Omega/4\pi \) & \( \gamma(0^\circ) \) & \( \gamma(90^\circ) \) & \( D \) \\
\hline
0.00 & 0. & 0. & 0. & 0. & 0. & 60.00 & 56.310 & 4.000 \\
0.05 & 9.343785 & 8.992900 & 9.164273 & 0.006382 & 58.203 & 55.812 & 5.984 \\
0.10 & 11.760977 & 11.326513 & 11.388033 & 0.010105 & 57.879 & 55.718 & 6.584 \\
0.15 & 13.516905 & 13.017912 & 13.261752 & 0.013334 & 57.691 & 55.663 & 6.994 \\
0.20 & 14.978535 & 14.421162 & 14.693491 & 0.016352 & 57.565 & 55.625 & 7.298 \\
0.25 & 16.278380 & 15.664040 & 15.964114 & 0.019283 & 57.476 & 55.599 & 7.531 \\
0.30 & 17.482702 & 16.810180 & 17.138532 & 0.022202 & 57.412 & 55.580 & 7.708 \\
0.35 & 18.631542 & 17.897690 & 18.255789 & 0.025166 & 57.366 & 55.556 & 7.839 \\
0.40 & 19.752850 & 18.952756 & 19.342913 & 0.028223 & 57.336 & 55.557 & 7.929 \\
0.45 & 20.868814 & 19.995684 & 20.421107 & 0.031423 * & 57.318 & 55.552 & 7.983 \\
0.50 & 21.999356 & 21.044155 & 21.509111 * & 0.034820 & 57.312 & 55.550 & 8.000 \\
0.60 & 24.387086 & 23.228099 & 23.790803 * & 0.042488 * & * * & * * \\
0.70 & 27.125427 & 26.574123 & 26.376087 & 0.052051 * & * * & * * \\
0.80 & 30.610867 & 28.683105 & 29.609944 & 0.065295 * & * * & * * \\
0.90 & 36.012306 & 33.085214 & 34.475423 & 0.087815 * & * * & * * \\
0.95 & 40.980637 & 36.837125 & 38.778501 & 0.110213 * & * * & * * \\
\hline
\end{tabular}
\caption{The characteristic angles of the Roche lobe as functions of \( \mu \)}
\end{table}

Note: The values of \( \gamma(0^\circ) \), \( \gamma(90^\circ) \) and \( D \) remain the same when \( \mu \) is replaced by \( 1 - \mu \).
Table 2 The position of the Lagrangian and boundary points and \( \nu_{cr} \)

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( x_L )</th>
<th>( r_0 )</th>
<th>( x_0(0^\circ) )</th>
<th>( x_0(90^\circ) )</th>
<th>( \nu_{cr} )</th>
<th>( x_{L3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.</td>
<td>0.</td>
<td>0.</td>
<td>0.</td>
<td>0.</td>
<td>2.</td>
</tr>
<tr>
<td>0.05</td>
<td>0.234775</td>
<td>0.161834</td>
<td>0.030717</td>
<td>0.030263</td>
<td>0.608682</td>
<td>1.970826</td>
</tr>
<tr>
<td>0.10</td>
<td>0.290965</td>
<td>0.202644</td>
<td>0.047950</td>
<td>0.047312</td>
<td>0.705248</td>
<td>1.941609</td>
</tr>
<tr>
<td>0.15</td>
<td>0.330260</td>
<td>0.231785</td>
<td>0.062612</td>
<td>0.061830</td>
<td>0.753672</td>
<td>1.912299</td>
</tr>
<tr>
<td>0.20</td>
<td>0.361924</td>
<td>0.255648</td>
<td>0.076115</td>
<td>0.075206</td>
<td>0.779316</td>
<td>1.882839</td>
</tr>
<tr>
<td>0.25</td>
<td>0.389257</td>
<td>0.276524</td>
<td>0.089065</td>
<td>0.088035</td>
<td>0.791023</td>
<td>1.853167</td>
</tr>
<tr>
<td>0.30</td>
<td>0.413870</td>
<td>0.295545</td>
<td>0.101817</td>
<td>0.100665</td>
<td>0.792969</td>
<td>1.823206</td>
</tr>
<tr>
<td>0.35</td>
<td>0.436705</td>
<td>0.313373</td>
<td>0.114627</td>
<td>0.113347</td>
<td>0.787437</td>
<td>1.792867</td>
</tr>
<tr>
<td>0.40</td>
<td>0.458382</td>
<td>0.330454</td>
<td>0.127709</td>
<td>0.126291</td>
<td>0.775776</td>
<td>1.762045</td>
</tr>
<tr>
<td>0.45</td>
<td>0.479358</td>
<td>0.347115</td>
<td>0.141269</td>
<td>0.139700</td>
<td>0.758814</td>
<td>1.730611</td>
</tr>
<tr>
<td>0.50</td>
<td>0.500000</td>
<td>0.363636</td>
<td>0.155528</td>
<td>0.153787</td>
<td>0.737024</td>
<td>1.698406</td>
</tr>
<tr>
<td>0.60</td>
<td>0.516180</td>
<td>0.377231</td>
<td>0.187247</td>
<td>0.185067</td>
<td>0.696867</td>
<td>1.630813</td>
</tr>
<tr>
<td>0.70</td>
<td>0.586130</td>
<td>0.43416</td>
<td>0.226020</td>
<td>0.223175</td>
<td>0.603106</td>
<td>1.556735</td>
</tr>
<tr>
<td>0.80</td>
<td>0.638076</td>
<td>0.475532</td>
<td>0.278487</td>
<td>0.274455</td>
<td>0.502255</td>
<td>1.471049</td>
</tr>
<tr>
<td>0.90</td>
<td>0.709035</td>
<td>0.531451</td>
<td>0.364954</td>
<td>0.358052</td>
<td>0.360289</td>
<td>1.359700</td>
</tr>
<tr>
<td>0.95</td>
<td>0.765225</td>
<td>0.572648</td>
<td>0.447896</td>
<td>0.436959</td>
<td>0.256948</td>
<td>1.278094</td>
</tr>
</tbody>
</table>

