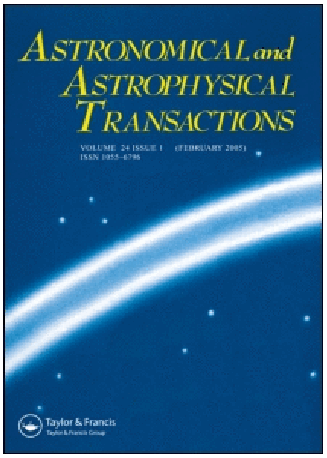

Astronomical & Astrophysical Transactions
The Journal of the Eurasian Astronomical Society
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713453505

Analytical approximations for some functions in the roche model

Ivan L. Andronov [a]
[a] Department of Astronomy, Odessa State University, Odessa, SU

Online Publication Date: 01 October 1992
To cite this Article: Andronov, Ivan L. (1992) 'Analytical approximations for some functions in the roche model', Astronomical & Astrophysical Transactions, 2:4, 341 - 345
To link to this article: DOI: 10.1080/10556799208205346
URL: http://dx.doi.org/10.1080/10556799208205346

Astronomical and Astrophysical Transactions, 1992,
Vol. 2, pp. 341–345
Reprints available directly from the publisher.
Photocopying permitted by license only

ANALYTICAL APPROXIMATIONS FOR SOME FUNCTIONS IN THE ROCHE MODEL

IVAN L. ANDRONOV

Department of Astronomy, Odessa State University T. G. Shevchenko Park, Odessa SU-270014

(Received November 21, 1990; in final form January 10, 1992)

Analytical approximations and tables are presented for the following functions in the Roche model: the 'barotropic' radius of the secondary companion r_e, the characterising angles of the Roche lobe seen from the center of the other star and from the inner Lagrangian point, the coordinates of the points at the border line separating the illuminated part of the Roche lobe and the dark one. By using Eggleton's (1983) form for the effective radius, $r = Aq^{2/3}(Bq^{2/3} + \log(1 + q^{1/3})$, we derived the values $A = 0.4990$, 0.4394 and $B = 0.5053$, 0.5333 for the function $\sin\theta$ in the y- and z- directions, respectively (where θ is the angular radius of the Roche lobe seen from the other star). For the mass ratio $q = M_2/M_1 \leq 4$, the maximal relative errors do not exceed 0.2 and 0.5 per cent, respectively. A more precise approximation for the 'inclination–eclipse duration' relation is derived.

KEY WORDS Stars, binaries–stars, cataclysmic.

Even though properties of the Roche lobe are well known (e.g. Kopal, 1978; Paczynski, 1971), recently some attempts were done to compute detailed tables of different functions in the Roche model (Chanan *et al.*, 1976; Mochnacki, 1984; Pennington, 1985; Todoran, 1990), especially of the 'effective' Roche radius r_e, which is usually defined as the radius r_v of the sphere whose volume is equal to the Roche lobe volume (Eggleton, 1983 and references therein). Andronov (1982) argued for a different definition of r_e, which may be assumed from the barothropic model, and is equal to the parameter r_0 of Kopal (1978). Here we tabulate also the angular dimensions of the star filling its Roche lobe as seen from its companion in the orbital plane $\theta(0°)$ and in the orthogonal plane θ (90°), and the coordinates x_0 and x_{90} at the Roche lobe corresponding to the boundary of the stellar surface illuminated by its compact companion. The best fit approximations for r_b, θ (0°), θ (90°) and the 'effective' angular dimension θ_e of the Roche lobe, are presented as well. Consider the reference frame with the origin at the secondary companion (the star filling its Roche lobe) of mass M_2. The mass of the compact primary is M_1. Using α (the distance between the centers of the stars), $M = M_1 + M_2$ and $U_0 = GM/\alpha$ as unit distance, mass and potential, respectively, the Jacobi integral can be written as

$$-\nu/r_1 - \mu/r_2 - ((x-\nu)^2 + y^2)/2 = C, \tag{1}$$

where $\mu = M_2/(M_1 + M_2)$, $\nu = 1 - \mu$, $r_1^2 = (1-x)^2 + y^2 + z^2$ and $r_2^2 = x^2 + y^2 + z^2$. Here $C = -c/2$ (where the constant c is the one used by Kopal (1978)) and thus C has the meaning of the potential as usually defined in mechanics (Landau and Lifshitz, 1973).

The 'barothropic' radius r_b can be defined as $r_b = -\mu/u$, where $u = v + v^2 - \mu/x_L - v/(1-x_L) - (x_L - v)^2/2$, and x_L is the coordinate of the inner Lagrangian point (cf. Andronov, 1982).

To obtain the boundary of the region at secondary's surface illuminated by the primary, we introduce the following spherical coordinates centered at the position of the primary: θ, the angle between the 'line of sight' and the 'line of centers', and the angle ψ between the orbital plane and the plane defined by the 'line of sight' and the 'line of centers'. At the boundary, θ is a function of ψ. In dimensionless units, the value of $\sin\theta$ is the distance between a point at the boundary and the center of the secondary companion. Neglecting variations of $\sin\theta$ with ψ (our computations show that $\sin\theta\,(90°)/\sin\theta\,(0°) = 0.9628 \pm 0.0009$ for $\mu \leqslant 0.5$), one may write the relation between μ, inclination i and the phase of the eclipse of the primary ϕ in the form:

$$\begin{aligned}\sin^2\theta &= \cos^2 i + \sin^2 i \cdot \sin^2\phi \\ &= \sin^2\phi + \cos^2\phi\cos^2 i = 1 - \sin^2 i\cos^2\phi \end{aligned} \tag{2}$$

(cf. Horne, 1980, Shafter, 1984; Downes *et al.*, 1986; Garnavich *et al.*, 1990). All these authors supposed that $\sin\theta = r_e$ and used the approximation of Eggleton (1983):

$$r_e = 0.49q^{2/3}/[0.6q^{2/3} + \log(1 + q^{1/3})]. \tag{3}$$

Here r_e is supposed to be equal to r_v. Martin *et al.* (1987) pointed out that r_v is by 3 per cent larger than r_e, and proposed a corrected value of r_e, neglecting the dependense $\theta(\psi)$. Some other relations can be obtained from spherical trigonometry:

$$\begin{aligned}\sin\theta\sin\psi &= \cos i,\\ \sin\theta\cos\psi &= \sin i\sin\phi,\\ \cos\theta &= \sin i\cos\phi.\end{aligned} \tag{4}$$

The value of ψ can be obtained from the following expression:

$$\sin^2\psi = \cos^2 i/(\cos^2 i + \sin^2 i \cdot \sin^2\phi), \tag{5}$$

while the values i and ϕ can be determined from observations. Numerical results can be approximated by

$$\sin\theta(\psi) = \sin\theta\,(0°) + [\sin\theta\,(90°) - \sin\theta\,(0°)]\sin^2\psi. \tag{6}$$

An exact value of $\sin\theta(\psi)$ is smaller than that given by this expression, but the relative error does not exceed $5 \cdot 10^{-4}$ for intermediate values of ψ. The value $\theta\,(0°)$ corresponds to the half-width of the eclipse of the compact companion when $i = 90°$; the eclipses occur when the inclination i exceeds the value $90° - \theta\,(90°)$.

To derive the $\phi - i$ relation for a fixed value of q, one may use the interpolating expression

$$\sin^2\theta(\psi) = \sin^2\theta\,(0°) + (\sin^2\theta\,(90°) - \sin^2\theta\,(0°))\sin^2\psi = \alpha + \beta\sin^2\psi. \tag{7}$$

Thus one may obtain from Eqs. (2), (5) and (7) the following expression:

$$\sin^2 i\cos^2\phi = (2 - \alpha - [\alpha^2 + 4\beta(1 - \sin^2 i)]^{1/2})/2, \tag{8}$$

which allows to compute ϕ for each appropriate value of i. In the previously published '$q - i$' and '$\psi - i$' diagrams (Horne, 1985; Shafter *et al.*, 1988), the approximations $a = r_v^2$, $\beta = 0$ were assumed. In this case the systematic error of Eq. (2) reaches a few percent, i.e. a few degrees in i or ϕ. Thus the more accurate Eq. (8) is recommended to be used when evaluating the '$i - \phi$' diagrams. Of course, to obtain the exact dependence $i(\phi)$ one should use precise values of $\theta(\psi)$ and to solve Eq. (2) numerically (cf. Chanan *et al.*, 1976). However, the good approximation provided by Eq. (7) allow to neglect the arbitrary deviations of a few 10^{-4}.

The solid angle Ω of the secondary companion can be evaluated as

$$\Omega = \int_0^{2\pi} (1 - \cos\theta(\psi))\, d\psi. \tag{9}$$

The total energy heating the secondary companion is $L \cdot \Omega/4\pi$, where L is the luminosity of the primary one. The numerical values of $\Omega/4\pi$ are given in Table 1. One may use the following approximation (cf. Andronov, 1986):

$$\Omega/4\pi = \begin{cases} 0.0473\mu^{2/3} & (m \leqslant 0.2), \\ 0.004 + 0.061\mu & (0.1 \leqslant \mu \leqslant 0.5). \end{cases} \tag{10}$$

For Ω itself, the corresponding best fit coefficients are 0.594, 0.051 and 0.766, respectively. The 'effective' angular radius of the Roche lobe θ_e may be thus defined from the expression $\Omega(\psi) = 2\pi[1 - \cos\theta_e(\psi)]$. It may be noted that the difference between $\sin\theta_e$ and r_b does not exceed 2.5 percent for $\mu \leqslant 0.7$.