\( 0.5195, \delta = 0.002, \) where \( \delta \) is the maximal relative difference between exact values and those approximated by the best fit expressions. These fits were obtained for \( \mu \leq 0.8 \), because for the larger values of \( \mu \) the accuracy becomes very poor and in addition, these values are not realistic for CV's. For \( r_e \), Eggleton (1983) obtained \( A = 0.49 \) and \( B = 0.6 \), which yields the values \( \approx 4-6 \) percent higher than those of \( r_e \). Masevich and Tutukov (1988) used a simpler approximation

\[ r_e = 0.52 \mu^{0.44}, \]  

which is applicable for a smaller subinterval of \( q \) from 0.3 to 2 and can be used for evolutionary computations.

However, for small values of \( \mu \), the form \( r_e = A \mu^{1/3} \) (cf. Paczynski, 1971; Patterson, 1981) is more accurate. For \( \mu \leq 0.45 \), we obtained \( A = 0.4418, 0.4477, 0.4309 \) and 0.4391 (\( \delta = 0.025, 0.038, 0.036, 0.037 \)) for \( r_e, \sin \theta (0^\circ) \), \( \sin \theta (90^\circ) \) and \( \sin \theta_e \), respectively. The mean squared relative deviations are 2-3 times smaller than \( \delta \).

In the vicinity of the inner Lagrangian point, the dimensionless Jacobi potential may be approximated by the following expression:

\[ U(x, y, z) = U(x_L, 0, 0) - \frac{1}{2}(2D + 1)(x - x_L)^2 + \frac{1}{2}(D - 1)y^2 + \frac{1}{2}Dz^2, \]  

where

\[ D = \mu/x_L^3 + \nu/(1 - x_L)^3 \]  

(Andronov, 1984 and references therein). Equation \( U(x, y, z) = U(x_L, 0, 0) \) defines the cone coinciding with the Roche lobe in the nearest vicinity of the inner Lagrangian point. The angle \( \gamma (\psi) \) between the line of centers and the corresponding line at the cone is the function of \( \psi \) and \( \mu \). However, for reasonable values of \( \mu \) ranging from 0.1 to 0.9, the value of \( \gamma (\psi) \) varies within a narrow range not exceeding 1 per cent (for fixed \( \psi \)). For fixed \( \mu \), the variations
with $\psi$ correspond to the well known difference of the Roche lobe dimensions in the $y$- and $z$-directions.

The values of $x_L$, $x_0$ and $x_{g0}$ give the limits of the Roche lobe illuminated by the compact primary companion. This region is shifted from the secondary's center towards the inner Lagrangian point $L_1$, thus the 'effective center of the reprocessed emission' has the radial velocity which is intermediate between those of the secondary and the inner Lagrangian point. The 'escape velocity' $v_e$ needed to reach the outer Lagrangian point $L_3$ (at $x_{L3}$) from the inner one is measured in units of the orbital velocity $v_{orb} = (GM/\alpha)^{1/2}$ and is very large as compared with the velocity of plasma ejection from the inner Lagrangian point.

References