We used Eggleton's (1983) form

$$r_e = Aq^{2/3}/[Bq^{2/3} + \log(1 + q^{1/3})], \tag{11}$$

where $q = \mu/(1-\mu)$, to obtain the coefficients for the abovementioned quantities: $r_b(A = 0.4660,\ B = 0.5929,\ \delta = 0.010)$, $\sin\theta\,(0°)$ $(A = 0.4990,\ B = 0.5053,\ \delta = 0.002)$, $\sin\theta\,(90°)$ $(A = 0.4394,\ B = 0.5333,\ \delta = 0.005)$, $\sin\theta_e$ $(A = 0.4441,\ B =$

Table 1 The characteristic angles of the Roche lobe as functions of μ

μ	θ (0°)	θ (90°)	θ_e	$\Omega/4\pi$	γ (0°)	γ (90°)	D
0.00	0.	0.	0.	0.	60.000	56.310	4.000
0.05	9.343785	8.992900	9.164273	0.006382	58.203	55.812	5.984
0.10	11.760972	11.326513	11.538803	0.010105	57.879	55.718	6.584
0.15	13.516905	13.017912	13.261752	0.013334	57.691	55.663	6.994
0.20	14.978535	14.421162	14.693491	0.016352	57.565	55.625	7.298
0.25	16.278380	15.664040	15.964114	0.019283	57.476	55.599	7.531
0.30	17.482702	16.810180	17.138532	0.022202	57.412	55.580	7.708
0.35	18.631542	17.897690	18.255789	0.025166	57.366	55.556	7.839
0.40	19.752850	18.952756	19.342913	0.028223	57.336	55.557	7.929
0.45	20.868814	19.995684	20.421107	0.031423	57.318	55.552	7.983
0.50	21.999356	21.044155	21.509111	0.034820	57.312	55.550	8.000
0.60	24.387086	23.228099	23.790803	0.042488	*	*	*
0.70	27.125432	25.674123	26.376087	0.052051			
0.80	30.611087	28.683105	29.609944	0.065295			
0.90	36.012306	33.085214	34.475423	0.087815			
0.95	40.980637	36.837125	38.778501	0.110213			

Note: The values of γ (0°), γ (90°) and D remain the same when μ is replaced by $1 - \mu$.

Table 2 The position of the Langrangian and boundary points and v_{cr}

μ	x_L	r_b	x_b (0°)	x_b (90°)	v_{cr}	x_{L3}
0.00	0.	0.	0.	0.	0.	2.
0.05	0.234775	0.161834	0.030717	0.030263	0.608682	1.970826
0.10	0.290965	0.202644	0.047950	0.047312	0.705248	1.941609
0.15	0.330260	0.231785	0.062612	0.061830	0.753672	1.912299
0.20	0.361924	0.255648	0.076115	0.075206	0.779316	1.882839
0.25	0.389257	0.276524	0.089065	0.088035	0.791023	1.853167
0.30	0.413870	0.295545	0.101817	0.100665	0.792969	1.823206
0.35	0.436705	0.313373	0.114627	0.113347	0.787437	1.792867
0.40	0.458382	0.330454	0.127709	0.126291	0.775778	1.762045
0.45	0.479358	0.347115	0.141269	0.139700	0.758814	1.730611
0.50	0.500000	0.363636	0.155528	0.153787	0.737024	1.698406
0.60	0.541618	0.397231	0.187247	0.185067	0.679687	1.630813
0.70	0.586130	0.433416	0.226020	0.223175	0.603106	1.556735
0.80	0.638076	0.475532	0.278487	0.274455	0.502255	1.471049
0.90	0.709035	0.531451	0.364954	0.358052	0.360928	1.359700
0.95	0.765225	0.572648	0.447896	0.436959	0.256948	1.278094

0.5195, $\delta = 0.002$), where δ is the maximal relative difference between exact values and those approximated by the best fit expressions. These fits were obtained for $\mu \leqslant 0.8$, because for the larger values of μ the accuracy becomes very poor and in addition, these values are not realistic for CV's. For r_v, Eggleton (1983) obtained $A = 0.49$ and $B = 0.6$, which yields the values ≈4–6 percent higher than those of r_b. Masevich and Tutukov (1988) used a simpler approximation

$$r_v = 0.52\mu^{0.44}, \tag{12}$$

which is applicable for a smaller subinterval of q from 0.3 to 2 and can be used for evolutionary computations.

However, for small values of μ, the form $r_c = A\mu^{1/3}$ (cf. Paczynski, 1971; Patterson, 1981) is more accurate. For $\mu \leqslant 0.45$, we obtained $A = 0.4418$, 0.4477, 0.4309 and 0.4391 ($\delta = 0.025$, 0.038, 0.036, 0.037) for r_b, $\sin\theta$ (0°), $\sin\theta$ (90°) and $\sin\theta_e$, respectively. The mean squared relative deviations are 2–3 times smaller than δ.

In the vicinity of the inner Lagrangian point, the dimensionless Jacobi potential may be approximated by the following expression:

$$U(x, y, z) = U(x_L, 0, 0) - \tfrac{1}{2}(2D + 1)(x - x_L)^2 + \tfrac{1}{2}(D - 1)y^2 + \tfrac{1}{2}Dz^2, \tag{13}$$

where

$$D = \mu/x_L^3 + \nu/(1 - x_L)^3 \tag{14}$$

(Andronov, 1984 and references therein). Equation $U(x, y, z) = U(x_L, 0, 0)$ defines the cone coinciding with the Roche lobe in the nearest vicinity of the inner Lagrangian point. The angle $\gamma(\psi)$ between the line of centers and the corresponding line at the cone is the function of ψ and μ. However, for reasonable values of μ ranging from 0.1 to 0.9, the value of $\gamma(\psi)$ varies within a narrow range not exceeding 1 per cent (for fixed ψ). For fixed μ, the variations with ψ correspond to the well known difference of the Roche lobe dimensions in the y- and z-directions.

The values of x_L, x_0 and x_{90} give the limits of the Roche lobe illuminated by the compact primary companion. This region is shifted from the secondary's center towards the inner Lagrangian point L_1, thus the 'effective center of the reprocessed emission' has the radial velocity which is intermediate between those of the secondary and the inner Lagrangian point. The 'escape velocity' v_e needed to reach the outer Lagrangian point L_3 (at x_{L3}) from the inner one is measured in units of the orbital velocity $v_{orb} = (GM/\alpha)^{1/2}$ and is very large as compared with the velocity of plasma ejection from the inner Lagrangian point.

References

Andronov, I. L. (1982). *Problems of Cosmic Physics (Kiev)* **17,** 106.
Andronov, I. L. (1984). *Astrofizika* **20,** 165.
Andronov, I. L. (1986). *Preprint UkrNIINTI (Kiev),* **358,** 64 pp.
Chanan, G. A., Middleditch, J. and Nelson, J. E. (1976). *ApJ* **208,** 512.
Downes, R. A., Mateo, M., Szkody, P., Jenner, D. C. and Margon, B. (1986). *ApJ* **301,** 240.
Eggleton, P. (1983). *ApJ* **268,** 368.
Garnavich, P. M., Szkody, P., Mateo, M., Feinswog, L., Booth, J., Goodrich, B., Miller, H. R., Carini, M. T. and Wilson, J. W. (1990). *ApJ* **365,** 696.
Horne, K. (1980). *ApJ* **242,** L167.
Horne, K. (1985). *MNRAS* **213,** 129.
Horne, K., Lanning, H. H. and Gomer, R. H. (1982). *ApJ* **252,** 681.
Kopal, Z. (1978). *Dynamics of Close Binary Systems,* Dordrecht, 512 pp.
Landau, L. D. and Lifshitz, E. M. (1973). *Mechanics* (in Russian), Moscow, 208 pp.
Martin, J. S., Jones, D. H. P. and Smith, R. C. (1987). *MNRAS* **224,** 1031.
Masevich, A. G. and Tutukov, A. V. (1988). *Evolution of Stars: Theory and Observations* (in Russian), Moscow, 'Nauka', 280 pp.
Mochnacki, S. W. (1984). *ApJ Suppl.* **55,** 551.
Paczynski, B. (1971). *Ann. Rev. Astron. Astrophys.* **9,** 183.
Patterson, J. (1981). *ApJ Suppl.* **45,** 517.
Pennington, R. (1985). In: *Interacting Binary Stars,* eds. J. E. Pringle and R. A. Wade, Cambridge Univ. Press, p. 197.
Shafter, A. W. (1984). *Astron. J.* **89,** N 1545, 1555.
Shafter, A. W., Hessman, F. V. and Zhang, E. H. (1988). *ApJ* **327,** 248.
Todoran, I. (1990). *Contr. Astron. Obs. Skalnate Pleso,* **20,** 19.